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Stability of Regional Traffic Networks Employing Maximum Throughput Demand Management

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Abstract— **This paper considers the stability and optimality properties of traffic demand management schemes, motivated by the integration of smart monitoring and control schemes in traffic networks. First, a suitable optimization problem is formulated that aims to obtain demand input values that maximize the throughput within traffic networks. We show that optimal solutions to this problem may lead to unstable behaviour, revealing a trade-off between stability and optimality. To address this issue, we analytically study the stability properties of traffic networks at the presence of constant demand input and provide suitable local conditions that guarantee stability when the system's equilibrium densities are strictly within the free-flow region, but not at the critical density. The latter case is significant, since the maximum throughput behaviour coincides in many cases with the local critical density. We resolve this by proposing a decentralized proportional demand control scheme and suitable local design conditions such that stability is guaranteed. Our analytic results are validated with numerical simulations in a 3-region system that demonstrate the effectiveness and practicality of the presented results.**

I. INTRODUCTION

Recent technological achievements enabled the evolution of traffic networks to smart networks. This was possible through the incorporation of real-time smart monitoring technologies (optical sensors, drones, etc.) and the integration of fast communication protocols that enabled the design and application of novel traffic-control schemes.

Despite these advancements, urban road networks frequently grapple with inherent congestion issues. This often results due to demand surpassing the available capacity in specific regions, giving rise to undesirable traffic conditions such as congestion [1]. To proactively address these challenges, a plethora of traffic management strategies [2], based on macroscopic traffic dynamics, have been proposed in existing research literature [3].

One such strategy is route guidance (RC) [4], which aims to redistribute vehicular loads across the network. By directing drivers along alternative pathways, this approach aims to optimize overall travel times [4]. Concurrently, perimeter control (PC) and gating strategies (GS) [5] focus on preemptively mitigating congestion. These methods regulate the inflow of vehicles at the outskirts of a designated, congestion-prone region. By doing so, the core area of this region is able to operate under congestion-free conditions, even though the congestion may subsequently be transferred to the periphery of the urban network [6]. A substantial number of macroscopic strategies rely on Model Predictive Control (MPC) frameworks [7], [8], [9], [10], [11], [12]. Despite their efficacy in handling traffic congestion, MPC methods have limitations. One notable drawback is their tendency to overlook stability issues, which may lead to unpredictable and suboptimal system performance.

Examples of MPC-based solutions include an optimal 2-region PC scheme [7], a three-level hierarchical PC scheme [8], a Lyapunov derived PC combined with a distributed MPC [9], a PC nonlinear MPC with stability by construction [10], a joint RG and demand management (DM) MPC [11], and a RG and DM MPC framework proposing two real-time solution approaches [12].

Alternative approaches, aiming to achieve congestion mitigation, include an optimal state-feedback PC strategy for a 2-region network [13], a PC strategy employing optimal multivariate PI-feedback generators and online adaptive optimization [14], a RG approach managing congestion at the microscopic level by route reservation in the spatial and temporal domains [15], a model-free data-driven adaptive PC with RG [16] and an approach at the microscopic level proposing a joint robust correlated equilibrium routing mechanism and a distributed optimization algorithm for congestion mitigation [17].

Despite the substantial technological advancements and the wide range of traffic management schemes available today, traffic congestion continues to be a pervasive issue. A drawback of several proposed schemes, which may hinder attaining efficient traffic network solutions, is their propensity to prioritize individual needs over achieving a system-wide optimum [18].

Recent works present a promising perspective, advocating for the joint integration of traffic and demand management (DM) strategies as an effective solution to mitigate congestion [11], [12], [15]. Traffic demand management strategies primarily focus on regulating the influx of vehicles into the network. They achieve this by encouraging drivers to opt for alternative travel times—either earlier or later than their initial plans—or to consider different modes of transportation altogether [19]. The joint integration of demand with traffic management enables the redistribution of traffic flows across both temporal and spatial dimensions, aiming to enhance the overall efficiency and performance of the road transportation system [20].

It is signified that the amount of works that employ DM in a macroscopic framework is limited [11], [12], [15], with all relying on MPC approaches and not considering the DM problem from a stability viewpoint. Considering the stability properties of DM schemes in traffic networks, and how these are affected by optimality considerations, is highly important to enable their large scale implementation.

Contribution: This paper studies the stability and optimal-

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ity properties of traffic networks at the presence of demand management schemes. Firstly, we formulate a problem that aims to maximize the throughput within a large-scale traffic network and discuss how such problem may be solved using standard optimization tools. However, we show numerically that optimal solutions may lead to unstable behaviour, when those coincide with local critical densities, which is frequently the case. In particular, we demonstrate that under optimal constant demand input, any deviation towards the congested region leads to gridlock behaviour. The latter reveals an inherent trade-off between stability and optimality of DM schemes and motivates the analytical study of the stability properties of such schemes in traffic networks.

Our stability analysis provides local conditions on the traffic network parameters that enable stability guarantees, when the local equilibrium densities lie strictly within the free flow region, i.e. being less than the critical density. To include the latter case, which frequently occurs when demand is optimized to maximize throughput, we propose a decentralized proportional feedback demand management scheme and provide local conditions on its gains such that stability is guaranteed. Our analytic results are validated with numerical simulations on a 3-region system which showcase the effectiveness and practicality of the proposed conditions and demand schemes.

The main contributions of this work are summarized below.

(i) A DM optimization problem that yields throughput-optimal equilibrium behaviour is formulated.
(ii) It demonstrates a trade-off between optimality and stability in DM schemes in traffic networks, showing that a maximum throughput behaviour may lead to instability.
(iii) It analytically studies the stability properties of large-scale traffic networks at the presence of constant demand and provides local conditions such that stability may be deduced. Moreover, it proposes a decentralized demand management scheme that offers enhanced stability properties.

Paper structure: The paper is organized as follows. In Section II the notation used throughout the paper and the traffic dynamics are provided. Section III defines the optimization problem for demand allocation, and presents the problem statement. A discussion on the inherent trade-off between optimality and stability concludes this section. Section IV presents our main stability results, concerning a constant demand input scheme and a developed decentralized proportional scheme, showcasing the enhanced stability properties of the latter. Section V demonstrates the practicality and applicability of the analytic results through simulations on a 3-region network. Conclusions are drawn in Section VI.

The proofs of the main results have been removed for compactness and will be provided in an extended version of this work.

II. Problem Formulation

A. Notation

Real numbers are denoted by $\mathbb{R}$, and $\mathbb{R}_+$ is the set of real non-negative numbers. Vectors are denoted by bold small letters where the i^{th} component of a vector $\mathbf{x}$ is denoted by x_i. The set of n-dimensional vectors with real entries is denoted by $\mathbb{R}^n$. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. Sets are denoted by capital calligraphic letters. The min-max operator is given by $[x]_a^b = \max(\min(x, b), a)$, where $a, b \in \mathbb{R}$ and $a \leq b$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. It is positive definite if $f(0) = 0$ and $f(x) > 0$ for every $x \neq 0$. It is negative definite if $f(0) = 0$ and $f(x) < 0$ for every $x \neq 0$. We write $\mathbf{0} \in \mathbb{R}^n$ to denote the $n \times 1$ vector with all elements equal to 0. A set $\mathcal{N}$ excluding an element i is denoted by $\mathcal{N} \setminus \{i\}$. The hat designation, $\hat{x}_i$, denotes that x_i is a primal or dual optimal element with respect to a given optimization problem whereas the right superscript x_i^* denotes a system equilibrium. For compactness, we write the units of the considered variables at the first instance only.

B. Model description

The behaviour of the traffic area under interest is modelled as a *network* of n homogeneous regions ($n \geq 2$) connected by *edges* between *nodes*. A directed graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$ captures the network structure where the sets of nodes and edges are denoted by $\mathcal{N} = \{1, 2, ..., n\}$, and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, respectively. Onward the term node and region are used interchangeably. The edge allowing region-j to receive vehicles from region-i, is denoted by $\epsilon_{i,j} = (i, j) \in \mathcal{E}$. Regions that can directly send vehicles to region-i belong to the set $\mathcal{P}_i = \{j \in \mathcal{N} : \epsilon_{j,i} \in \mathcal{E}\}$ and are termed predecessors, while regions that directly receive vehicles from region-i belong to the set $\mathcal{S}_i = \{l \in \mathcal{N} : \epsilon_{i,l} \in \mathcal{E}\}$ and are termed successors. Vehicle inflow to the multi-region traffic network occurs via origin regions. All regions are considered to be both origin and destination regions, where it is assumed that every vehicle that enters the network, does so to reach a destination region that belongs to the network.

Homogeneity conditions dictate that the behaviour of the internal traffic flow[1], $q_i(t)$ [veh/h], in any region-i is described by a fundamental diagram (FD)

$$q_i(t) = f_i(\rho_i(t)), i \in \mathcal{N}. \tag{1}$$

This work employs triangular FDs, i.e., a piecewise-linear concave function that is equal to the product between vehicle density of region-i, $\rho_i(t)$ [veh/km], and vehicle speed, $v_i(t)$ [km/h] of the same region, i.e.

$$q_i(t) = \rho_i(t) v_i(t), i \in \mathcal{N}. \tag{2}$$

A triangular Macroscopic fundamental diagram (MFD) [3], [21], [22], see Fig. 1, defines the aggregated traffic behaviour of region-i by connecting the region-i internal flow, $q_i(t)$, with the total inter-regional flow of region-i, $g_i(\rho_i(t))$ [veh/h], resulting in homogeneous behaviour.

Two traffic states characterize the total inter-regional flow of region-i, $g_i(\rho_i(t))$, see Fig. 1. They are defined by the critical density threshold of region-i, ρ_i^{C} [veh/km], and the jam density threshold of region-i, ρ_i^{J} [veh/km], as

$$g_i(\rho_i) = \begin{cases} r_i f_i(\rho_i(t)) = r_i v_i^{\mathrm{f}} \rho_i(t), & \rho_i(t) \leq \rho_i^{\mathrm{C}} \\ r_i f_i(\rho_i(t)) = \frac{b_i^{\mathrm{C}} - v_i^{\mathrm{C}} \rho_i(t)}{r_i^{-1}}, & \rho_i(t) > \rho_i^{\mathrm{C}} \end{cases}, \tag{3}$$

for all $i \in \mathcal{N}$ and where, $r_i \in \mathbb{R}_+$ is the region-i trip completion ratio given by

$$r_i = L_i l_i^{-1}, \tag{4}$$

[1] Internal traffic flow is observed/described in a 1 [Km] road stretch.

where L_i [km] is the length of region-i, and l_i [km] is the average trip length of a vehicle in the same region.

Moreover, v_i^{f} [km/h] is the free-flow speed of region-i given by

$$v_i^{\mathrm{f}} = q_i^{\mathrm{C}}(\rho_i^{\mathrm{C}})^{-1}, i \in \mathcal{N}, \tag{5}$$

where q_i^{C} [veh/h] is the capacity flow[2] of region-i. The backward congestion propagation speed v_i^{C} [km/h], and the constant b_i^{C} [veh/h], for region-i are defined in a manner that ensures that the function $g_i(\rho_i)$ is continuous as follows

$$v_i^{\mathrm{C}} = q_i^{\mathrm{C}}(\rho_i^{\mathrm{J}} - \rho_i^{\mathrm{C}})^{-1}, i \in \mathcal{N}, \tag{6a}$$

$$b_i^{\mathrm{C}} = \rho_i^{\mathrm{J}} q_i^{\mathrm{C}}(\rho_i^{\mathrm{J}} - \rho_i^{\mathrm{C}})^{-1}, i \in \mathcal{N}. \tag{6b}$$

Flows exiting the network through destination regions and transfer flows between regions are calculated according to the MFDs total inter-regional flow, $g_i(\rho_i(t))$, see (3). Moreover transfer flows between regions are further limited by their inter-boundary capacity. This is modelled via the inter-boundary region-i to region-l capacity function, $c_{il} : \mathbb{R}_+ \to \mathbb{R}_+$, that is given by

$$c_{il}(\rho_l) = \begin{cases} c_{il}^{\max}, & \rho_l(t) \le \rho_{il}^- \\ r_l(b_l^{\mathrm{C}} - v_l^{\mathrm{C}}\rho_l(t)), & \rho_l(t) > \rho_{il}^- \end{cases}, \tag{7}$$

with $i \in \mathcal{N}, l \in \mathcal{S}_i$, and where $c_{il}^{\max} = r_l(b_l^{\mathrm{C}} - v_l^{\mathrm{C}}\rho_{il}^-) \in \mathbb{R}_+$ is the maximum inter-boundary capacity flow from region-i to region-l, and $\rho_{il}^- \in \mathbb{R}_+$ is the critical inter-boundary density threshold between region-i and region-l. The transfer flow from region-i to region-l, $l \in \mathcal{S}_i$, is given by

$$g_{il}(\rho_i, \rho_l) = \min(w_{il} g_i(\rho_i(t)), c_{il}(\rho_l(t))), \tag{8}$$

with $i \in \mathcal{N}, l \in \mathcal{S}_i$, and where $w_{il} \in \mathcal{W}_i$ are the region-i outflow split constants with the set defined as

$$\mathcal{W}_i = \Big\{ w_{il}, c_i \in \mathbb{R}_+ : l \in \mathcal{S}_i, c_i + \sum_{l \in \mathcal{S}_i} w_{il} = 1 \Big\}, \forall i \in \mathcal{N}, \tag{9}$$

where $c_i \in \mathcal{W}_i$ in (9) is the rate of vehicles that end their trip in region-i. The continuous-time evolution of the vehicle density state of each region-i, $\rho_i(t)$, is given by

$$\begin{aligned} \dot{\rho}_i(t) = & \frac{1}{L_i}\Big(- c_i g_i(\rho_i(t)) - \sum_{l \in \mathcal{S}_i} g_{il}(\rho_i(t), \rho_l(t)) \\ & + \sum_{j \in \mathcal{P}_i} g_{ji}(\rho_j(t), \rho_i(t)) + u_i(t) \Big), \forall i \in \mathcal{N}. \end{aligned} \tag{10}$$

The first term in the right hand side of (10) is the flow exiting the network through destination region-i, the second term is the flow towards successor nodes, the third term is the flow from predecessor nodes and the last term, $u_i(t)$ [veh/h], is the serviced demand admitted to the network through the origin region-i and it is considered a control variable.

We proceed the analysis under the following assumption:

Assumption 1: For each transfer flow from region-i to region-j, the following relation holds, $c_{ij}(\rho_j^{\mathrm{C}}) > w_{ij} g_i(\rho_i^{\mathrm{C}})$.

Remark 1: In the density intervals $\rho_i(t) \in [0, \rho_i^{\mathrm{C}}], \rho_j(t) \in [0, \rho_j^{\mathrm{C}}]$, Assumption 1 results in (8) always yielding inter-regional transfer flows from region-i to region-j satisfying $g_{ij}(\rho_i(t), \rho_j(t)) = w_{ij} g_i(\rho_i(t))$.

[2]Capacity flow is the maximum flow that can be supported by region-i; it is yielded by the FD at the critical density point, ρ_i^{C}.

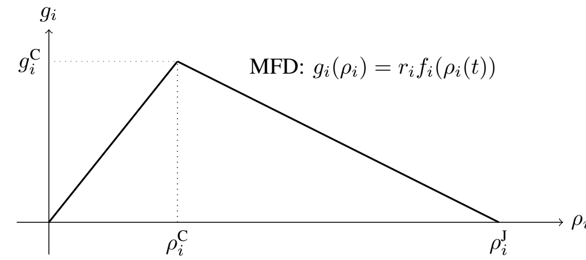

Fig. 1: Region-i total inter-regional flow $g_i(\rho_i(t))$, given by a triangular macroscopic fundamental diagram (MFD).

Assumption 1 plays a central role in the subsequent analysis. For conciseness the traffic dynamics are defined in a compact form as follows:

Traffic Model 1: The traffic dynamics of the considered n-region connected traffic network are described by (3), (7), (8), and (10).

III. Problem Statement and Optimality Considerations

A. Optimization problem

To enable the design of efficient demand management schemes, an optimization problem for demand allocation is formulated yielding *throughput-optimal equilibria*, i.e. a system state characterized by maximum vehicle network throughput or maximum serviced vehicle demand with respect to the network boundaries.

Optimization Problem 1: For an n-region traffic network with traffic dynamics described by Traffic Model 1, solve:

$$\max_{(\boldsymbol{\rho}, \mathbf{u})} \sum_{i \in \mathcal{N}} u_i \tag{11a}$$

$$\text{s.t.} : 0 \le \rho_i \le \rho_i^{\mathrm{C}}, \quad i \in \mathcal{N} \tag{11b}$$

$$u_i^{\min} \le u_i \le u_i^{\max}, \quad i \in \mathcal{N} \tag{11c}$$

$$\begin{aligned} & \frac{1}{L_i}\Big(\sum_{j \in \mathcal{P}_i} g_{ji}(\rho_j, \rho_i) + u_i \\ & - c_i g_i(\rho_i) - \sum_{l \in \mathcal{S}_i} g_{il}(\rho_i, \rho_l) \Big) = 0, \quad i \in \mathcal{N} \end{aligned} \tag{11d}$$

where $u_i^{\min}, u_i^{\max} \in \mathbb{R}_+$ are selected in a manner that the feasibility of the optimization problem is ensured[3].

Remark 2: The constraint (11b) stems from practical/physical considerations/observations i.e. an equilibrium point characterized by free-flow conditions is desirable. In (11c) we allow $u_i^{\min}$ to take values that serve the vehicle demand objective; it ensures that demand takes non-negative values since $u_i \in \mathbb{R}_+$ and enables the formulation of different problems that may either aim to maximize the total throughput or suitably constraint the problem based on some minimum practical value serving realistic regional demand. The justification for $u_i^{\max}$ follows in a similar fashion, noting also that its value in practice may also depend on the structure and parameters of the traffic network.

[3]An approach that enables feasibility assurances involves obtaining a solution of (11a), (11b), (11d), given by $\hat{u}$, and then design $u_i^{\min}$, and $u_i^{\max}$ such that $u_i^{\min} \le \hat{u}_i \le u_i^{\max}$ for all $i \in \mathcal{N}$.

B. Problem Statement

As already mentioned in Section I, urban transportation networks frequently suffer from congestion since some regions of the network attract more vehicles than others. Hence this work aims to address the following problem:

Problem 1: For the Traffic Model 1 and under Assumption 1:

1) define an optimal throughput DM solution, based on Optimization Problem 1,
2) design a DM scheme that enables convergence guarantees to free-flow steady state values and is applicable to any (connected) traffic network configuration.

The first condition is associated with the efficiency of DM schemes, aiming to obtain suitable solutions to Optimization Problem 1. The second condition is the main goal of the DM approach, i.e., to ensure that system states attain a desired equilibrium characterised by free-flow conditions, and also that the controller is supplemented by stability guarantees. Additionally, it is required that the DM controller is applicable to any connected traffic network configuration.

C. Optimal Solution

Assumption 1 results in inter-regional transfer flows from region-i to region-j equal to $g_{ij}(\rho_i(t), \rho_j(t)) = w_{ij} g_i(\rho_i(t))$. As a result, the equality constraint (11d) simplifies at steady state conditions to

$$\frac{1}{L_i}\Big(\sum_{j\in\mathcal{P}_i} w_{ji} g_j(\rho_j) + u_i \\ - c_i g_i(\rho_i) - \sum_{l\in\mathcal{S}_i} w_{il} g_i(\rho_i)\Big) = 0, i \in \mathcal{N}, \tag{12}$$

and Optimization Problem 1 is given by (11a), (11b), (11c) and (12), and has linear constraints. Moreover, it is also convex and due to advancements in optimization theory and computing, obtaining solutions of convex, linear optimization problems is a relatively easy task serviced by a plethora of commercial solvers that yield solutions in a mater of seconds, even for large-scale problems.

The solution of Optimization Problem 1 with (12) yields optimal equilibria, $(\hat{\boldsymbol{\rho}}, \hat{\mathbf{u}})$, characterized by maximum serviced vehicle demand fulfilling the network inter-boundary demand allocation task.

D. Stability Issues

Simulation results (see Figs. 2-4 and the discussion in Section V) show that operating the system at optimal equilibrium points may result in stability issues. In particular, when an optimal equilibrium point coincides with the local critical density, as is frequently the case, then an arbitrary small increase in density yields substantial deviations in density trajectories, and in many cases leads to gridlock behaviour. As a result, an inherent trade-off between optimality and stability is recognized since when the system operates in a suboptimal way, avoiding equilibrium points at the critical density, then its operation is more robust to disturbances. This offers motivation to rigorously explore the stability properties of the traffic network system.

IV. Traffic Network Stability

Motivated by the discussion in the previous section, this section explores the stability properties of traffic networks under constant demand and a proposed proportional feedback demand management scheme.

A. Equilibria

Below we provide a definition of the equilibrium points of Traffic Model 1 under interest, to facilitate the analysis later on.

Definition 1: For the Traffic Model 1 under control action $\mathbf{u}^* \in \mathbb{R}^n_+$, an equilibrium point $\boldsymbol{\rho}^*$, satisfies

$$\Big(- c_i g_i(\rho_i^*) - \sum_{l\in\mathcal{S}_i} g_{il}(\rho_i^*, \rho_l^*) \\ + \sum_{j\in\mathcal{P}_i} g_{ji}(\rho_j^*, \rho_i^*) + u_i^*\Big) = 0, \forall i \in \mathcal{N}. \tag{13}$$

Next, an investigation is conducted to identify conditions that result in well behaved dynamics such that the solutions of Traffic Model 1 converge to an equilibrium point, $\boldsymbol{\rho}^*$.

B. Stability Analysis under Constant Demand

The behaviour of Traffic Model 1, when its equilibria lie strictly within the free-flow region of operation are investigated next. More explicitly, by means of Lyapunov analysis, sufficient conditions for stability for the Traffic Model 1, are derived.

Theorem 1: Consider the Traffic Model 1, let Assumption 1 hold, let $u_i(t) = u_i^*$, $i \in \mathcal{N}$, and consider an equilibrium point $\boldsymbol{\rho}^*$, such that $\rho_i^* \in [0, \rho_i^c), \forall i \in \mathcal{N}$. If

$$c_i > \frac{1}{r_i v_i^{\mathrm{f}}}\Big(\sum_{j\in\mathcal{P}_i} \frac{a_{ji}}{4}\Big), \forall i \in \mathcal{N}, \tag{14a}$$

$$a_{ji} = w_{ji} r_j v_j^{\mathrm{f}}, j \in \mathcal{P}_i, i \in \mathcal{N}, \tag{14b}$$

then, the solutions to Traffic Model 1 locally converge to $\boldsymbol{\rho}^*$. Moreover, if $u_i^* = \hat{u}_i, \forall i \in \mathcal{N}$, where $\hat{u}_i$ is a solution to Optimization Problem 1, then the aforementioned equilibrium point globally solves Optimization Problem 1.

The significance of the newly derived criterion, (14), is that it provides locally-verifiable sufficient conditions for stability which enable an assessment of the network stability properties at the presence of some constant demand input. However, since we do not conciser the case $\rho_i^* = \rho_i^{\mathrm{C}}, i \in \mathcal{N}$, this does not addresses the issue discussed in Section III-D, which has been observed at particularly this case. Nevertheless, the presented analysis facilitates the solution to this issue, as demonstrated below.

C. Stabilizing Demand Management Scheme

In the last paragraph of Section III-D, an inherent trade-off between optimality and stability was recognized. In particular, simulations revealed that the system is sensitive to arbitrary small deviations from the optimal/critical equilibrium to the congested region of operation. This unstable behaviour (see Figs. 2-4 and the discussion in Section V), motivates the development of a stabilizing controller that enables stability guarantees even at the congested region of operation. We propose a controller described by

$$u_i(t) = [\bar{u}_i - k_{p_i}\rho_i(t)]_0^{u_i^{\max}}, i \in \mathcal{N}, \tag{15}$$

where $k_{p_i} \in \mathbb{R}_+ \setminus \{0\}$ and $\bar{u}_i$ is a design constant, that may be selected by taking into account optimality considerations and satisfies the following condition by design

$$0 < \bar{u}_i - k_{p_i}\rho_i^* < \bar{u}_i^{\max}. \tag{16}$$

The selection of the constants $\bar{u}_i$ and k_{p_i} rely on knowledge of the local equilibrium density $\rho_i^*, i \in \mathcal{N}$. When feasible, this value may be provided centrally, through historical data, or by solving Optimization Problem 1. Alternatively, the design could satisfy (16) for a range of local equilibrium values, e.g. for all $\rho_i \in [0, \rho_i^{\mathrm{C}}], i \in \mathcal{N}$.

Theorem 2: Consider Traffic Model 1 under the action of (15), (16), and let Assumption 1 hold. Then, if

$$k_{p_i} > r_i v_i^{\mathrm{f}} + \sum_{j \in \mathcal{P}_i} \frac{\alpha_{ji}}{4} + \sum_{j \in \mathcal{S}_i} \alpha_{ij}, i \in \mathcal{N}, \tag{17a}$$

$$\alpha_{ji} = w_{ji} r_j v_j^{\mathrm{L}}, i \in \mathcal{N}, j \in \mathcal{P}_i, \mathcal{S}_i, \tag{17b}$$

$$v_j^{\mathrm{L}} = \max(v_j^{\mathrm{f}}, v_j^{\mathrm{C}}), j \in \mathcal{N} \tag{17c}$$

then, the solutions of Traffic Model 1 locally converge to an equilibrium point $\boldsymbol{\rho}^*$. Moreover, if $u_i^* = \hat{u}_i, \forall i \in \mathcal{N}$, where $\hat{u}_i$ is a solution to Optimization Problem 1, then the aforementioned equilibrium point globally solves Optimization Problem 1.

Theorem 2 yields a stabilizing controller and a sufficient condition for gain selection, i.e. (17a), that restores system stability even at the congested region of operation. In addition, when the controller coincides with the solution to Optimization Problem 1, then the equilibrium point enables a globally optimal throughput behaviour. Moreover, the proposed analysis is applicable to any network configuration. Hence, all objectives of Problem 1 are met.

Next, we validate the analytic results presented in this paper with numerical simulations.

V. Numerical Results

To illustrate the developments of this work and without loss of generality a representative simulation of a 3-region traffic network is conducted aiming to showcase three important points: a) the stable network behaviour under constant control action, illustrating the validity of Theorem 1, b) the unstable network behavior when a small deviation from the optimal/critical equilibrium occurs to the congested region of operation, and c) the corrective action of the developed control law, (15), steering the network states back to the optimal/critical equilibrium.

Two simulation cases are considered; the first case uses a constant control action close to the optimal control value, $u_i(t) = \hat{u}_i - 0.1, i \in \mathcal{N}$ (see Table I for $\hat{u}_i$), where $\hat{u}_i$ is the solution to Optimization Problem 1 for region i, to be referred to as benchmark network (black dashed line). The second case employs the derived control law (15) (solid blue line); where $\bar{u}_i = \hat{u}_i + k_{p_i}\rho_i^*, i \in \mathcal{N}$ (see Table I for the values of $\hat{u}_i, \rho_i^*, k_{p_i}$).

Network parameters are given in Table I. It is signified that the rates that vehicles end their trip in region-i, determined by the parameter $c_i \in \mathcal{W}_i$, satisfy the conditions of Theorem 1 and the gain values k_{p_i} were selected to satisfy the gain selection condition (17a) of Theorem 2. An 80 [min] simulation scenario is conducted. At $t = 30$ [min] and approximately for 1 [min] a small deviation from the optimal/critical equilibrium is introduced in each network in the form of $u_1(t) = u_1(t) + 0.1\hat{u}_1$.

In Figs. 2-4, the green-shaded interval corresponds to the interval before the appearance of the control input deviation, see Fig. 2 $t \in [0, 30]$ [min]. In both cases the density state at each region achieves steady state, see Fig. 3, demonstrating the validity of Theorem 1. However state convergence

TABLE I: Network parameters, throughput-optimal equilibria and control gains

Region 1	Region 2	Region 3
$v_1^{\mathrm{f}} = 30$	$v_2^{\mathrm{f}} = 35$	$v_3^{\mathrm{f}} = 32$
$L_1 = 1.2$	$L_2 = 1$	$L_3 = 0.85$
$\rho_1^{\mathrm{J}} = 118$	$\rho_2^{\mathrm{J}} = 125$	$\rho_3^{\mathrm{J}} = 98$
$l_1 = 0.6$	$l_2 = 0.45$	$l_3 = 0.35$
$\rho_1^{\mathrm{C}} = 26.3$	$\rho_2^{\mathrm{C}} = 28.2$	$\rho_3^{\mathrm{C}} = 24.4$
$c_1 = 0.25$	$c_2 = 0.35$	$c_3 = 0.2$
$w_{21} = 0.5, w_{31} = 0.25$	$w_{12} = 0.15, w_{32} = 0.5$	$w_{13} = 0.1, w_{23} = 0.7$
$\hat{\rho}_1 = 26.3$	$\hat{\rho}_2 = 28.2$	$\hat{\rho}_3 = 24.4$
$\hat{u}_1 = 620.71$	$\hat{u}_2 = 76.97$	$\hat{u}_3 = 405.06$
$k_{p_1} = 109.86$	$k_{p_2} = 149.5$	$k_{p_3} = 153.4$

is slower in the benchmark network (see dashed lines in Fig. 3). This is expected since it lacks the proportional-term operating on the alternative case that speeds up convergence (see solid lines in Fig. 3). At the same interval both networks operate in free-flow conditions, see Fig. 4.

At $t = 30$ [min] a deviation from steady state appears in both networks in the form of $u_1(t) = u_1(t) + 0.1\hat{u}_1$ (beginning of the red-shaded interval) lasting about 1 [min] (see Fig. 2, first row). For the benchmark network (dashed lines), this deviation results in density drop for regions-2,3 and in region-1 getting congested as its density veers from the optimal-throughput equilibrium and attains gridlock status[4], i.e., $\rho_1(t) = \rho_1^{\mathrm{J}}$, see Fig. 3. The convergence to a sub-optimal equilibrium is evident as the speed of region-1 decreased to zero, as demonstrated in Fig. 4.

An inherent trade-off between optimality and stability is recognized, in the sense that a traffic coordinator can select a sub-optimal control action $u_i(t) = \mu\hat{u}_i, \mu \in (0, 1)$ (thus operating in sub-optimal throughput conditions) to ensure stability, i.e., that system uncertainty and disturbances are unable to steer the system to congestion.

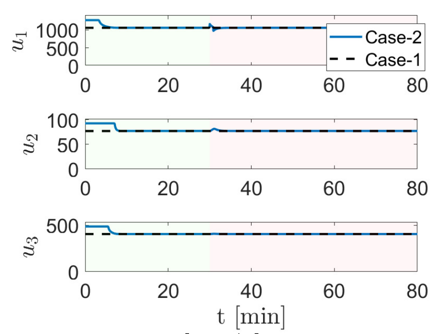

Fig. 2: Admitted demand [veh/h]. Case-1 operates at constant demand except for a 1 [min] variation at $t = 30$ [min]. Case-2 demand levels saturate (see $t \in [0, 10]$ [min]) due to the proportional term. Due to demand variation at $t = 30$ [min], stabilizing adjustments are observed for Case-2.

In contrast to the above, the network employing the control law with the proportional term, described by (15), is not affected by the deviation (solid lines) and the states remain to the optimal-throughput equilibrium in all regions, as shown in Fig. 3. Only a very minor drop in the speed of region-1 (for a very short time) is observed at $t = 30$ [min], see first row in Fig. 4. A minor increase in the admitted demand in region-2 (for a very short time) is also observed. This is attributed to the corrective action of the proportional term

[4]Gridlock is a condition where a region cannot admit any more vehicles and congestion is so bad that all vehicles come to standstill.

(solid lines, Fig. 2, second row). The network operates in free-flow conditions, see Fig. 4. Hence, the effectiveness of the control law is clearly demonstrated.

In conclusion, the simulation results clearly demonstrate the validity of the results presented in Section III and Section IV.

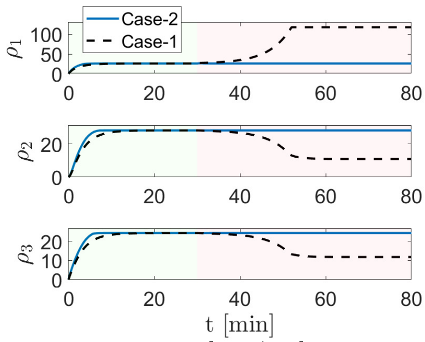

Fig. 3: Network density [veh/km]. Initially, both cases achieve steady state. A demand variation at $t = 30$ [min] results in instability for Case-1. Case-2 is stable due to the derived control law.

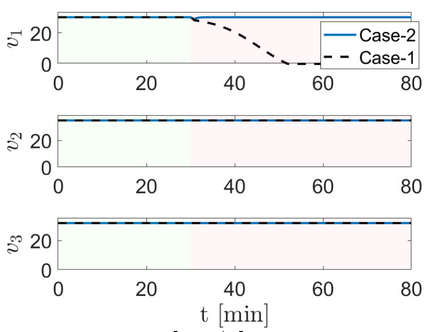

Fig. 4: Network speed [km/h]. Initially, vehicles travel at free-flow speed. Due to a demand variation at $t = 30$ [min] region-1 (Case-1) gets congested and eventually attains grid-lock status.

VI. Conclusions

This work considered the stability and optimality properties of traffic demand management schemes in traffic networks. First, a suitable optimization problem was formulated that aimed to obtain demand input values that maximize the throughput within traffic networks. A trade-off between stability and optimality was revealed by showing that optimal solutions to this problem may lead to unstable behaviour. To address this issue, we analytically studied the stability properties of traffic networks at the presence of constant demand input and provided suitable local conditions that guarantee stability when the system's equilibrium densities are strictly within the free-flow region, but not at the critical density. However, the critical density case is significant, since the maximum throughput behaviour frequently coincides with the local critical density. To resolve this issue, we proposed a decentralized proportional demand control scheme and suitable local design conditions such that stability is guaranteed. Our analytic results were validated with numerical simulations in a 3-region system that demonstrated the effectiveness and practicality of the presented results.

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