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Data-Driven H_∞ Control for Unknown Piecewise Affine Systems with Bounded Disturbances

Kaijian Hu, Tao Liu

Abstract— **This paper studies the data-driven control problem for piecewise affine (PWA) systems with bounded disturbances, in which both the system model and disturbances are unknown. Due to the unknown disturbances, different PWA systems generate the same input-state-output data, making data-based system identification difficult. In view of this issue, a set containing all systems that could generate the given input-state-output data is constructed in terms of quadratic matrix inequalities (QMIs). The matrix S-lemma is then used to design an H_∞ controller for all these systems. The proposed data-driven H_∞ control method guarantees the internal stability and prescribed performance of the closed-loop system only based on input-state-output data. The effectiveness of the proposed methods is illustrated by a single-link robot arm control system.**

I. Introduction

Direct data-driven control (DDC) is becoming increasingly popular within the control community. Compared with traditional model-based control methods, DDC approaches aim to directly design the controller without relying on a precise system model. This is motivated by the challenges associated with obtaining accurate models and the convenience of collecting system data [1]. In recent years, several DDC methods have been developed to address different control scenarios, encompassing model-free adaptive control [2], model-free reinforcement learning [3], behavioral system control [4]–[10] and informativity approach [11]–[13]. The appeal of DDC lies in its broad applicability to different types of systems, including linear systems [4]–[6], [12], [13], switched linear systems [7], [10], linear parameter-varying systems [8], and certain special nonlinear systems [3], [9].

On the other hand, piecewise affine (PWA) systems, as an important type of hybrid system, have received considerable attention. A PWA system consists of multiple subsystems that operate within distinct polyhedral state space regions, offering a powerful tool for studying nonlinear systems due to their ability to approximate a wide range of nonlinear behaviors [14]. Over the past few years, extensive research has been conducted on the stability analysis and controller design for PWA systems [15]–[18]. Fundamental control issues such as stabilization, state/output feedback control, and observer-based control problems are addressed in [15], [16]. Additionally, an indirect adaptive control method and an output feedback model reference adaptive control method are developed in [17] and [18], respectively.

An important control issue of PWA systems is how to deal with unknown but bounded disturbances. For example, these disturbances can be used to denote the unmodeled dynamics encountered when using PWA systems to approximate nonlinear systems. The control targets are to ensure internal stability and meanwhile achieve the desired control performance for the closed-loop system. Recently, several methods have been proposed to solve this issue, such as H_∞ state and output feedback control [19], [20]. All of these methods are model-based and rely on accurate nominal system models. In the case of unknown PWA systems, the traditional approach involves using data to identify the system model through system identification [14], followed by designing the controller using the aforementioned model-based methods. Nevertheless, the presence of unknown disturbances may lead to different PWA systems generating the same input-state-output data. Consequently, the nominal system may be difficult to uniquely determine, rendering these model-based methods unsuitable for the control design of unknown PWA systems with unknown disturbances.

In view of these issues, this paper aims to design a data-driven H_∞ controller for unknown PWA systems with unknown bounded disturbances. For a given input-state-output data set of a PWA system, the controller is designed to achieve the two control targets for all systems that can generate the given data. Specifically, for each subsystem within the PWA system, we construct a set that contains all systems, including the true subsystem, capable of generating the given data when subjected to a bounded disturbance. The set is described by a data-based quadratic matrix inequality (QMI). Then, a matrix S-lemma [12] is used to design an H_∞ controller for all the systems that satisfy the QMIs.

The remainder of this paper is organized as follows. Section II describes the problem to be solved. Section III constructs the data-based QMIs for PWA systems using the input-state-output data. Based on these QMIs, a data-driven state feedback controller is designed in Section IV. The effectiveness of the proposed methods is illustrated by a single link robot arm in Section V. Finally, Section VI gives some concluding remarks.

Notation: Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ represent the sets of natural numbers, integers, and real numbers, respectively. I_n denotes the identity matrix with dimensions $n\times n$, and $0_{n\times m}$ denotes a zero matrix with dimensions $n \times m$. The subscripts n and $n \times m$ are omitted when the context makes the dimensions

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K. Hu and T. Liu are with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Hong Kong SAR, and the HKU Shenzhen Institute of Research and Innovation, Shenzhen, China. (e-mail: kjhu@eee.hku.hk; taoliu@eee.hku.hk).

evident. The notation $A < B$ ($A \leq B$) means that the matrix $A - B$ is negative (semi-)definite. Given a matrix M, $M^\top$ stands for its matrix transposition; M^{-1} denotes its inverse if it is nonsingular. $\text{diag}(M_1, \ldots, M_n)$ denotes the block diagonal matrix composed of the matrices M_i, $i = 1, \ldots, n$. Given a signal $z : \mathbb{Z} \to \mathbb{R}^n$, define $\|z\|_2 = \sqrt{z^\top z}$, $z_{[k,k+T]} = [z(k)^\top, \ldots, z(k+T)^\top]^\top$, $k \in \mathbb{Z}$ and $T \in \mathbb{N}$. Denote $z_{[k,k+T]}$ as $1_{[k,k+T]}$ if $z(i) = 1$, $\forall i = k, \ldots, k+T$.

II. Problem formulation

Consider the following PWA system with s subsystems

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + b_i + \omega_E(k), \\ y(k) &= C_i x(k) + D_i u(k) + d_i + \omega_F(k), \\ & \qquad x(k) \in \mathscr{X}_i, i = 1, \ldots, s, \end{aligned} \tag{1}$$

where $x(k) \in \cup_{i=1}^s \mathscr{X}_i \subseteq \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the system state, input and output, respectively; $\omega_E(k) \in \mathbb{R}^n$ and $\omega_F(k) \in \mathbb{R}^p$ are the unknown disturbance and measurement noise, respectively; $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $b_i \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{p\times m}$, and $d_i \in \mathbb{R}^p$ are system matrices; $\mathscr{X}_i$ is a polyhedral partition constructed by the intersection of a finite number of half-spaces. Different partitions do not have intersections except for their boundaries. We assume there exists at least one partition $\mathscr{X}_i$, $i \in \mathcal{S} = \{1, \ldots, s\}$ containing the origin. For those $i \in \mathcal{S}$ with $0 \in \mathscr{X}_i$, we have $b_i = 0$ and $d_i = 0$.

Define $\omega(k) = [\omega_E(k)^\top, \omega_F(k)^\top]^\top \in \mathbb{R}^q$ with $q = n+p$, $E = [I_n, 0_{n\times p}] \in \mathbb{R}^{n\times q}$, and $F = [0_{p\times n}, I_p] \in \mathbb{R}^{p\times q}$. The system (1) can be expressed as

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + b_i + E\omega(k), \\ y(k) &= C_i x(k) + D_i u(k) + d_i + F\omega(k). \end{aligned} \tag{2}$$

For clarity, the tuple $(u_i(k), x_i(k), y_i(k), \omega_i(k))$ is used to represent the data $(u(k), x(k), y(k), \omega(k))$ of the subsystem i, $\forall i \in \mathcal{S}$. Let $(u_{i,[0,T-1]}, x_{i,[0,T]}, y_{i,[0,T-1]})$, $\forall i \in \mathcal{S}$ be an input-state-output trajectory of the i-th subsystem of (2) under the unknown disturbance trajectory $\omega_{i,[0,T-1]}$. We define the following data sequences

$$X_i = [x_i(0), \ldots, x_i(T-1)], X_{i,+} = [x_i(1), \ldots, x_i(T)], \tag{3a}$$
$$U_i = [u_i(0), \ldots, u_i(T-1)], Y_i = [y_i(0), \ldots, y_i(T-1)], \tag{3b}$$
$$W_i = [\omega_i(0), \ldots, \omega_i(T-1)]. \tag{3c}$$

Without loss of generality, the following three assumptions are made for each subsystem i, $i \in \mathcal{S}$ of the system (2).

Assumption 1: The system matrices A_i, B_i, b_i, C_i, D_i, and d_i are unknown. Dimensions n, m, and p are known.

Assumption 2: The disturbance sequence W_i is bounded by a known $q \times q$ matrix Υ, denoted as

$$W_i W_i^\top \leq \Upsilon. \tag{4}$$

Assumption 3: There exists a (degenerate) ellipsoid $\mathcal{X}_i$ defined by (5) such that $\mathscr{X}_i \subseteq \mathcal{X}_i$.

$$\mathcal{X}_i = \{x(k) | \|H_i x(k) + h_i\|_2 \leq 1\}, \tag{5}$$

where $H_i \in \mathbb{R}^{\ell_i \times n}$, and $h_i \in \mathbb{R}^{\ell_i}$. Moreover, if $0 \notin \mathscr{X}_i$, then $0 \notin \mathcal{X}_i$.

All the assumptions are widely adopted in the related literature. The first two assumptions are commonly used in the study of the DDC problem for unknown systems (e.g., [4], [5], [21]). In particular, Assumption 2 provides the necessary information on the disturbance to design the DDC for systems subject to disturbances [21]. Assumption 3 is typically made when studying the control problem for PWA systems [22], which is used to incorporate the partition information into the controller design for PWA systems.

Remark 1: The partition information is essential in collecting data for each subsystem, which can be calculated in practical scenarios using partition estimation methods, such as clustering technique [23].

Since the PWA system (2) is unknown, the traditional model-based control methods such as those in [19], [20] cannot be directly employed. Moreover, due to the existence of the unknown disturbance $\omega(k)$, any input-state-output trajectory of (2) may be generated by different systems. It implies that it is challenging to uniquely determine the system matrices A_i, B_i, C_i, D_i, b_i and d_i, $i = 1, \ldots, s$ using even a persistently exciting input-state-output sequence. In view of these problems, this paper will design a data-driven controller for (2) without explicitly relying on its system model. The problem to be solved is formulated as follows.

Problem 1: Consider the unknown PWA system (2) under Assumptions 1-3. Given a pre-collected input-state-output trajectory of the system (i.e., $(u_{i,[0,T-1]}, x_{i,[0,T]}, y_{i,[0,T-1]})$, $\forall i \in \mathcal{S}$), and an H_∞ performance index σ, design a data-driven state feedback controller to 1) stabilize the PWA system when the disturbance $\omega(k) = 0$, and 2) fulfill the following H_∞ performance requirement for the system with $x(0) = 0$ and $\omega(k) \neq 0$

$$\sum_{k=0}^{\infty} \|y(k)\|_2^2 < \sigma^2 \sum_{k=0}^{\infty} \|\omega(k)\|_2^2. \tag{6}$$

III. Data-Based QMIs

To address Problem 1, we can design a controller to achieve the two targets for all the systems that are compatible with the given data (3a)-(3b). With these data, this section will derive sets Ω_i, $\forall i = 1, \ldots, s$, where each set Ω_i contains all possible systems of the i-th subsystem in the system (2). Additionally, these sets are described using data-based QMIs.

All subsystems of the system (2) are divided into two types depending on whether the origin is in $\mathscr{X}_i$ or not,

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + E\omega(k) \\ y(k) &= C_i x(k) + D_i u(k) + F\omega(k) \end{aligned}, \quad i \in \Pi_0 \tag{7a}$$

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + b_i + E\omega(k) \\ y(k) &= C_i x(k) + D_i u(k) + d_i + F\omega(k) \end{aligned}, \quad i \in \Pi_1 \tag{7b}$$

where Π_0 denotes the index set of subsystems with $0 \in \mathscr{X}_i$, while Π_1 is the index set of those with $0 \notin \mathscr{X}_i$.

First, for each subsystem i, $\forall i \in \Pi_1$, (3a)-(3c) satisfy

$$X_{i,+} = A_i X_i + B_i U_i + b_i 1_{[0,T-1]}^\top + E W_i, \tag{8a}$$
$$Y_i = C_i X_i + D_i U_i + d_i 1_{[0,T-1]}^\top + F W_i. \tag{8b}$$

We abuse the notations and still use A_i, B_i, b_i, C_i, D_i, and d_i to denote all the possible system matrices satisfying (8). For simplicity, we denote $(A_i, B_i, b_i, C_i, D_i, d_i)$ as sys_i, $\forall i \in \Pi_1$. In addition, it is worth noting that the data sequences $(X_i, X_{i,+}, Y_i, U_i)$, and matrices E and F are known, and $(A_i, B_i, b_i, C_i, D_i, d_i)$ and W_i are unknown.

Since both W_i and the system matrices sys_i are unknown, for a given set of input-state-output data, there could be multiple sys_i that satisfy (8) with some $W_i \in \Omega^\omega$, where $\Omega^\omega = \{W_i | (4) \text{ is satisfied}\}$. All these systems can be encompassed within a set Ω_i, which is defined as

$$\Omega_i = \{sys_i | (8) \text{ and } W_i \in \Omega^\omega \text{ hold}\}, \forall i \in \Pi_1. \tag{9}$$

Obviously, Ω_i contains the unknown true system. Therefore, by designing a controller for all systems in Ω_i, we can achieve the control targets for the true system.

Under Assumption 3, (5) can be rewritten as

$$\begin{bmatrix} x(k) \\ 1 \end{bmatrix}^\top \begin{bmatrix} H_i^\top H_i & H_i^\top h_i \\ h_i^\top H_i & h_i^\top h_i - 1 \end{bmatrix} \begin{bmatrix} x(k) \\ 1 \end{bmatrix} \leq 0, \forall i \in \mathcal{S}. \tag{10}$$

The partition information, represented by (10) with $i \in \Pi_1$, will be integrated into the controller design by utilizing the S-procedure [24], where the pair (H_i, h_i) are handled similarly to the pairs (A_i, b_i) and (C_i, d_i). More precisely, we introduce the following identity

$$\bar{H}_i \bar{X}_i = H_i X_i + h_i 1_{[0,T-1]}^\top, \forall i \in \Pi_1, \tag{11}$$

where $\bar{H}_i = [h_i, H_i]$ and $\bar{X}_i = [1_{[0,T-1]}, X_i^\top]^\top$. Since (11) always holds for all X_i, it is independent of the choice of system matrices. As a result, Ω_i can be redefined as

$$\Omega_i = \{sys_i | (8) \text{ and } (11) \text{ and } W_i \in \Omega^\omega \text{ hold}\}, \forall i \in \Pi_1. \tag{12}$$

Subsequently, we construct an equivalent set of Ω_i, by eliminating the unknown disturbance sequence W_i from (8) under Assumption 2. Combining (8) and (11), we have

$$G_i W_i = [I_{\ell_i+n+p}, Z_i] L_i, \tag{13}$$

where

$$G_i = \begin{bmatrix} 0_{\ell_i \times q} \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & I_p \end{bmatrix}, Z_i = \begin{bmatrix} h_i & H_i & 0_{\ell_i \times m} \\ b_i & A_i & B_i \\ d_i & C_i & D_i \end{bmatrix}, \tag{14}$$

$$L_i = [\bar{X}_i^\top \bar{H}_i^\top, X_{i,+}^\top, Y_i^\top, -\bar{X}_i^\top, -U_i^\top]^\top.$$

Left and right multiplying both terms of (4) by G_i and $G_i^\top$ gives

$$G_i W_i W_i^\top G_i^\top \leq G_i \Upsilon G_i^\top. \tag{15}$$

Since G_i has full column rank, (4) is equivalent to (15) [25]. As a result, the set Ω_i can be equivalently described as

$$\Omega_i = \{sys_i | (13) \text{ and } (15) \text{ hold}\}, \forall i \in \Pi_1.$$

Moreover, substituting (13) into (15) yields the QMI

$$[I_{\ell_i+n+p}, Z_i] N_i [I_{\ell_i+n+p}, Z_i]^\top \leq 0, \tag{16}$$

where

$$N_i = L_i L_i^\top - \text{diag}(G_i \Upsilon G_i^\top, 0_{(n+m+1)\times(n+m+1)}). \tag{17}$$

Hence, Ω_i is redefined as $\Omega_i = \{sys_i | (16) \text{ holds}\}$, $\forall i \in \Pi_1$.

On the other hand, when the subsystem i belongs to Π_0, we define

$$G_i = \begin{bmatrix} E \\ F \end{bmatrix}, Z_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, L_i = [X_{i,+}^\top, Y_i^\top, -X_i^\top, -U_i^\top]^\top.$$

Using a similar process from (8) to (16) yields

$$[I_{n+p}, Z_i] N_i [I_{n+p}, Z_i]^\top \leq 0, \tag{18}$$

where

$$N_i = L_i L_i^\top - \text{diag}(G_i \Upsilon G_i^\top, 0_{(n+m)\times(n+m)}). \tag{19}$$

Then, for each subsystem $i \in \Pi_0$, the set Ω_i is defined as

$$\Omega_i = \{(A_i, B_i, C_i, D_i) | (18) \text{ holds}\}, \forall i \in \Pi_0.$$

IV. Controller Design

This section proposes a data-driven state feedback controller to solve Problem 1 using the data-based QMIs (16) and (18).

Let $u(k) = K_i x(k)$, $\forall i \in \mathcal{S}$, where $K_i \in \mathbb{R}^{m \times n}$. Then, the closed-loop system becomes

$$\begin{array}{ll} x(k+1) & = A_{i,cl} x(k) + E\omega(k) \\ y(k) & = C_{i,cl} x(k) + F\omega(k) \end{array}, \quad i \in \Pi_0 \tag{20a}$$

$$\begin{array}{ll} x(k+1) & = A_{i,cl} x(k) + b_i + E\omega(k) \\ y(k) & = C_{i,cl} x(k) + d_i + F\omega(k) \end{array}, \quad i \in \Pi_1 \tag{20b}$$

where $A_{i,cl} = A_i + B_i K_i$ and $C_{i,cl} = C_i + D_i K_i$.

To design such a control law for the closed-loop PWA system (20), the multiple Lyapunov functions method proposed in [26] is employed. First, a piecewise Lyapunov function candidate is selected as follows:

$$V(x(k)) = \sum_{i=1}^{s} \xi_i(k) V_i(x(k)), \tag{21}$$

where $\xi_i(k) = 1$ if the subsystem i is active, otherwise $\xi_i(k) = 0$; $V_i(x(k)) = x(k)^\top P_i x(k)$ with $P_i \in \mathbb{R}^{n \times n} > 0$.

Consider the scenario where the system state $x(k)$ is in $\mathscr{X}_i$, $i \in \mathcal{S}$ at time k, and the succeeding state $x(k+1)$ is in $\mathscr{X}_j$, $j \in \mathcal{S}$. In this context, the difference between $V(x(k))$ and $V(x(k+1))$ can be defined as

$$\Delta V(x(k)) = V_j(x(k+1)) - V_i(x(k)), \forall i, j \in \mathcal{S}. \tag{22}$$

Before proceeding with the main content, we present three lemmas. The first one is about the internal stability of the closed-loop PWA system (20) [19]. The second one is matrix S-lemma [12]. The third one is a matrix inequality result.

Lemma 1 (Lemma 1 in [19]): Consider the system (20). If the following inequality

$$\Delta V(x(k)) < -y(k)^\top y(k) + \sigma^2 \omega(k)^\top \omega(k) \tag{23}$$

holds for all $x(k) \neq 0$, then the nominal system of (20) (i.e., $\omega(k) = 0$) is asymptotically stable, and meanwhile, the H_∞ performance requirement (6) is fulfilled when $x(k) = 0$.

Lemma 2 ([12]): Let $M, N \in \mathbb{R}^{(n+m)\times(n+m)}$ be symmetric matrices. If there exists a scalar $\tau \geq 0$ such that $M - \tau N < 0$, then for all $Z \in \mathbb{R}^{m \times n}$ satisfying $[I, Z] N [I, Z]^\top \leq 0$, the inequality $[I, Z] M [I, Z]^\top < 0$ holds.

Lemma 3: If $1-h^\top h<0$ with $h\in\mathbb{R}^\ell$, then there exists at least one non-singular matrix $\Phi\in\mathbb{R}^{\ell\times\ell}$ such that $I_\ell-hh^\top<\Phi$ and $(I_\ell-hh^\top)^{-1}>\Phi^{-1}$.

Proof: The proof is straightforward, and thus it is omitted. ∎

Under Assumption 3, it can be derived that $1-h_i^\top h_i<0$ for any $i\in\Pi_1$. Leveraging Lemma 3, there exists a matrix $\Phi_i\in\mathbb{R}^{\ell_i\times\ell_i}$, $\forall i\in\Pi_1$ such that $I_{\ell_i}-h_ih_i^\top<\Phi_i$ and $(I_{\ell_i}-h_ih_i^\top)^{-1}>\Phi_i^{-1}$. These matrices will play a crucial role in the subsequent controller design process.

With the help of Lemmas 1-3, the solution to Problem 1 is summarized in the following theorem.

Theorem 1: Consider the PWA system (2). Suppose Assumptions 1-3 hold. Given a set of data ($u_{i,[0,T-1]}$, $x_{i,[0,T]}$, $y_{i,[0,T-1]}$), $\forall i\in\mathcal{S}$, matrices Φ_i, $\forall i\in\Pi_1$, and a performance index σ, the state feedback controller $u(k)=K_ix(k)$ with $K_i=S_i\Gamma_i^{-1}$ stabilizes the system (2) with $\omega(k)=0$ and fulfills the H_∞ performance requirement (6) with zero initial condition, if there exist scalars $\tau_{ij}\geq 0$, $\lambda_{ij}>0$, $\forall i,j\in\mathcal{S}$, positive definite matrices $\Gamma_i\in\mathbb{R}^{n\times n}$, and matrices $S_i\in\mathbb{R}^{m\times n}$, $\forall i\in\mathcal{S}$ such that the following linear matrix inequalities (LMIs)

$$\mathcal{M}_{ij}-\tau_{ij}\mathcal{N}_i<0,\forall i,j\in\mathcal{S}, \tag{24}$$

where the matrices $\mathcal{M}_{ij}$ and $\mathcal{N}_i$ depend on whether i belongs to Π_0 or Π_1. Specifically, if $i\in\Pi_0$, then $\mathcal{M}_{ij}=\text{diag}(Q_j,R_i)$, and $\mathcal{N}_i=\text{diag}(N_i,0_{n\times n})$ with N_i defined in (17); if $i\in\Pi_1$, then $\mathcal{M}_{ij}=\text{diag}(\lambda_{ij}^{-1}(I_{\ell_i}-\Phi_i),Q_j,-\lambda_{ij}^{-1},R_i)$, and $\mathcal{N}_i=\text{diag}(N_i,0_{n\times n})$ with N_i defined in (19). The notations Q_i and R_i are defined as

$$Q_j=\begin{bmatrix}-\Gamma_j+\frac{1}{\sigma^2}EE^\top & \frac{1}{\sigma^2}EF^\top\\ \frac{1}{\sigma^2}FE^\top & -I_p+\frac{1}{\sigma^2}FF^\top\end{bmatrix},\forall j\in\mathcal{S}, \tag{25}$$

$$R_i=\begin{bmatrix}\Gamma_i & S_i^\top & 0_{n\times n}\\ S_i & 0_{m\times m} & S_i\\ 0_{n\times n} & S_i^\top & -\Gamma_i\end{bmatrix},\forall i\in\mathcal{S}.$$

Proof: Based on Lemma 1, our goal is to prove that the inequality (23) holds if the LMIs (24) are feasible. At any given time k, there will be only two types of switching, i.e., 1) switching from $i\in\Pi_0$ to $j\in\mathcal{S}$ and 2) switching from $i\in\Pi_1$ to $j\in\mathcal{S}$. Therefore, the proof is divided into two cases: Case 1 when $i\in\Pi_1$ and Case 2 when $i\in\Pi_0$.

Case 1: When LMIs (24) is feasible with $i\in\Pi_1$, applying Schur complement [24] to (24) gives

$$M_{ij}-\tau_{ij}N_i<0,\forall i\in\Pi_1,j\in\mathcal{S}, \tag{26}$$

where $M_{ij}=\text{diag}(\lambda_{ij}^{-1}(I_{\ell_i}-\Phi_i),Q_j,-\lambda_{ij}^{-1},\bar{R}_i)$ with

$$\bar{R}_i=\begin{bmatrix}\Gamma_i & S_i^\top\\ S_i & S_i\Gamma_i^{-1}S_i^\top\end{bmatrix},$$

which, together with $K_i=S_i\Gamma_i^{-1}$, $i\in\Pi_1$, gives

$$\bar{R}_i=\begin{bmatrix}\Gamma_i & \Gamma_iK_i^\top\\ K_i\Gamma_i & K_i\Gamma_iK_i^\top\end{bmatrix}=\begin{bmatrix}I_n\\ K_i\end{bmatrix}\Gamma_i\begin{bmatrix}I_n\\ K_i\end{bmatrix}^\top. \tag{27}$$

By applying Lemma 2 to (26), we can deduce that for any $i\in\Pi_1$ and $j\in\mathcal{S}$, the following QMI

$$[I_{\ell_i+n+p},Z_i]M_{ij}[I_{\ell_i+n+p},Z_i]^\top<0 \tag{28}$$

holds for all Z_i such that (16), where Z_i is defined in (14). In other words, inequality (28) holds for all systems in Ω_i.

Substituting the definition of M_{ij} into (28) gives

$$\begin{bmatrix}\lambda_{ij}^{-1}(I_{\ell_i}-\Phi_i) & 0\\ 0 & Q_j\end{bmatrix}+Z_i\begin{bmatrix}-\lambda_{ij}^{-1} & 0\\ 0 & \bar{R}_i\end{bmatrix}Z_i^\top<0. \tag{29}$$

Substituting (25), (27), and the definition of $A_{i,cl}$, $C_{i,cl}$ and Z_i into (29) gives

$$\begin{bmatrix}\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top-\Phi_i) & -\lambda_{ij}^{-1}h_ib_i^\top & -\lambda_{ij}^{-1}h_id_i^\top\\ -\lambda_{ij}^{-1}b_ih_i^\top & -\Gamma_j-\lambda_{ij}^{-1}b_ib_i^\top & -\lambda_{ij}^{-1}b_id_i^\top\\ -\lambda_{ij}^{-1}d_ih_i^\top & -\lambda_{ij}^{-1}d_ib_i^\top & -I_p-\lambda_{ij}^{-1}d_id_i^\top\end{bmatrix}$$
$$-\begin{bmatrix}H_i\Gamma_i & 0_{\ell_i\times q}\\ A_{i,cl}\Gamma_i & \frac{1}{\sigma^2}E\\ C_{i,cl}\Gamma_i & \frac{1}{\sigma^2}F\end{bmatrix}\begin{bmatrix}-\Gamma_i^{-1} & 0\\ 0 & -\sigma^2I_q\end{bmatrix}\begin{bmatrix}H_i\Gamma_i & 0_{\ell_i\times q}\\ A_{i,cl}\Gamma_i & \frac{1}{\sigma^2}E\\ C_{i,cl}\Gamma_i & \frac{1}{\sigma^2}F\end{bmatrix}^\top<0. \tag{30}$$

Since $\text{diag}(-\Gamma_i^{-1},-\sigma^2I_q)<0$ and $\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top-\Phi_i)<0$, applying Schur complement to (30) twice gives

$$\mathscr{H}_{ij}-\tilde{\phi}_{ij}\big(\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top-\Phi_i)\big)^{-1}\tilde{\phi}_{ij}^\top<0, \tag{31}$$

where $\tilde{\phi}_{ij}=[-\lambda_{ij}^{-1}h_ib_i^\top,-\lambda_{ij}^{-1}h_id_i^\top,H_i\Gamma_i,0_{\ell_i\times q}]^\top$, and

$$\mathscr{H}_{ij}=\begin{bmatrix}-\Gamma_j-\lambda_{ij}^{-1}b_ib_i^\top & -\lambda_{ij}^{-1}b_id_i^\top & A_{i,cl}\Gamma_i & \frac{1}{\sigma^2}E\\ -\lambda_{ij}^{-1}d_ib_i^\top & -I_p-\lambda_{ij}^{-1}d_id_i^\top & C_{i,cl}\Gamma_i & \frac{1}{\sigma^2}F\\ \Gamma_iA_{i,cl}^\top & \Gamma_iC_{i,cl}^\top & -\Gamma_i & 0\\ \frac{1}{\sigma^2}E^\top & \frac{1}{\sigma^2}F^\top & 0 & -\frac{1}{\sigma^2}I_q\end{bmatrix}.$$

Let $\Theta_i=I_{\ell_i}-h_ih_i^\top$. By applying the matrix inversion lemma [24], we get

$$(\Theta_i-\Phi_i)^{-1}=\Theta_i^{-1}-\Theta_i^{-1}(-\Phi_i^{-1}+\Theta_i^{-1})^{-1}\Theta_i^{-1}.$$

Since $-\Phi_i^{-1}+(I_{\ell_i}-h_ih_i^\top)^{-1}>0$, i.e., $-\Phi_i^{-1}+\Theta_i^{-1}>0$, we get $\Theta_i^{-1}(-\Phi_i^{-1}+\Theta_i^{-1})^{-1}\Theta_i^{-1}\geq 0$. It implies that $(\Theta_i-\Phi_i)^{-1}\leq\Theta_i^{-1}$, which is equivalent to

$$(I_{\ell_i}-h_ih_i^\top-\Phi_i)^{-1}\leq(I_{\ell_i}-h_ih_i^\top)^{-1}. \tag{32}$$

Then, combining (32) and (31) yields

$$\mathscr{H}_{ij}-\tilde{\phi}_{ij}\big(\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top)\big)^{-1}\tilde{\phi}_{ij}^\top<0. \tag{33}$$

Let $P_j=\Gamma_j^{-1}$, $\forall j\in\mathcal{S}$. Pre- and post-multiplying (33) with $\text{diag}(P_j,I_p,P_i,\sigma^2I_q)$, respectively, yields

$$\begin{bmatrix}-P_j-\lambda_{ij}^{-1}P_jb_ib_i^\top P_j & -\lambda_{ij}^{-1}P_jb_id_i^\top & P_jA_{i,cl} & P_jE\\ -\lambda_{ij}^{-1}d_ib_i^\top P_j & -I_p-\lambda_{ij}^{-1}d_id_i^\top & C_{i,cl} & F\\ A_{i,cl}^\top P_j & C_{i,cl}^\top & -P_i & 0\\ E^\top P_j & F^\top & 0 & -\sigma^2I_q\end{bmatrix}$$
$$-\hat{\phi}_{ij}\big(\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top)\big)^{-1}\hat{\phi}_{ij}^\top<0, \tag{34}$$

where $\hat{\phi}_{ij}=[-\lambda_{ij}^{-1}h_ib_i^\top P_j,-\lambda_{ij}^{-1}h_id_i^\top,H_i,0_{\ell_i\times q}]^\top$.

Applying the matrix inversion lemma to each term in $\hat{\phi}_{ij}\big(\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top)\big)^{-1}\hat{\phi}_{ij}^\top$ yields

$$\begin{aligned}&\hat{\phi}_{ij}\big(\lambda_{ij}^{-1}(I_{\ell_i}-h_ih_i^\top)\big)^{-1}\hat{\phi}_{ij}^\top\\ &=\Psi_{ij}+\phi_{ij}\big(\lambda_{ij}(1-h_i^\top h_i)\big)^{-1}\phi_{ij}^\top,\end{aligned} \tag{35}$$

where $\phi_{ij}=[b_i^\top P_j,d_i^\top,-\lambda_{ij}h_i^\top H_i,0_{1\times q}]^\top$ and

$$\Psi_{ij}=\Big(\begin{bmatrix}-\lambda_{ij}^{-1}P_jb_ib_i^\top P_j & -\lambda_{ij}^{-1}P_jb_id_i^\top\\ -\lambda_{ij}^{-1}d_ib_i^\top P_j^\top & -\lambda_{ij}^{-1}d_id_i^\top\end{bmatrix},\lambda_{ij}H_i^\top H_i,0_{q\times q}\Big).$$

Substituting (35) into (34) gives

$$\begin{bmatrix} -P_j & 0 & P_jA_{i,cl} & P_jE \\ 0 & -I_p & C_{i,cl} & F \\ A_{i,cl}^\top P_j & C_{i,cl}^\top & -P_i - \lambda_{ij}H_i^\top H_i & 0 \\ E^\top P_j & F^\top & 0 & -\sigma^2 I_q \end{bmatrix} - \phi_{ij}\left(\lambda_{ij}(1-h_i^\top h_i)\right)^{-1}\phi_{ij}^\top < 0. \quad (36)$$

Since $\lambda_{ij}(1 - h_i^\top h_i) < 0$ and $\mathrm{diag}(-P_j, -I_p) < 0$, applying Schur complement to (36) twice gives

$$\tilde{M}_{ij} - \lambda_{ij}\Xi_i < 0, \quad (37)$$

where

$$\tilde{M}_{ij} = \begin{bmatrix} A_{i,cl}^\top \\ E^\top \\ b_i^\top \end{bmatrix} P_j \begin{bmatrix} A_{i,cl}^\top \\ E^\top \\ b_i^\top \end{bmatrix}^\top + \begin{bmatrix} C_{i,cl}^\top \\ F^\top \\ d_i^\top \end{bmatrix}\begin{bmatrix} C_{i,cl}^\top \\ F^\top \\ d_i^\top \end{bmatrix}^\top - \begin{bmatrix} P_i & 0 & 0 \\ 0 & \sigma^2 I_q & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Xi_i = \begin{bmatrix} H_i^\top H_i & 0 & H_i^\top h_i \\ 0 & 0_{q\times q} & 0 \\ h_i^\top H_i & 0 & h_i^\top h_i - 1 \end{bmatrix}.$$

On the other hand, the (degenerate) ellipsoid (5), i.e., $\mathcal{X}_i$, can be written as

$$\xi(k)^\top \Xi_i \xi(k) \le 0, \quad (38)$$

where $\xi(k) = [x(k)^\top, \omega(k)^\top, 1]^\top$.

By applying S-procedure [24], (37) is equivalent to that

$$\xi(k)^\top \tilde{M}_{ij}\xi(k) < 0 \quad (39)$$

holds for all $\xi(k)$ such that (38). Since $\mathscr{X}_i \subseteq \mathcal{X}_i$ under Assumption 3, the inequality (39) holds when $x(k) \in \mathscr{X}_i$, $\forall i \in \Pi_1$.

Substituting (20b) into (39) yields that

$$\Delta V(x(k)) = x(k+1)^\top P_j x(k+1) - x(k)^\top P_i x(k) < -y(k)^\top y(k) + \sigma^2 \omega(k)^\top \omega(k) \quad (40)$$

holds when $x(k) \in \mathscr{X}_i, \forall i \in \Pi_1$ and $x(k+1) \in \mathscr{X}_j, \forall j \in \mathcal{S}$.

For Case 2, we can prove that if the LMIs (24) are feasible for all $i \in \Pi_0$ and $j \in \mathcal{S}$, then the inequality (23) holds when $x(k) \in \mathscr{X}_i$, $\forall i \in \Pi_0$ and $x(k+1) \in \mathscr{X}_j$, $\forall j \in \mathcal{S}$. The proof of Case 2 is similar to that of Case 1, and thus it is omitted.

We conclude that if the LMIs (24) are feasible, then the inequality (23) holds. Therefore, the proof is completed by using Lemma 1. ■

Remark 2: When considering subsystem i, $\forall i \in \Pi_0$, we have $b_i = 0$, $d_i = 0$, and $h_i^\top h_i - 1 \le 0$. Consequently, the LMI (37) becomes infeasible for all $i \in \Pi_0$. Given this issue, the designed controller does not consider the partition information when $i \in \Pi_0$. This is also why the QMI (18) for those subsystems in Π_0 does not contain the partition information.

Remark 3: For each pair (i, j), $i, j \in \mathcal{S}$, the proposed controller needs to ensure that (23) holds when switching from any system in Ω_i to any system in Ω_j. It is conservative compared to model-based controllers, which only need to ensure that (23) holds when switching from subsystem i to subsystem j. To mitigate this conservativeness, the size of Ω_i, $i \in \mathcal{S}$ should be reduced. According to the deviation process of Ω_i in Section III, the size of Ω_i is heavily influenced by the collected data (3a)-(3b) and the disturbance bound Υ. A more informative data sequence and a tighter disturbance bound can lead to a smaller size of Ω_i, which may reduce the conservativeness of the designed controller.

Remark 4: The controller design method can be extended to a broader class of PWA systems. Compared to the system (2), the disturbances $\omega(k)$ in the extended system can have any dimension, and the corresponding matrices E and F can be any known value. The only requirement of E and F is that the matrix $[E^\top, F^\top]^\top$ has full column rank.

V. Case Study

In this section, we demonstrate the effectiveness of the proposed control method using a single-link robot arm control system as an example. For simplicity, we adopt a discrete-time PWA model of the system from [27], which has the same form as (2), where $i = 1, 2, 3$; $x(k) = [x_1(k), x_2(k)]^\top \in \mathscr{X}_1 \cup \mathscr{X}_2 \cup \mathscr{X}_3$ denotes the angle position and angular velocity of the arm, respectively; $u(k) \in \mathbb{R}$ denotes the torque supplied by the motor; $y(k) \in \mathbb{R}$ denotes the angle position of the arm; the sampling time is 0.1 s. The control target is to stabilize the angle position to the origin. The related system matrices are

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.9168 & 0.95 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.2834 & 0.95 \end{bmatrix},$$
$$B_1 = B_2 = B_3 = [0, 0.1]^\top, E = [0, 0.1]^\top,$$
$$b_1 = [0, 0]^\top, b_2 = -b_3 = [0, 0.3980]^\top,$$
$$C_1 = C_2 = C_3 = [1, 0], D_1 = D_2 = D_3 = 0, F = 0.3.$$

The polyhedral partitions $\mathscr{X}_i$, $i = 1, 2, 3$ are given by

$$\mathscr{X}_1 = \{x(k) | -\pi/5 \le x_1(k) \le \pi/5, -5 \le x_2(k) \le 5\},$$
$$\mathscr{X}_2 = \{x(k) | -3\pi/5 \le x_1(k) \le -\pi/5, -5 \le x_2(k) \le 5\},$$
$$\mathscr{X}_3 = \{x(k) | \pi/5 \le x_1(k) \le 3\pi/5, -5 \le x_2(k) \le 5\}.$$

The system matrices A_i, B_i, C_i, D_i, b_i, and d_i, $i = 1, 2, 3$ are assumed to be unknown in the control design process, and the disturbance $\omega(k)$ is bounded by $[-0.2, 0.2]$.

Following Remark 2, we exclude the partition $\mathscr{X}_1$, which contains the origin, from the controller design. Therefore, we only provide outer approximations for the polyhedral partitions that do not contain the origin, i.e., $\mathscr{X}_2$ and $\mathscr{X}_3$. The definitions of the corresponding outer approximations are $\mathcal{X}_i = \{x(k) | \|H_i x(k) + h_i\|_2 \le 1\}$, $i = 2, 3$, where $H_2 = H_3 = \mathrm{diag}(1.7383, 0.1000)$, $h_2 = -h_3 = [1.7321, 0]^\top$.

By simulating the system subject to a disturbance $\omega(k)$, we collect an input-state-output trajectory with length $T = 10$ for each subsystem and denote it as $(u_{i,[0,9]}, x_{i,[0,10]}, y_{i,[0,9]})$, $i = 1, 2, 3$. The input signals u_i and disturbances ω are randomly generated in the interval $[-1, 1]$ and $[-0.2, 0.2]$, respectively. Then, referring to Section III, we can calculate a data-based QMI for each subsystem.

We select the disturbance bound $\Upsilon = 0.4$, the performance index $\sigma = 0.5$ and $\Phi_2 = \Phi_3 = \mathrm{diag}(-0.005, 1.005)$. Then, the state feedback controller $u(k) = K_i x(k)$, $i = 1, 2, 3$ is designed by solving the LMIs (24) in Theorem 1 with

$K_1 = [-6.0402, -10.8067]$, $K_2 = [-6.3116, -10.8534]$ and $K_3 = [-5.2280, -11.1176]$.

Let the initial state be $x(0) = [0, 0]^\top$. we define $\delta(\sigma, L) = \sum_{k=0}^{L} \|y(k)\|_2^2 - \sigma^2 \|\omega(k)\|_2^2$. By simulation, we get $\delta(\sigma, 100) = -0.2256$. It is seen that the H_∞ performance (6) holds in the simulation time interval.

The closed-loop output results with the initial condition $x(0) = [-\frac{\pi}{3}, \frac{\pi}{2}]^\top$ are given in Fig. 1. The first subgraph shows the bounded disturbance, and the last two subgraphs show the switched subsystems and output results of the closed-loop system with our proposed DDC law and an H_∞ model-based control (MBC) law, respectively. The model-based controller is designed using the same procedure as the data-driven controller, except that the former directly uses explicit system model information. It can be observed that our proposed data-driven controller can achieve a control performance that is marginally lower than the model-based controller. The discrepancy observed could be attributed to the conservativeness of the proposed controller, as mentioned in Remark 3.

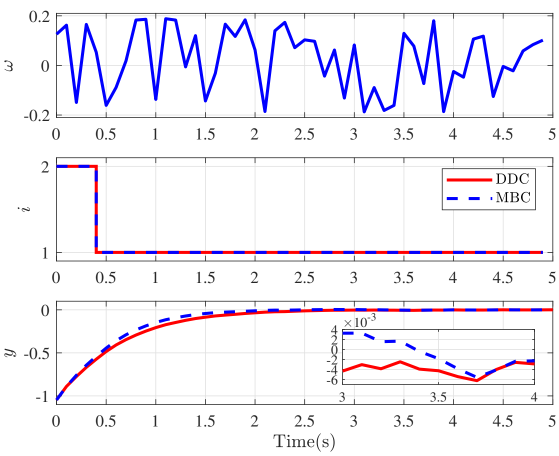

Fig. 1. The disturbance, switched subsystems, and output trajectories of the closed-loop system with DDC law and MBC law, respectively.

VI. Conclusion

This paper has studied the problem of designing a data-driven H_∞ controller for unknown PWA systems with bounded disturbances. First, a group of data-based QMIs has been constructed to describe all possible systems that may generate the given input-state-output data. Then, a data-driven H_∞ controller has been designed for all systems satisfying QMIs. A single-link robot arm control system has been used to show the effectiveness of the proposed method.

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