2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Distributed Backstepping Control for Nonlinear Switched Interconnected Systems: An Application of Water Transport Systems

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Abstract— **In this paper, we propose a distributed model-based backstepping control approach for a class of nonlinear switched interconnected systems with a strict-feedback structure. The proposed control architecture effectively addresses the technical challenges that arise from strong interconnection terms without imposing restrictive assumptions on them, which have not been reported in existing literature. Specifically, we utilize a distributed control architecture to handle the switched strongly interconnected systems. The model-based distributed controller is then constructed by combining common Lyapunov functionals and the backstepping scheme. This controller ensures the uniform boundedness of all the closed-loop variables, while the tracking error converges to a tunable small compact set that includes the origin. Finally, the applicability of the proposed controller is demonstrated through the numerical simulations of a water testbed representing a water transport network.**

Index Terms: **Distributed control, Backstepping, Interconnected systems, Control applications, Water transport systems, Testbed.**

I. INTRODUCTION

In the past decades, investigation on control of interconnected systems using a decentralized method has garnered significant attention due to its wide applications in different contexts, such as water transport systems [1], power distribution systems [2], etc. In a decentralized control, each subsystem is locally controlled without the need for data transmissions with its neighbors. This yields key advantages, including reduced computational burden compared to centralized schemes and enhanced reliability. However, it is essential to note that a fully decentralized controller design also has limitations, proving completely effective only in the weakly interconnected systems, see, for example, [3]–[5].

Regarding the applications of strongly interconnected nonlinear systems in real-world, such as the water testbed in [6] which serves as a benchmark example of a water transport system, researchers have been studying decentralized control methods in recent years to address the challenges that have arisen in the stability analysis of such systems, e.g., [7]–[11]. However, all of these results are obtained under conservative assumptions, such as boundedness of interconnected terms with known linear functions, satisfying the linear growth condition for interconnected terms along with prior information about their growth rates, or considering strongly connected digraphs. On the other hand, in these results, the local controller of each subsystem is frequently forced to generate control input with large amplitudes to compensate the effects of unknown interconnections, which may necessitate the use of high-gain feedback, (i.e., robust control action). To address these restrictions, efforts have been made in *distributed control design* for strongly interconnected nonlinear systems with unknown dynamics, [12]–[15]. Unlike the decentralized methodology, in distributed control design, each subsystem can utilize local information from its neighbors for the purpose of controller design. Therefore, in [12]–[15], the control algorithms compensate for the effects of unknown interconnections based on the online approximation theorem and shared information between neighbors, all while avoiding the limited assumptions found in [7]–[11]. However, the proposed schemes in [12]–[15] are only applicable to *non-switched interconnected* systems with only *matched dynamics*.

It is well known that many practical systems can be represented as a switched interconnected system in strict-feedback form, (i.e., systems with mismatched dynamics). An example of such systems is water transport systems, the dynamics of which are evident in [6]. Hence, controller design and closed-loop stability analysis for such systems are crucial in theory and applications. Accordingly, some decentralized control strategies have been presented for switched interconnected systems under different contexts, such as sampled date control [16], small gain theorem [17], and output-feedback design [18]. It is important to note that these studies do not take strong interconnection dynamics into consideration.

Inspired by the preceding discussions, controller design for a class of switched strongly interconnected systems is of interest for practical applications. Nevertheless, few results have been obtained in this field. In this paper, our focus is on designing and analysing a distributed model-based controller for nonlinear switched strongly interconnected systems with a strict-feedback structure. Our main contributions are: First, for the considered class of interconnected systems, the tracking control problem is investigated based on the data transmissions between neighboring controllers in a distributed manner. By utilizing this shared data, the effects of mismatched nonlinearities, including the strongly interconnected terms, are compensated through the design of appropriate virtual and actual control laws using a backstepping approach. Second, since many practical applications involve slowly switched systems, such as water transport systems, the closed-loop stability of the proposed controller is evaluated by using a common Lyapunov function. Third, the validity of the

This paper is supported in part by funding from the European Research Council (ERC) under grant agreement No. 951424 (Water Futures) and the European Union's Horizon 2020 research and Innovation programme under grant agreement No. 739551 (KIOS CoE).

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proposed controller is verified through a numerical example on a water testbed which emulates a water transport system.

II. SYSTEM DESCRIPTION AND BASIC ASSUMPTIONS

In this paper, we consider a switched strongly interconnected system consisting of N nonlinear strict-feedback subsystems. The dynamics of the i-th subsystem, where $i \in \mathcal{N} \triangleq \{1, 2, ..., N\}$, are described by

$$\begin{aligned}\dot{x}_{i,k} =& x_{i,k+1} + f_{i,k}^{\sigma_i}(\bar{x}_{i,k}) + \sum_{j\in\mathcal{N}_i} \delta_{ij,k}^{\sigma_i}(\bar{x}_{j,n_j}) + d_{i,k}^{\sigma_i},\\ k =& 1, 2, ..., n_i - 1,\\ \dot{x}_{i,n_i} =& u_i^{\sigma_i} + f_{i,n_i}^{\sigma_i}(\bar{x}_{i,n_i}) + \sum_{j\in\mathcal{N}_i} \delta_{ij,n_i}^{\sigma_i}(\bar{x}_{j,n_j}) + d_{i,n_i}^{\sigma_i},\\ y_i =& x_{i,1},\end{aligned} \tag{1}$$

where $\bar{x}_{i,n_i} = [x_{i,1}, ..., x_{i,n_i}]^{\mathrm{T}} \in \mathbb{R}^{n_i}$ is the measurable state vector of i-th subsystem, $\bar{x}_{i,k} = [x_{i,1}, ..., x_{i,k}]^{\mathrm{T}} \in \mathbb{R}^k$, $y_i \in \mathbb{R}$ is the output of the i-th subsystem, and $u_i^{\sigma_i} \in \mathbb{R}$ is the control input of i-th subsystem. $\sigma_i : \mathbb{R}^+ \mapsto \mathbb{Q}_i = \{1, 2, ..., q_i\}$ is a time-dependent piecewise right continuous switching signal for i-th subsystem, and q_i is the number of operation modes in i-th subsystem. The notation $\mathcal{N}_i$ describes the set of all the neighboring subsystems for the i-th subsystem, which denotes all the subsystems that the dynamics of the i-th subsystem are affected by their states. The nonlinear terms $f_{i,s}^{\sigma_i}(\bar{x}_{i,s}) : \mathbb{R}^{n_s} \mapsto \mathbb{R}$ and $\delta_{ij,s}^{\sigma_i}(\bar{x}_{j,n_j}) : \mathbb{R}^{n_j} \mapsto \mathbb{R}$ for $s = 1, 2, ..., n_i$ and $j \in \mathcal{N}_i$ represent known mappings that denote the local dynamics of the i-th subsystem and the effects of the neighbors of the i-th subsystem on its dynamics, (i.e., strong interconnections), respectively. Both of these nonlinear dynamics satisfy the smoothness property. The term $d_{i,s}^{\sigma_i} : \mathbb{R}^+ \mapsto \mathbb{R}$ is the time-dependent external disturbance in the i-th subsystem.

The main control objective of this paper is to design a distributed model-based backstepping control approach for a class of nonlinear switched strict-feedback system with strong interconnected terms such that all the closed-loop signals remain uniformly bounded.

To attain the previously mentioned control objective, we employ the following Assumptions:

Assumption 1: For all $i \in \mathcal{N}$, the external disturbances are assumed to be unknown but bounded, with an unknown bound, i.e., $|d_{i,s}^{\sigma_i}| \le \bar{d}_i^{\sigma_i}$, where $\bar{d}_i^{\sigma_i}$ is an unknown constant.

Assumption 2: The desired trajectory $x_{i,d}$ and its time-derivatives up to order n_i for the i-th subsystem are known, smooth, and bounded.

III. THEORETICAL RESULTS

In this section, we provide the design details of the distributed model-based backstepping architecture for controlling nonlinear systems in (1). The stability of the overall closed-loop system will be demonstrated utilizing the common Lyapunov approach. Hereafter, the notation $\sigma_i = l$ indicates the activation of the l-th mode in the i-th subsystem. For the l-th activated mode of i-th subsystem, the n_i-step distributed backstepping control design is based on the following common change of coordinate [19]:

$$e_{i,s} = x_{i,s} - \omega_{i,s-1}^l, \quad s = 1, 2, ..., n_i, \tag{2}$$

where $e_{i,s}$ is the error surface, $\omega_{i,s-1}^l$ is the virtual control law to be specified later, and $\omega_{i,0}^l = x_{i,d}$.

Initial step: The derivative of the error surface $e_{i,1}$ in (2) along with (1) results in

$$\dot{e}_{i,1} = x_{i,2} + f_{i,1}^l(\bar{x}_{i,1}) + \sum_{j\in\mathcal{N}_i} \delta_{ij,1}^l(\bar{x}_{j,n_j}) + d_{i,1}^l + \dot{x}_{i,d}. \tag{3}$$

The common Lyapunov candidate function is introduced as $\mathcal{V}_{i,k} = \frac{1}{2}e_{i,k}^2$, whose derivative using (2) is given by

$$\begin{aligned}\dot{\mathcal{V}}_{i,1} = e_{i,1}\Big(&e_{i,2} + \omega_{i,1}^l + f_{i,1}^l(\bar{x}_{i,1}) + \sum_{j\in\mathcal{N}_i} \delta_{ij,1}^l(\bar{x}_{j,n_j})\\ &+ d_{i,1}^l - \dot{x}_{i,d}\Big).\end{aligned} \tag{4}$$

To proceed, we design the following feasible model-based distributed virtual control law as follows

$$\omega_{i,1}^l = -c_{i,1}e_{i,1} - f_{i,1}^l(\bar{x}_{i,1}) - \sum_{j\in\mathcal{N}_i} \delta_{ij,1}^l(\bar{x}_{j,n_j}) + \dot{x}_{i,d}, \tag{5}$$

where $c_{i,1} > 0$ is a control gain.

Then, by substituting (5) into (4), and employing Young's inequality as $e_{i,1}(e_{i,2} + d_{i,1}^l) \le e_{i,1}^2 + \frac{1}{2}e_{i,2}^2 + \frac{1}{2}d_{i,1}^2 2$ along with the Assumption 1, the following can be obtained

$$\dot{\mathcal{V}}_{i,1} \le -(c_{i,1} - 1)e_{i,1}^2 + \frac{1}{2}e_{i,2}^2 + \frac{1}{2}\bar{d}_{i,1}^2. \tag{6}$$

Step k for $(k = 2, 3, \ldots, n_i)$: From (1) and (2), one has

$$\begin{aligned}\dot{e}_{i,k} =& e_{i,k+1} + \omega_{i,k}^l + f_{i,k}^l(\bar{x}_{i,k}) + \sum_{j\in\mathcal{N}_i} \delta_{ij,k}^l(\bar{x}_{j,n_j}) + d_{i,k}^l\\ &- \dot{\omega}_{i,k-1}^l,\end{aligned} \tag{7}$$

where $u_i^l = \omega_{i,n_i}^l$ and $e_{i,n_i+1} = 0$.

Select the common Lyapunov candidate function as $\mathcal{V}_{i,k} = \frac{1}{2}e_{i,k}^2$. From (7) the derivative of $\mathcal{V}_{i,k}$ is computed by

$$\begin{aligned}\dot{\mathcal{V}}_{i,k} = e_{i,k}\Big(&e_{i,k+1} + \omega_{i,k}^l + f_{i,k}^l(\bar{x}_{i,k}) + \sum_{j\in\mathcal{N}_i} \delta_{ij,k}^l(\bar{x}_{j,n_j})\\ &+ d_{i,k}^l - \dot{\omega}_{i,k-1}^l\Big).\end{aligned} \tag{8}$$

The following feasible model-based distributed control law is designed

$$\begin{aligned}\omega_{i,k}^l =& -c_{i,k}e_{i,k} - f_{i,k}^l(\bar{x}_{i,k}) - \sum_{j\in\mathcal{N}_i} \delta_{ij,k}^l(\bar{x}_{j,n_j}) + \dot{\omega}_{i,(k-1)q}^l\\ &- \frac{1}{2}e_{i,k}\Big(\sum_{m=1}^{k-1}\Big(\frac{\partial \omega_{i,(k-1)}^l}{\partial x_{i,m}}\Big)^2 - \sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}\Big(\frac{\partial \omega_{i,(k-1)}^l}{\partial x_{j,m}}\Big)^2\Big),\end{aligned} \tag{9}$$

where $c_{i,k} > 0$ is a control gain and $\dot{\omega}_{i,(k-1)q}^l = \sum_{m=0}^{k-1} \frac{\partial \omega_{i,(k-1)}^l}{\partial x_{i,d}^{(m)}} x_{i,d}^{(m+1)} + \sum_{m=1}^{k-1} \frac{\partial \omega_{i,(k-1)}^l}{\partial x_{i,m}}(\dot{x}_{i,m} - d_{i,m}) + \sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j} \frac{\partial \omega_{i,(k-1)}^l}{\partial x_{j,m}}(\dot{x}_{j,m} - d_{j,m})$.

It is important to highlight that the designed control laws in (5) and (9) are feasible since both are independent of prior knowledge of disturbances.

Then, by substituting (9) into (8), and applying the Young's inequality for

$$e_{i,k}(e_{i,k+1}+d^l_{i,k}) \leq e^2_{i,k}+\frac{1}{2}e^2_{i,k+1}+\frac{1}{2}d^2_{i,k}, \tag{10}$$

$$2e_{i,k}\sum_{m=1}^{k-1}\frac{\partial\omega^l_{i,(k-1)}}{\partial x_{i,m}}d_{i,m} \leq e^2_{i,k}\sum_{m=1}^{k-1}\Big(\frac{\partial\omega^l_{i,(k-1)}}{\partial x_{i,m}}\Big)^2+\sum_{m=1}^{k-1}d^2_{i,m}, \tag{11}$$

$$2e_{i,k}\sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}\frac{\partial\omega^l_{i,(k-1)}}{\partial x_{j,m}}d_{j,m} \leq e^2_{i,k}\sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}\Big(\frac{\partial\omega^l_{i,(k-1)}}{\partial x_{j,m}}\Big)^2 + \sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}d^2_{j,m}, \tag{12}$$

along with the Assumption 1, the following can be obtained

$$\dot{\mathcal{V}}_{i,k} \leq -\big(c_{i,k}-1\big)e^2_{i,k}+\frac{1}{2}e^2_{i,k+1}+\frac{1}{2}\bar{d}^2_{i,k}+\frac{1}{2}\sum_{m=1}^{k-1}\bar{d}^2_{i,m} + \frac{1}{2}\sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}\bar{d}^2_{j,m}. \tag{13}$$

The designed fully distributed controller is summarized below.

Theorem 1: Under Assumptions 1-3, consider the closed-loop switched interconnected system including the plant (1), and the distributed control laws (5) and (9), respectively. For the bounded initial conditions there exists control gain $c_{i,s} > 1.5$ such that: 1) all the closed-loop signals are globally uniformly bounded, and 2) the tracking error vector $e_1(t) = [e_{1,1}(t), ..., e_{N,1}(t)]^{\mathrm{T}}$ converges to the following compact set:

$$\Psi \triangleq \left\{ e_1(t)\in\mathbb{R}^N \,\middle|\, \lim_{t\to\infty}||e_1(t)||^2 \leq 2\frac{\mathcal{W}}{\mathcal{C}} \right\}, \tag{14}$$

where $\mathcal{C} > 0$ and $\mathcal{W} > 0$ are two constants, which will be specified later in the proof.

Proof: To consider the overall closed-loop stability of the interconnected system subject to the switching dynamics, we define the following total Lyapunov function candidate as $\mathcal{V} = \sum_{i=1}^N\sum_{s=1}^{n_i}\mathcal{V}_{i,s}$. Then, the time-derivative of $\mathcal{V}$ by following (6) and (13) is obtained by

$$\begin{aligned}\dot{\mathcal{V}} \leq& \sum_{i=1}^N\left(\sum_{s=1}^{n_i}\Big(-(c_{i,s}-1.5)e^2_{i,s}+\frac{1}{2}\bar{d}^2_{i,s}\Big) + \frac{1}{2}\sum_{p=2}^{n_i}\sum_{m=1}^{p-1}\bar{d}^2_{i,m}+\frac{1}{2}(n_i-1)\sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}\bar{d}^2_{j,m}\right),\\ \leq& -\mathcal{C}\mathcal{V}+\mathcal{W},\end{aligned} \tag{15}$$

where $\mathcal{C} = \min_{\substack{i=1,2,...,N\\ s=1,2,...,n_i}}\{2(c_{i,s}-1.5)\}$ and $\mathcal{W} = \sum_{i=1}^N\Big(\sum_{s=1}^{n_i}\frac{1}{2}\bar{d}^2_{i,s}+\frac{1}{4}\sum_{p=2}^{n_i}\sum_{m=1}^{p-1}\bar{d}^2_{i,m}+\frac{1}{4}(n_i-1)\sum_{j\in\mathcal{N}_i}\sum_{m=1}^{n_j}\bar{d}^2_{j,m}\Big)$. By selecting control gains as $c_{i,s} > 1.5$, we can easily obtain from (15) that

$$\begin{aligned}\mathcal{V}(t) \leq& \Big(\mathcal{V}(t_0)-\frac{\mathcal{W}}{\mathcal{C}}\Big)\exp(-\mathcal{C}(t-t_0))+\frac{\mathcal{W}}{\mathcal{C}}, \quad \forall t\geq t_0,\\ \leq& \mathcal{V}(t_0)+\frac{\mathcal{W}}{\mathcal{C}},\end{aligned} \tag{16}$$

which implies $\mathcal{V}(t)$ is uniformly bounded. As a result, $\lim_{t\to\infty}\sum_{i=1}^N e^2_{i,1}(t) \leq 2\frac{\mathcal{W}}{\mathcal{C}}$ can be deduced. This reveals that the tracking errors converge to the tunable ultimate bound, as given in Theorem 1, which the size of this ultimate bound can be reduced by increasing the control gains, (i.e., increasing the $\mathcal{C}$). To tune the control gains, we apply the trail and error.

To implement the proposed distributed control laws, it is assumed that there exist ideal communication links between neighboring subsystems for data transmission, which is common in existing literatures such as [12]–[15].

Assumption 3: The interconnection terms in the plant (1) are bounded over a compact set Ω, i.e., $|\delta^{\sigma_i(t)}_{ij,s}(\bar{x}_{j,n_j})| \leq \vartheta_{ij,s}$ for all $\bar{x}_{j,n_j}\in\Omega\subset\mathbb{R}^{n_j}$, where $\vartheta^{\sigma_i}_{ij,s}$ is a known parameter.

In the following theorem, the fully decentralized control scheme for controlling plant (1) will be introduced.

Theorem 2: Under Assumptions 1-4, consider the closed-loop switched interconnected system including the plant (1), and the decentralized model-based virtual and the actual control laws as follows:

$$\omega^l_{i,1} = -c_{i,1}e_{i,1}-f^l_{i,1}(\bar{x}_{i,1})-\tanh\Big(\frac{e_{i,1}}{\kappa_{i,1}}\Big)\sum_{j\in\mathcal{N}_i}\vartheta^l_{ij,1}+\dot{x}_{i,d} \tag{17}$$

$$\omega^l_{i,k} = -c_{i,k}e_{i,k}-f^l_{i,k}(\bar{x}_{i,k})-\tanh\Big(\frac{e_{i,k}}{\kappa_{i,k}}\Big)\sum_{j\in\mathcal{N}_i}\vartheta^l_{ij,k} - \frac{1}{2}e_{i,k}\sum_{m=1}^{k-1}\Big(\frac{\partial\omega^l_{i,(k-1)}}{\partial x_{i,m}}\Big)^2+\dot{\omega}^l_{i,(k-1)q}, \tag{18}$$

where $\dot{\omega}^l_{i,(k-1)q} = \sum_{m=0}^{k-1}\frac{\partial\omega^l_{i,(k-1)}}{\partial x^{(m)}_{i,d}}x^{(m+1)}_{i,d} + \sum_{m=1}^{k-1}\frac{\partial\omega^l_{i,(k-1)}}{\partial x_{i,m}}(\dot{x}_{i,m}-d_{i,m})$. Then, for the bounded initial conditions within the compact set Ω there exist control gain $c_{i,s} > 1.5$ and design constant $\kappa_{i,s} > 0$ such that 1) all the closed-loop signals are semi-globally uniformly bounded; and 2) the tracking error vector converges to a compact set, similar to the one given in (14).

Proof: This Theorem can be easily proved under the given design procedure of the backstepping controller in Theorem 1, the introduced total Lyapunov candidate function in (13), and employing the following inequality for interconnected term $e_{i,s}\sum_{j\in\mathcal{N}_i}\delta^l_{ij,n_i}(\bar{x}_{j,n_j}) \leq e_{i,s}\tanh\Big(\frac{e_{i,s}}{\kappa_{i,s}}\Big)\sum_{j\in\mathcal{N}_i}\vartheta^l_{ij,s} + 0.2785\kappa_{i,s}$. The general form of this inequality can be founded in [20]. The remaining details of the proof are omitted here, as they are similar to the proof of Theorem 1.

In many real applications, such as water transport systems, maintaining the system states between predefined safety limits is more important than tracking a set-point with high accuracy. Therefore, the proposed distributed and decentralized controls are modified as in the following Propositions.

Proposition 1: Under Assumptions 1-3, consider the closed-loop switched interconnected system including the plant (1), the first distributed virtual control in (5), the kth distributed virtual controls in (9) for $k = 2, 3, \ldots, n_i - 1$, and the distributed actual control as follows:

$$u_i^l = \begin{cases} -c_{i,n_i} e_{i,n_i} - f_{i,n_i}^l(\bar{x}_{i,n_i}) - \sum_{j \in \mathcal{N}_i} \delta_{ij,n_i}^l(\bar{x}_{j,n_j}) \\ \mathcal{A} + \dot{\omega}_{i,(n_i-1)q}^l, \qquad |e_{i,1}| > \phi, & (19) \\ 0 \qquad\qquad\qquad |e_{i,1}| \leq \phi, & (20) \end{cases}$$

where $\mathcal{A} = -\frac{1}{2} e_{i,k} \Big(\sum_{m=1}^{k-1} \Big(\frac{\partial \omega_{i,(k-1)}^l}{\partial x_{i,m}} \Big)^2 - \sum_{j \in \mathcal{N}_i} \sum_{m=1}^{n_j} \Big(\frac{\partial \omega_{i,(k-1)}^l}{\partial x_{j,m}} \Big)^2 \Big)$ and $\dot{\omega}_{i,(n_i-1)q}^l$ is given under equation (9) by selecting $k = n_i$.Then, for the bounded initial conditions, there exist the control gain $c_{i,s} > 0$ and the predefined user constant $\phi > 0$ such that: 1) all the closed-loop signals are uniformly bounded; and 2) the tracking error vector converges to its ultimate bound as $\lim_{t \to \infty} ||e_1(t)|| \leq \phi \sqrt{N}$.

Proof: This proposition is easy to prove according to the presented proof of Theorem 1, therefore, it is omitted.

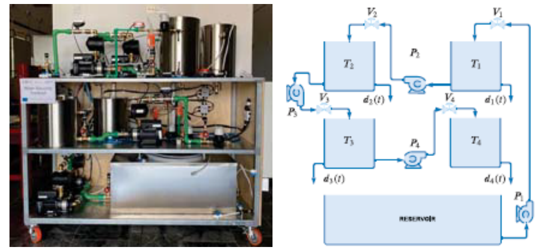
Proposition 2: Under Assumptions 1-4, the results of Proposition 1 remains valid for the decentralized controller if we replace the applied distributed control laws in Proposition 1 with decentralized ones introduced in (17) and (18).

IV. WATER TESTBED SYSTEM AND SIMULATION RESULTS

We consider the water testbed system which has been developed at the KIOS Research and Innovation Center of Excellence at the University of Cyprus, to support the advancement of smart water networks research, [6]. The fundamental components of this water testbed, depicted in Fig. 1(a) are the reservoir, the cylindrical water tanks, the pipes, the centrifugal pumps, the valves, and the sensors. These sensors are employed to measure the water level, water flow, and water pressure. The reservoir serves a dual purpose. The first one is to serve as infinite water source for the main pumping station, while the second one is is to be the sink for all outflows and emulated consumer demands. Water tanks are interconnected through the network of pipes, which enable water flow between the neighboring tanks and the reservoir. Water level in each tank is directly affected by water consumption, as well as water inflow and outflow. The topology of the water testbed is reconfigurable; in this work we consider the tank-in-series configuration where the output of each tank is the inflow to its neighbor. The control inputs of the examined water testbed are remotely controlled valves which operate between the interval $[0, 1]$, indicating a completely close or open valve respectively.

For the numerical simulation, consider series configuration of the testbed system, as illustrated in Fig. 1(b), which is comprised of 4 switched interconnected tanks. The dynamics of i-th tank, $i \in \mathcal{N} \triangleq \{1, 2, 3, 4\}$, are given by [6]

$$\begin{cases} \dot{h}_i = \frac{1}{\mathcal{A}_i} \Big[q_i - q_{i+1} - f_i(h_i) \Big], \\ \dot{q}_i = \begin{cases} \lambda_i u_i - \alpha_i q_i, & 0.1 \leq u_i < 1, \\ -\zeta_i q_i, & 0 \leq u_i < 0.1, \end{cases} \end{cases} \tag{21}$$



(a) (b)

Fig. 1: (a) The KIOS Water Testbed System; (b) Simplified illustration of the series configuration.

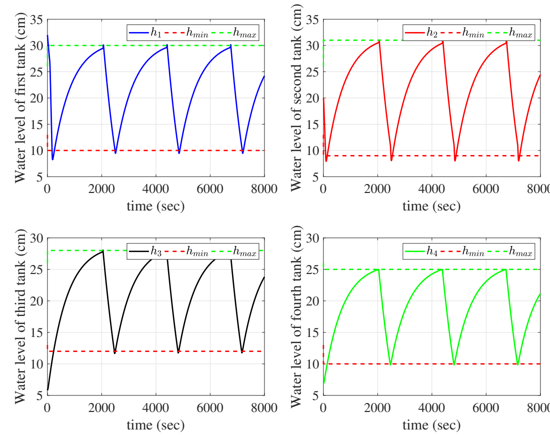


Fig. 2: Tank water levels.

where h_i (cm) is the water level of i-th tank (tank's output), q_i (cm^3/s) and q_{i+1} (cm^3/s) are the inflow and outflow rate of i-th tank subsystems, and $\mathcal{A}_i$ (cm^2) is the tank base area. The variable $u_i \in [0, 1]$ is the control input of i-th tank, and λ_i, ζ_i, and α_i are some positive constant parameters, which are assumed to be completely known. All of these parameters are identified utilizing the experimental data, and their numerical values can be found in [6]. Moreover, the variable $f_i(h_i)$ $(cm^3/$s) denotes the water consumption which is described by

$$f_i(h_i) = \gamma_i(t) \sqrt{2 g h_i}, \tag{22}$$

where $\gamma_i(t) > 0$ is a term representing the opening percentage of the consummation of the outflow. The dynamics of water flow evolve within the first-order switched differential equation for the i-th tank subsystem according to (21). Hence, it is clear that the represented model for the water testbed system in (21) is completely matched to the general form of switched strongly interconnected system in (1) since the interconnection term is not an output of the subsystems. Note that $f_i(h_i)$ is a non-Lipschitz function at $h_i = 0$, but this non-Lipschitz point is not in the operation range for tanks. Therefore, the proposed control approaches given in this paper are applicable for the considered water testbed.

The main control objective when designing the distributed model-based backstepping controller is to regulate the water level in each tank between predefined minimum and maximum

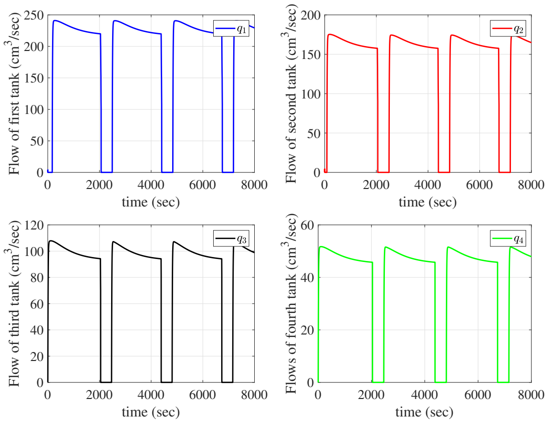


Fig. 3: Water flows (tank inflows).

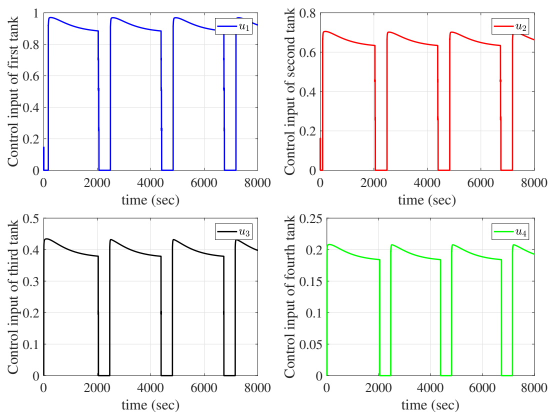


Fig. 4: Control inputs (value opening).

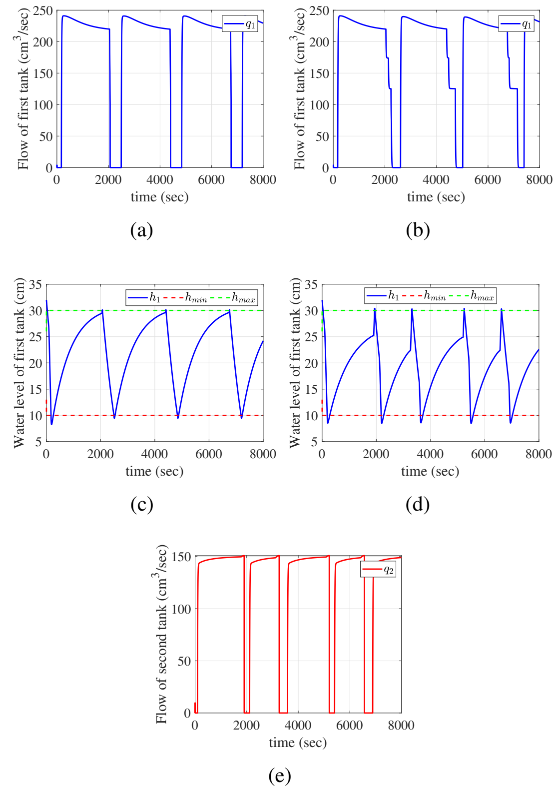


Fig. 5: Water flow of the first tank when controller is designed using (a): the switched model; (b): the non-switched model. Water level in the first tank under. (c): the distributed control; (d): the decentralized control; (c): Water flows of the second tank under the decentralized control.

water levels, denoted by $h_{i,\min}$ and $h_{i,\max}$ respectively, for all $i \in 1, 2, 3, 4$. We consider the 4-tank system of Fig. 1(b) for a simulation example.

It is well known that when designing controllers for switched systems, it is essential to be aware of which mode of the switched system is active at each instant. In the water testbed model represented in (21), the switching signal depends on the control input. Therefore, we must compute the control input at each instant to realize which subsystem is active. To accomplish this, we modify (21) as follows:

$$\begin{cases} \dot{h}_i = \frac{1}{A_i}\Big[q_i - q_{i+1} - f_i(h_i)\Big], \\ \dot{q}_i = \begin{cases} \lambda_i u_i - \alpha_i q_i, & 0.1 \leq u_i < 1, \\ \lambda_i u_i - \zeta_i q_i + d_i, & 0 \leq u_i < 0.1, \end{cases} \end{cases} \tag{23}$$

where $d_i = -\lambda_i u_i$ is a bounded disturbance like term.

We cannot apply the proposed controllers in Theorems 1 and 2 in this context directly, since the output of each subsystem needs to be regulated within specific minimum and maximum water level thresholds. To achieve this control objective, similar to Proposition 1, we use the following control structure: 1) we select $x_{i,d} = h_{i,\max}$ and compute the regulation error $e_{i,1} = h_i - h_{i,\max}$; 2) update the distributed and the decentralized non-switched virtual control respectively based on (24) and (25) as follows:

$$\omega_i = -c_{i,1}e_{i,1} + f_i(h_i) + q_{i+1}, \tag{24}$$

$$\omega_i = -c_{i,1}e_{i,1} + f_i(h_i) - \tanh\Big(\frac{e_{i,1}}{\kappa_{i,1}}\Big)\vartheta_{i+1}, \tag{25}$$

where we assume that the $|q_{i+1}| \leq \vartheta_{i+1}$.

3) compute the regulation error $e_{i,2} = q_i - \omega_{i,1}$ and update the distributed actual control law, based on (19), as follows:

3.1: if $h_i(t) \leq h_{i,\min}$, then

$$u_i = \frac{1}{\lambda_i}\big[-c_{i,2}e_{i,2} + \alpha_i q_i + \dot{\omega}_i\big], \quad 0.1 \leq u_i < 1, \tag{26}$$

$$u_i = \frac{1}{\lambda_i}\big[-c_{i,2}e_{i,2} + c_i q_i + \dot{\omega}_i\big], \quad 0 \leq u_i < 0.1, \tag{27}$$

3.2: elseif $h_i(t) \in (h_{i,\min},\ h_{i,\max})$ and $\dot{h}_i(t) \geq 0$, then

$$u_i = \frac{1}{\lambda_i}\big[-c_{i,2}e_{i,2} + \alpha_i q_i + \dot{\omega}_i\big], \quad 0.1 \leq u_i < 1, \tag{28}$$

$$u_i = \frac{1}{\lambda_i}\big[-c_{i,2}e_{i,2} + c_i q_i + \dot{\omega}_i\big], \quad 0 \leq u_i < 0.1, \tag{29}$$

3.3: else

$$u_i = 0. \tag{30}$$

From practical perspective, to guarantee the non-occurrence water overflowing in a tank, the maximum filling height (i.e.,

the desired value $x_{i,d}$) is selected in a safe zone.
To implement the simulation, we select the minimum and maximum of the water level thresholds as $h_{1,\min} = 10(cm)$, $h_{2,\min} = 9(cm)$, $h_{3,\min} = 12(cm)$, $h_{4,\min} = 10(cm)$, $h_{i,\max} = 30(cm)$, $h_{i,\max} = 31(cm)$, $h_{3,\max} = 28(cm)$ and $h_{4,\max} = 25(cm)$. The tank initial conditions were set as $h_1(0) = 32(cm)$, $h_2(0) = 20(cm)$, $h_1(0) = 6(cm)$, $h_1(0) = 7(cm)$, and $q_i(0) = 0(cm^3/s)$ along with the control gains $c_{i,1} = c_{i,2} = 1.7$. The simulation results are shown in Fig. 2-Fig. 5. The water regulation between the minimum and maximum levels in each tank is shown in Fig. 2. In 3 and Fig. 4, the water flow and control input of each tank is plotted, respectively. It is clear from Fig. 4 that the control input is limited to the bands of zero to one. Moreover, the boundedness of water flow in each tank (second state of subsystems) is demonstrated. In both of these figures the effects of switched dynamics, i.e., ON and OFF in the actuators, are obvious. Fig. 5(a) and Fig. 5(b) display the responses of water inflows for the first tank of the water testbed system when considering a model with and without switched dynamics, respectively. It can be deduced that the controller designed considering switched dynamics exhibits better performance in a distributed control scheme in terms of the number of pump/mode switchings. This leads to both a reduction in energy consumption and a decrease in wear on the actuator. Since the remaining three tanks has a similar pattern in water regulation, they are omitted here. In Fig. 5(c) and Fig. 5(d), the performance of distributed and decentralized controllers, i.e., the modified controllers introduced in Propositions 1 and 2, respectively, (i.e., relations (24)-(30)) are compared for the water regulation in the first tank. The performance of remaining three tanks are not plotted since they have similar pattern. Fig. 5(c), demonstrates the effective regulation of water levels within acceptable time durations for the actuator's ON and OFF modes, ensuring they remain between their minimum and maximum thresholds. This aligns with the slow switching characteristics inherent to the dynamic behavior of the water testbed. However, the water level regulation response under the proposed decentralized control scheme in Fig. 5(d) exhibits fast switching behavior. Fast switchings in the first tank, as seen in Fig. 5(d) and 5(e), occur whenever their water outflows (q_2) drop to zero. This undesirable phenomenon occurs due to the presence of robust terms, i.e., $\tanh\left(\frac{e_{1,1}}{\kappa_{1,1}}\right)\vartheta_2$, in the virtual controls of the first tank, denoted as ω_1 in (25). These robust terms are applied in the virtual control of the first tank to compensate for the effects of the water outflow functions q_2. Form Fig. 5(e), it is evident that in some time durations water outflows q_2 drop to zero in the first tank, however, the term $\tanh\left(\frac{e_{1,1}}{\kappa_{1,1}}\right)\vartheta_2$ is continuously updated in ω_i to compensate for the q_2.

V. CONCLUSIONS

A model-based distributed control problem for switched strongly interconnected systems with mismatched nonlinearities has been solved in this work. First, a model-based distributed controller was designed based on the backstepping strategy in the switching mechanism. The closed-loop stability was analyzed with common Lyapunov function framework. In our future work, we aim to explore adaptive techniques to alleviate some of the assumptions.

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