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An Event-Triggered Adaptive Quantized Feedback Control for LTI Systems*

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Abstract— **This study's topic is stabilizing a linear time-invariant plant subjected to bounded disturbance using quantized state feedback. We propose an adaptive quantization scheme consisting of two terms, a natural decay term, and a positive coupling proportional to the norm of the state. The proposed procedure ensures the utilization of a lower quantizer resolution if the system is steered away from the origin due to disturbances. In conjunction with dynamic quantization, the event-triggered approach is motivated by the need to utilize computing resources efficiently. The proposed control law and dynamic quantization ensure the system is input-to-state stable (ISS). We ensure that Zeno's phenomenon is absent by showing that the inter-event times are lower bound. We compare the average number of bits the controller requires between the proposed reactive control law and a controller with constant quantization error on a numerical example.**

I. Introduction

Traditional control techniques assume that the state of the system and the desired control input are available with infinite precision. However, in practical applications, this is not satisfied. For example, information quantization occurs before transmission over a digital communication network. Mathematically analyzing the effects of truncation and limitations in data is to consider a quantizer present in the feedback control loop. A quantizer is a function that maps a signal into a set of finite bins.

Quantized feedback systems have attracted much attention since the work of Kalman, demonstrating that a fixed quantizer in the system can cause limit cycles and chaotic behaviour in [1]. Constructing controllers for fixed quantizers that guarantee input-to-state stability is possible, as shown in [4]. On the other hand, if the quantizer is dynamic, the parameters of the quantizer can be modified to guarantee asymptotic stability as shown in [2], [3]. This methodology of modifying the parameters of the quantizer has been further utilized to guarantee the stability of quantized feedback systems with disturbance [5], switched systems [6], nonlinear systems in [7] and uncertain linear systems in [8], for multi-agent systems in [9].

On a parallel development, event-triggered control has drawn interest to reduce resource consumption. Applying corrective control only when the performance deviates from satisfactory is the central tenet of event-triggered control; see ([10], [11]) and the references therein. An event-triggered approach for controlling quantized feedback systems is of interest to us due to its advantages in reducing energy consumption, bandwidth usage, and computational resources.

The authors in [12] investigate the combination of an event-triggered controller and dynamic quantization. In [13], the authors design sliding-mode control via an event-triggered approach for a quantized feedback system. The work in [14] deals with designing a dynamic quantizer, where the quantizer resolution update depends on a triggering condition based on quantizer saturation. The authors in [15] use quantization events to trigger sampling for control. In earlier works, updating the quantizer resolution is done as a preset rule, thus utilizing a finer quantizer resolution, an undesirable behaviour if the system's state is steered away from the origin. Our work tackles this issue by adding positive feedback on the norm of the quantized state in the quantizer update rule.

The contribution of the paper is as follows. We present a novel adaptive event-triggered-based control law combining the two approaches (event-triggering and adaptive dynamic quantization) that rely solely on the system's quantized feedback. We show using Lyapunov stability that the closed-loop system under the proposed quantized state-feedback control law is ISS. By showing that the inter-event times are lower bound, we also demonstrate that the proposed controller does not exhibit Zeno behaviour.

The structure of the paper is as follows: Section II collects the mathematical preliminaries. Section III discusses relevant material on event-triggered control and quantized feedback systems. Section IV defines the problem, proposes the control law, and demonstrates how we avoid Zeno behavior. Section V provides a tuning strategy, and the control law is validated using a numerical example, the reduction in the average number of bits the controller uses. The paper concludes with the conclusion in Section VI.

II. Notations

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}^n_{\geq 0}$, $\mathbb{Z}$ and $\mathbb{N}$ represent the set of real numbers, non-negative real numbers, non-negative real vectors, the set of integers and natural numbers respectively. The identity matrix of order n is denoted by I_n. Given $x \in \mathbb{R}^n$, $B(x,r) \subset \mathbb{R}^n$ denotes the open ball of radius $r > 0$ centered at x. For the sets $A \subset \mathbb{R}^n$, we denote by $\mathrm{co}(A)$, $\overline{\mathrm{co}}(A)$, $A^c, \partial A$ the convex hull, closure of the convex hull, complement and the boundary of the set A respectively. $\lceil x \rceil$ denotes the least integer greater than x.

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For a matrix A, $A \succ 0$ denotes a symmetric positive-definite matrix.

III. Background

A. Quantized feedback systems

Assume that the space $\mathbb{R}^n$ is partitioned into a finite number of disjoint, open, and connected domains $D_i, i \in \{1,2,\ldots,m\}$ whose unions cover $\mathbb{R}^n$. A dynamic quantizer, as defined in [4], is given by the map, $q:\mathbb{R}^n \to Q$, where Q is a finite subset of $\mathbb{R}^n$. Let $S = \bigcup_{i=1}^m \partial D_i$ denote the union of all boundaries of the domains and is of measure zero. The parameters $\Delta \in (0,\infty)$, $M \in \mathbb{Z}_{>0}$ are known as the quantization error and the saturation value, respectively. The quantizer further satisfies

$$\begin{cases} \|q(x)-x\| \leq \Delta & \|x\| \leq M\Delta \\ \|q(x)-x\| > M\Delta - \Delta & \|x\| > M\Delta. \end{cases}$$

Similar to [5], we consider the approach that it is possible to change the quantization error (Δ) of the quantizer dynamically but not the saturation value (M). Note that the quantization error is inversely proportional to the quantizer resolution that is obtained. We consider a time-invariant linear system

$$\dot{x} = Ax + Bu,\; x \in \mathbb{R}^n,\; u \in \mathbb{R}^m \tag{1}$$

where the pair (A,B) is stabilizable. Instead of the state feedback with infinite precision, only the quantized information about the state is accessible. This requires us to replace the linear stabilizing feedback law $u = Kx$ by $u = Kq(x)$, leading to a closed-loop system that is discontinuous on the set S. The solutions to such a system are to be understood in terms of Filippov.

B. Filippov Set-Valued maps and solutions

Consider the closed-loop system (1) with quantized feedback control $u = Kq(x)$

$$\dot{x} = X(x) := Ax + BKq(x),\; x(0) = x_0 \tag{2}$$

As (2) is discontinuous on the set S, the corresponding Filippov set-valued map $F[X](x): \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

$$F[X](x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{co}\{X(B(x,\delta)\backslash S)\}. \tag{3}$$

Thus the discontinuous system (2) is posed as a differential inclusion

$$\dot{x} \in F[X](x). \tag{4}$$

A Filippov solution to the differential equation (2) is a Carathéodory solution to the differential inclusion (4). A Carathéodary solution of (4) with initial condition $x(0) = x_0$ on the interval $[0,\tau]$ is an absolutely continuous mapping $x: [0,\tau] \to \mathbb{R}^n$ satisfying (4) for almost all $t \in [0,\tau]$ and $x(0) = x_0$. Since the Filippov set-valued map (3), by construction, is upper semi-continuous with non-empty, convex, and compact values, locally bounded, it follows that Filippov solutions to (2) exists [16].

C. Event Triggered Control

In event-triggered control, the control law is updated in a non-periodic fashion based on triggering conditions. Following the simplified relative thresholding scheme proposed in [10], for a fixed $K \in \mathbb{R}^{m\times n}$ the control law takes the form

$$u(t) = Kx(t_k),\; \text{if } t \in [t_k, t_{k+1}) \tag{5}$$

and the corresponding closed-loop system (1)-(5) is given by,

$$\dot{\eta}(t) = A\eta(t) + BKx(t_k), \eta(t_k) = x(t_k), t \in [t_k, t_{k+1})$$

where $t_k, k \in \mathbb{N}, K \in \mathbb{R}^{m\times n}$ denotes the k^{th} update or triggering instant given by

$$\tau_k = \inf\left\{t \in \mathbb{R}_{\geq 0} \mid \|\eta(t) - x(t_k)\| = \sigma\|\eta(t)\|\right\}. \tag{6}$$

The real number τ_k defines the inter event time given by

$$\tau_k = t_k - t_{k-1}, k \geq 1$$
$$t_0 = 0.$$

We use the relative thresholding condition given by $\|e(t)\| = \sigma\|x(t)\|$, where $e(t) = x(t_k) - x(t)$ known as the relative error. When dealing with controllers that are used in switching systems we have to ensure that Zeno behaviour is avoided. Control laws that demand instantaneous switching or infinitely numerous switches in a finite interval, both of which are undesirable, are characteristics of Zeno behaviour. These are known as Type-1 and Type-2 Zeno behaviour respectively [4]. If the switching times are given by $t_k, k \in \mathbb{N}$ and the inter-event times are given by τ_k, then it is sufficient to show the inter-event times are lower bounded $\tau_k \geq \tau_* > 0, \forall\; k \in \mathbb{N}$ to avoid Zeno Behaviour.

IV. Problem Formulation and Methodology

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t),\;\; w(t) \in \mathbb{R}^n. \tag{7}$$

with bounded disturbance $\|w(t)\| \leq W, \forall\, t \geq 0$. The goal is to design functions $G: \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $H: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $T: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, that depend on the quantized state-feedback $q(x(t), \Delta(t))$, with the control law given by (8), such that the closed-loop system (7)-(8) is ISS.

$$\begin{aligned} u(t) &= G(q(x(t_k)), \Delta(t)) && \text{if } t \in [t_k, t_{k+1}) \\ \Delta(t) &= \Delta_k = H(t, q(x(t_k), \Delta_{k-1}), \Delta_{k-1}) && \text{if } t \in [t_k, t_{k+1}) \\ t_k &= T(t_{k-1}, q(x(t))),\; k \in \{1,2,\ldots\}. \end{aligned} \tag{8}$$

We propose a control law that incorporates event-triggering to modify the resolution of the quantizer. The proposed control law has an inverse mechanism that adapts the quantizer resolution based on the norm of the quantized state-feedback that indirectly captures the disturbance acting on the system. Without loss of generality, we assume $x(0) \neq 0$. The proposed control law acts in two stages: Initialization and stabilization. In the initialization stage, a one-off step: the system is left in open-loop until a suitable starting condition is met. Once this is met a zero order hold (ZOH) control law is utilized to stabilize the system. The control law is given as follows,

$$u = \begin{cases} 0 & 0 < t < T_0 \\ Kq(x(t_k), \Delta_{k-1}), & T_0 \le t_k \le t < t_{k+1} \end{cases} \tag{9}$$

$$\Delta(t) = \Delta_k = \max\{\Omega\Delta_{k-1} + \gamma \|q(x(t_k), \Delta_{k-1})\|, \Delta_{\min}\}, \tag{10}$$

$$T_0 = \inf\left\{ t \ge 0 \,\middle|\, \max\left\{ \frac{k_1\Omega}{(1-k_1\gamma)} + r_w(1+\sigma), \frac{\frac{\sigma}{(1+\sigma)} - \gamma}{(1+\Omega+d)} \right\} < \frac{\|q(x(t), \Delta_1)\|}{\Delta_1} \right\} \tag{11}$$

where T_0 given by (11) defines the time duration of the initialization stage subject to the open-loop dynamics $\dot{x}(t) = Ax(t)$ and $d > 0$. $\Delta_1 > 0$ is the initial quantization error. The t_k's, known as the triggering instances are defined using a modified relative thresholding scheme,

$$\tau_k = \inf\{\tau \in \mathbb{R}_{\ge 0} | \, \|q(\eta(\tau), \Delta_k) - q(\eta(t_k), \Delta_{k-1})\| \ge \sigma \|q(\eta(\tau), \Delta_k)\|\} \tag{12}$$
$$t_{k+1} = \tau_k + t_k, \; k \in \mathbb{N}, \; t_1 = T_0$$

and $\eta(\tau)$ is the solution to the differential equation

$$\dot{\eta}(\tau) = A\eta(\tau) + BKq(x(t_k), \Delta_{k-1}), \eta(\tau) \in \mathbb{R}^n, \tau \in \mathbb{R}_{\ge 0}$$

with the initial condition $\eta(0) = q(x(t_k), \Delta_{k-1})$. We present the main contributions of the letter through the following theorem.

Theorem 1: The system (7) is rendered input-to-state stable using the ZOH feedback control law (9)-(10). The various parameters defined in the control law satisfy the following constraints,

C1. $A_c = (A + BK)$ is Hurwitz and there exists $P, Q \succ 0$ such that $PA_c + A_c^T P = -Q$.

C2. σ, Ω satisfy

$$\sigma \in (0, \frac{a}{b}) \bigcap (0, 1), \quad 0 < \Omega < \frac{1}{1+\sigma} \tag{13}$$

where $a := \lambda_{min}(Q), b := \|K^T B^T P + PBK\|, \beta := \frac{b(1+\sigma)}{(1-b\sigma)}$, $k_1 := (1+\sigma)(1+\beta)$, $r_w := \frac{cW}{(a-b\sigma)}$, $c := \|K^T B_w^T P + PB_w K\|$.

C3. The constant γ satisfies

$$0 < \gamma < \min\left\{ \frac{(\frac{\sigma}{1+\sigma})(1 - \Omega(1-\sigma))}{((1+d)(1-\sigma)+1)}, \left(\frac{1}{k_1} - \frac{\Omega}{1+\beta} \right) \right\}.$$

C4. $\Delta_{\min}$ denotes the minimum resolution of the quantizer given by

$$\Delta_{\min} = \frac{r_w(\sigma + 1 - k_1\gamma)}{(1+\beta)(1 - k_1\gamma - \Omega(1+\sigma))}$$

and the saturation of the quantizer satisfies

$$M > 1 + \frac{1}{\gamma(1-\sigma)}.$$

Further the control law does not exhibit Zeno behaviour as the inter event time is lower bounded.

Proof: As the matrix A_c is Hurwitz the linear feedback control $u = Kx$ is a stabilizing controller. The closed-loop control system (7)-(8) with quantized feedback control in $t \in [t_k, t_{k+1})$ is written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKq(x(t_k), \Delta_{k-1}) + B_w w(t) \\ &= A_c x(t) + BKs(t) + BKe(t) + B_w w(t) \end{aligned} \tag{14}$$

where, $s(t) = q(x(t), \Delta_k) - x(t)$ is called the quantization error and $e(t) = q(x(t_k), \Delta_{k-1}) - q(x(t), \Delta_k)$ is the quantized state error. Consider the candidate Lyapunov function $V = x^T Px$, the derivative of V along the trajectories of the system (14) during the time window $t \in [t_k, t_{k+1})$ is

$$\begin{aligned} \dot{V} &= x^T(A_c^T P + PA_c)x + x^T(K^T B^T P + PBK)s + \\ &\quad x^T(K^T B^T P + PBK)e + x^T(K^T B_w^T P + PB_w K)w \\ &\le -\lambda_{\min}(Q)\|x\|^2 + \|x\|\|K^T B^T P + PBK\|\|s\| \\ &\quad + \|x\|\|K^T B^T P + PBK\|\|e\| + \|x\|\|K^T B_w^T P + PB_w K\|\|w\|. \end{aligned}$$

Using the fact that $\|s(t)\| \le \Delta_k$, $t \in [t_k, t_{k+1})$, it follows

$$\begin{aligned} \dot{V} &\le -a\|x\|^2 + \|x\|b\Delta_k + \|x\|b\sigma\|q(x)\| + \|x\|c\|w\| \\ &\le -a\|x\|^2 + \|x\|b\Delta_k + \|x\|b\sigma(\|x\| + \Delta_k) + \|x\|cW \\ &= -(a-b\sigma)\|x\| \left(\|x\| - \frac{b(1+\sigma)\Delta_k}{(a-b\sigma)} - \frac{cW}{(a-b\sigma)} \right). \end{aligned}$$

Thus, $\dot{V}$ is negative definite in the closed region given by (15)

$$R = \left\{ x \in \mathbb{R}^n \,\middle|\, \|x\| \ge \frac{b(1+\sigma)\Delta_k}{(a-b\sigma)} + r_w = \beta\Delta_k + r_w \right\}. \tag{15}$$

Then V along the system trajectories is decreasing outside the region R^c. The triggering condition is given by

$$\|q(x(t_{k+1}), \Delta_k) - q(x(t_k), \Delta_{k-1})\| = \sigma \|q(x(t_{k+1}), \Delta_k)\| \tag{16}$$

and using *Lemma.*1, (16) can be rewritten as

$$\begin{aligned} &\|q(x(t_{k+1}), \Delta_k) - \frac{1}{1-\sigma^2} q(x(t_k), \Delta_{k-1})\| \\ &\qquad = \frac{\sigma}{1-\sigma^2} \|q(x(t_k), \Delta_{k-1})\|. \end{aligned} \tag{17}$$

The condition (17) gives us the region R_1 defined as

$$R_1 = \left\{ x \in \mathbb{R}^n \,\middle|\, \|q(x(t_{k+1}), \Delta_k) - \frac{1}{1-\sigma^2} q(x(t_k), \Delta_{k-1})\| \ge \frac{\sigma}{1-\sigma^2} \|q(x(t_k), \Delta_{k-1})\| \right\}. \tag{18}$$

As the triggering occurs on the boundary of the region R_1^c and as $\dot{V}$ is negative outside the region R^c, if we ensure that $R^c \cap R_1^c = \emptyset$, we are guaranteed that V is always decreasing between two triggering instances. This can be ensured if the sum of the radii of the regions R^c, R_1^c is lesser than the center-to-center distance between R^c, R_1^c, giving us

$$\begin{aligned}\frac{1}{1-\sigma^2}\|q(x(t_k),\Delta_{k-1})\| &> \frac{\sigma}{1-\sigma^2}\|q(x(t_k),\Delta_{k-1})\| \\ &\quad + \beta\Delta_k + \Delta_k + r_w \\ \|q(x(t_k),\Delta_{k-1})\| &> k_1\Delta_k + r_w(1+\sigma) \\ &> k_1\Omega\Delta_{k-1} + k_1\gamma\|q(x(t_k),\Delta_{k-1})\| \\ &\quad + r_w(1+\sigma) \\ \implies \|q(x(t_k),\Delta_{k-1})\| &> \frac{k_1\Omega}{1-k_1\gamma}\Delta_{k-1} + \frac{r_w(1+\sigma)}{1-k_1\gamma}. \quad (19)\end{aligned}$$

This ensures that Lyapunov function is decreasing along the trajectories of the closed-loop system (7)-(9). To show that the control law avoids Zeno Behaviour, we consider the transformation in the time interval $[t_k, t_{k+1})$ as

$$\zeta(t) := x(t) - \frac{1}{1-\sigma^2}q(x(t_k),\Delta_{k-1}), \text{ resulting in}$$

$$\dot{\zeta}(t) = A\zeta(t) + (BK + \frac{A}{1-\sigma^2})\, q(x(t_k),\Delta_{k-1}) + B_w w(t). \quad (20)$$

If the inter-event times are given by $\tau_k = t_{k+1} - t_k$ the solution of the system (20) in the time interval $[t_k, t_{k+1}]$ is given as

$$\begin{aligned}\zeta(t_{k+1}) &= e^{A\tau_k}\zeta(t_k) \\ &\quad + \int_0^{\tau_k} e^{A\tau}\left(BK + \frac{A}{1-\sigma^2}\right)q(x(t_k),\Delta_{k-1})\,d\tau \\ &\quad + \int_0^{\tau_k} e^{A\tau}B_w w(t)\,d\tau.\end{aligned}$$

Taking the norm of the transformed state $\zeta(t)$ and using the triangle inequality we get,

$$\begin{aligned}\|\zeta(t_{k+1})\| &\le e^{\|A\|\tau_k}\|\zeta(t_k)\| \\ &\quad + \int_0^{\tau_k} e^{\|A\|\tau}\|BK + \frac{1}{1-\sigma^2}A\|\|q(x(t_k),\Delta_{k-1})\|\,d\tau \\ &\quad + \int_0^{\tau_k} e^{\|A\|\tau}\|B_w\|W\,d\tau.\end{aligned}$$

Rearranging the inequality we obtain a bound on τ_k,

$$\tau_k \ge \frac{1}{\|A\|}\ln\left(\frac{\frac{\|\zeta(t_{k+1})\|}{\|q(x(t_k),\Delta_{k-1})\|} + \frac{\|BK+\frac{1}{1-\sigma^2}A\|}{\|A\|} + \frac{\|B_w\|W}{\|A\|\|q(x(t_k),\Delta_{k-1})\|}}{\frac{\|\zeta(t_k)\|}{\|q(x(t_k),\Delta_{k-1})\|} + \frac{\|BK+\frac{1}{1-\sigma^2}A\|}{\|A\|} + \frac{\|B_w\|W}{\|A\|\|q(x(t_k),\Delta_{k-1})\|}}\right).$$

In order to obtain bounds on the various fractions, first consider $\frac{\|\zeta(t_k)\|}{\|q(x(t_k),\Delta_{k-1})\|}$,

$$\begin{aligned}\|\zeta(t_k)\| &= \|x(t_k) - \frac{1}{1-\sigma^2}q(x(t_k),\Delta_{k-1})\| \\ &= \|x(t_k) - q(x(t_k),\Delta_{k-1}) - \frac{\sigma^2}{1-\sigma^2}q(x(t_k),\Delta_{k-1})\| \\ &\le \Delta_{k-1} + \frac{\sigma^2}{1-\sigma^2}\|q(x(t_k),\Delta_{k-1})\|.\end{aligned}$$

Dividing by $\|q(x(t_k),\Delta_{k-1})\|$ we get,

$$\frac{\|\zeta(t_k)\|}{\|q(x(t_k),\Delta_{k-1})\|} \le \frac{\Delta_{k-1}}{\|q(x(t_k),\Delta_{k-1})\|} + \frac{\sigma^2}{1-\sigma^2} = \Gamma_2 > 0.$$

Similarly, carrying out the same operation on $\zeta(t_{k+1})$ yields,

$$\begin{aligned}\|\zeta(t_{k+1})\| &= \|x(t_{k+1}) - \frac{1}{1-\sigma^2}q(x(t_k),\Delta_{k-1})\| \\ &= \|x(t_{k+1}) - q(x(t_{k+1}),\Delta_k) + q(x(t_{k+1}),\Delta_k) \\ &\quad - \frac{1}{1-\sigma^2}q(x(t_k),\Delta_{k-1})\| \\ &\ge \frac{\sigma}{1-\sigma^2}\|q(x(t_k),\Delta_{k-1})\| - \Delta_k \\ &= \frac{\sigma}{1-\sigma^2}\|q(x(t_k),\Delta_{k-1})\| - \Omega\Delta_{k-1} \\ &\quad - \gamma\|q(x(t_k),\Delta_{k-1})\|\end{aligned}$$

$$\frac{\|\zeta(t_{k+1})\|}{\|q(x(t_k),\Delta_{k-1})\|} > \frac{\sigma}{1-\sigma^2} - \frac{\Omega\Delta_{k-1}}{\|q(x(t_k),\Delta_{k-1})\|} - \gamma = \Gamma_1.$$

The difference $\Gamma_1 - \Gamma_2$ is,

$$\Gamma_1 - \Gamma_2 = (\frac{\sigma}{(1+\sigma)} - \gamma) - (1+\Omega)\frac{\Delta_{k-1}}{\|q(x(t_k),\Delta_{k-1})\|}.$$

To ensure that $\Gamma_1 - \Gamma_2 > 0$ it suffices for the following strict inequality to hold,

$$\frac{\Delta_{k-1}}{\|q(x(t_k),\Delta_{k-1})\|} < \frac{(\frac{\sigma}{(1+\sigma)} - \gamma)}{(1+\Omega+d)}. \quad (21)$$

Accordingly,

$$\Gamma_1 - \Gamma_2 \ge \left(\frac{\sigma}{(1+\sigma)} - \gamma\right)\frac{d}{(1+\Omega+d)} > 0. \quad (22)$$

Using the fact that the triggering happens outside the region R^c, we have

$$\|q(x(t_k),\Delta_k\| > \frac{k_1\Omega}{1-k_1\gamma}\Delta_{\min} + r_w(1+\sigma).$$

Rearranging,

$$\frac{\|B_w\|W}{\|A\|\|q(x(t_k),\Delta_{k-1})\|} < b_0 := \frac{\|B_w\|W}{\|A\|(\frac{k_1\Omega}{1-k_1\gamma}\Delta_{\min} + r_w(1+\sigma))}. \quad (23)$$

We obtain a bound on the fraction $\frac{\|B_w\|W}{\|A\|\|q(x(t_k),\Delta_{k-1})\|}$. Therefore it follows that from (22), (23) that

$$\tau_k \ge \tau_* = \frac{1}{\|A\|}\ln\left(\frac{\Gamma_1 + \frac{\|BK+\frac{1}{1-\sigma^2}A\|}{\|A\|} + b_0}{\Gamma_2 + \frac{\|BK+\frac{1}{1-\sigma^2}A\|}{\|A\|} + b_0}\right) > 0.$$

This proves the avoidance of Zeno Behaviour. We need to ensure that after the triggering, once the resolution of the quantizer is updated, it is necessary to ensure that the conditions (19) and (21) hold for the time interval $[t_{k+1}, t_{k+2})$. Using the fact that the resolution $\Delta(t)$ is lower-bounded by $\Delta_{\min}$ we get

$$\begin{aligned}\Delta_k \ge \Delta_{min} &= \frac{r_w(\sigma+1-k_1\gamma)}{(1+\beta)(1-k_1\gamma-\Omega(1+\sigma))} \\ \implies (1+\beta)\Delta_k + r_W &> \frac{k_1\Omega}{(1-k_1\gamma)}\Delta_k + \frac{r_W(1+\sigma)}{1-k_1\gamma}.\end{aligned}$$

As the triggering happens outside the region R^c given by (15) we have

$$\|q(x(t_{k+1}),\Delta_k)\| > (1+\beta)\Delta_k + r_w > \frac{k_1\Omega}{(1-k_1\gamma)}\Delta_k + \frac{r_W(1+\sigma)}{1-k_1\gamma}.$$

Similarly we have to ensure that the condition (21) holds for the next time interval $[t_{k+1},t_{k+2})$ as well. Using the update equation for $\Delta(t)$,

$$\frac{\Delta_k}{\|q(x(t_k),\Delta_{k-1})\|} = \Omega\frac{\Delta_{k-1}}{\|q(x(t_k),\Delta_{k-1})\|} + \gamma < \frac{\Delta_{k-1}}{\|q(x(t_k),\Delta_{k-1})\|} + \gamma < \frac{\frac{\sigma}{1+\sigma}-\gamma}{(1+\Omega+d)} + \gamma.$$

We also have from the modified triggering condition (17),

$$\frac{1}{1+\sigma} < \frac{\|q(x(t_{k+1}),\Delta_k)\|}{\|q(x(t_k),\Delta_{k-1})\|} < \frac{1}{1-\sigma}.$$

Using which we get

$$\frac{\Delta_k}{\|q(x(t_{k+1}),\Delta_k)\|} = \frac{\Delta_k/\|q(x(t_k),\Delta_{k-1})\|}{\|q(x(t_{k+1}),\Delta_k)\|/\|q(x(t_k),\Delta_{k-1})\|} < \left(\frac{\Omega(\frac{\sigma}{1+\sigma})-\gamma}{(1+\Omega+d)} + \gamma\right)(1-\sigma) < \frac{\frac{\sigma}{1+\sigma}-\gamma}{(1+\Omega+d)}$$

where in the previous step we have used the inequality given in C3. Further we have to show that the quantizer does not saturate at any of the triggering instances. During the time period $[t_{k+1},t_{k+2})$, from the condition on the range of the quantizer $\frac{1}{\gamma(1-\sigma)}+1 < M$, we get,

$$0 < \left((M-1)\gamma - \frac{1}{1-\sigma}\right)\|q(x(t_k),\Delta_{k-1})\| + (M-1)\Omega\Delta_{k-1}$$

$$\frac{1}{1-\sigma}\|q(x(t_k),\Delta_{k-1})\| < (M-1)(\gamma\|q(x(t_k),\Delta_{k-1})\| + \Omega\Delta_{k-1})$$

$$\|q(x(t_{k+1}),\Delta_k)\| < (M-1)\Delta_k \implies \|x(t_{k+1})\| < M\Delta_k$$

which shows that the quantizer is not saturated in the time interval $[t_k,t_{k+1})$. Define the region

$$R_{\min} = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| \geq (1+\beta)\Delta_{\min} + r_w \right\}. \quad (24)$$

This ensures that Lyapunov function is decreasing along the trajectories of the system under the control law in the region R_{min} given by (24) which guarantees input-to-state stability. ■

The condition (11) is a consequence of ensuring the twin-objective of Lyapunov function decreasing between triggering instances and the avoidance of Zeno behaviour. The condition C4 on the range of the quantizer (M) ensures that the quantizer is not saturated at any of the triggering instances.

V. Simulation and Results

A. Tuning the Controller

The controller has the parameters $\sigma,\Omega,d,\gamma,\Delta_0$, satisfying the constraints given in C2, C3 which need to be tuned. This section summarises the effect of each of the parameter on the transient behaviour of the system. We propose the following procedure to tune the parameters, resulting in a feasible solution.

1) Choose σ satisfying constraint C2. A larger value of σ results in a more relaxed event-triggering and thereby a larger inter-event time.
2) Choose Ω satisfying constraint C2 (which depends on the value of σ). A larger value of Ω results in a smaller decay rate of the quantizer resolution.
3) Choose $d > 0$. Notice that a large value of d reflects in a large minimum-inter-event time. However, a large value of d lowers the maximum allowable value of γ. A lower γ results in a weak dependence of Δ_k on $q(x(t),\Delta_{k-1})$ in (10).
4) Choose γ satisfying constraint C3. A larger value of γ results in a stronger coupling between the incoming disturbance and the adaptive change in the quantizer resolution.
5) Given a value of Δ_1 The control law ensures that given a value of Δ_1 the initial condition on the resolution is met.

Using the proposed tuning methodology, and using additional performance measures that based on one's application a suitable control law can be designed.

B. Numerical Example

Consider the following toy example, which we numerically simulate, to verify our control law.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0.75 & 0.9\\ 0.0 & 0.45\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}20\\ 20\end{bmatrix}u_1 + \begin{bmatrix}1.0\\ 1.0\end{bmatrix}w(t). \quad (25)$$

Using the proposed tuning methodology, the constants used in the simulation are: $\sigma = 0.0164, \Omega = 0.0197, d = 0.01, W = 30.0, \gamma = 0.008, \Delta_{\min} = 0.0966, \tau_* = 0.0021, K = \begin{bmatrix}0 & 0\\ -0.0936 & -0.0414\end{bmatrix}$. A mid-tread uniform step quantizer, similar to [5], is utilized for the simulation. The simulation is done for a total of $30s$, and a disturbance of magnitude 30 is provided for $10s$ from $t = 2.5s$ to $t = 12.5s$. The numerical simulation obtained is shown in Fig.1. The following observations are made

1) The proposed control law stabilizes the origin of the closed-loop system (25) in the sense of Input-to-State stability. The states are shown to converge to a neighbourhood of the origin in Fig 1, Plot 1.
2) The resolution of the quantizer is decreased ($\Delta(t)$ of the quantizer is increased) as the system states are steered away from the origin due to the disturbance $w(t)$. This is depicted in Fig 1, Plot 2.
3) The inter-event times are lower bounded by $0.0121s > \tau_*$ as seen in Fig 1, Plot 4. As given in Section III-C, this eliminates the possibility of Zeno behaviour. A

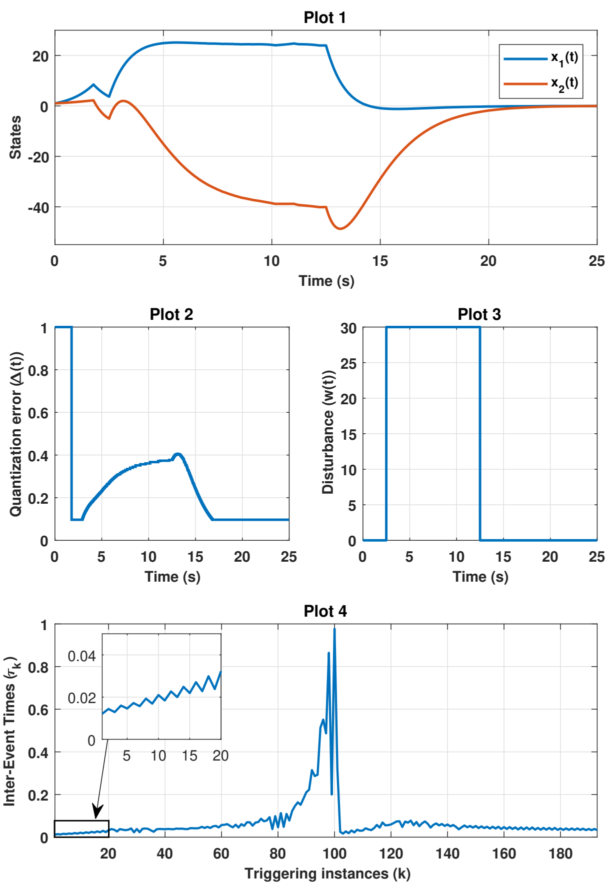

Fig. 1. (i) Plot 1 shows the evolution of the states of the system, (ii) Plot 2 depicts the variation of the quantization error $\Delta(t)$. We can observe that it increases in conjunction with the disturbance. (iii) Plot 4 shows that the inter-event times at various triggering instances.

total of 196 event-triggers have occurred in the $25s$ time interval.

Consider an ADC with a saturation value of $M > 0$ and a quantization error of Δ. The number of bits required for a binary representation of any value between $[0,M]$ is given by $B = \lceil \log_2(\frac{M}{\Delta}) \rceil$. Thus the average number of bits required during the stabilization of the system can be given as $B_{\text{avg}} = (\sum_{i=1}^{N} \lceil \log_2(\frac{M}{\Delta_k}) \rceil \tau_k)/(\sum_{i=1}^{N} \tau_k)$, τ_k are the inter event times and N is number of event triggers. In the absence of a reactive controller, the average number of bits that would be required is given by $B'_{avg} = \lceil \log_2(\frac{M}{\Delta_{min}}) \rceil$. For our numerical simulation we get $B_{\text{avg}} = 12.75, B'_{\text{avg}} = 14.00$, thus if $B_{\text{avg}} < B'_{\text{avg}}$, our control law has provided a better utilization of resources.

VI. Conclusion

An event-triggered control algorithm is proposed to stabilize the class of linear time-invariant systems with quantized feedback with disturbance. A novel quantizer-resolution update equation is proposed that includes the quantized state feedback. Using a numerical example, we demonstrate the reduction in average number of bits used by the controller. The authors will extend the results derived in this article to non-linear systems with quantized feedback as part of future work.

Appendix

Lemma 1: Consider $a \in \mathbb{R}^n, b \in \mathbb{R}^n, 0 \leq \sigma < 1$, then,

$$\|a-b\| = \sigma\|a\| \iff \|a - \frac{1}{1-\sigma^2}b\| = \frac{\sigma}{(1-\sigma^2)}\|b\| \quad (26)$$

Proof: On squaring and rewriting, we have

$$\sum_{i=1}^{n}(a_i - b_i)^2 = \sigma^2 \sum_{i}^{n} a_i^2$$

$$\implies \sum_{i=1}^{n}(a_i - \frac{1}{1-\sigma^2}b_i)^2 = \frac{\sigma^2}{(1-\sigma^2)^2}b_i^2$$

which gives us the lemma. ■

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