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Constrained synchronization of multi-agent systems using distributed MPC and invariant families of terminal sets

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Abstract— **In this paper, we present a synchronizing DMPC scheme that employs two ingredients: (*i*) a cost function that penalizes the deviation of the MPC control input from an unconstrained synchronization control law based on algebraic graph theory and (*ii*) an invariant family of constraints admissible terminal sets for MASs in closed-loop with an unconstrained synchronization control law. We prove that the developed DMPC scheme with a shrinking prediction horizon guarantees finite-time controllability to a family of invariant terminal sets and recursive feasibility. Compared to existing LMI methods for computing a family of constraints admissible invariant sets, we reduce conservatism by exploiting specific graph properties common to MASs. The developed DMPC algorithm for achieving constrained synchronization is tested in different benchmark examples, including balancing capacitor voltages for modular multilevel converters and harmonic oscillators, yielding faster synchronization.**

I. Introduction

Controlling multi-agent systems (MASs) with cooperative agents is a challenging problem, especially when constraints on the control inputs and states are presented. Such systems can be found in many critical applications, e.g., energy system management (EMS), microgrids or modular multilevel converters (MMCs). In these cases, distributed model predictive control (DMPC) is a well-established methodology to control the constrained agents, see, e.g. [1], [2]. However, there are two key challenges in DMPC: (*i*) guaranteeing local closed-loop stability and (*ii*) distributed computation of local terminal ingredients such as terminal cost and sets. This paper addresses these challenges in a specific control problem of cooperative MASs, i.e., output synchronization.

Regardless of the control problem, the mentioned DMPC challenges are predominantly handled using two methodologies: (*i*) synthesis of structured global terminal costs, e.g., [3] and (*ii*) distributed synthesis of terminal costs, e.g. [4], [5]. In this paper, we study the distributed synthesis of terminal costs due to tractable computational complexity Among these methods, the results from [4], [5] have special relevance. They use Lyapunov theory to calculate local time-varying terminal sets computed in a cooperative framework, i.e., the influence of the neighbour agents is considered but not as a disturbance. In these cases, the time-varying terminal sets are computed in an online fashion, which can limit the controller implementation when synchronization of the outputs is needed. Moreover, in [4], even when updating the time-varying terminal sets is not computationally costly, a global MPC problem needs to be solved at each sample time.

This paper considers the synchronization problem defined in [6], i.e., the challenge of steering all the agents' outputs to a common output trajectory. Achieving synchronization is a distributed cooperative control task since the agents have a common objective but a local controller with limited information about their neighbours' states. DMPC methods have been successfully used in this task, e.g., [7]–[11]. However, in these applications, local stability and recursive feasibility are challenging to guarantee, too. The first DMPC that guarantees synchronization and recursive feasibility was introduced in [11]. In that case, the synchronization is ensured using local cost functions to minimise the distance of the agent output to those of their neighbours while simultaneously tracking the corresponding target steady-state and input pair. The terminal ingredients are computed non-cooperatively, resulting in time-varying terminal sets corresponding to the target steady-states. Albeit stable and recursive feasible, in [10], it was shown that the DMPC from [11] may yield conservative solutions, i.e, the synchronization rate is slower than for other DMPCs such as [7]–[10].

This paper proposes a synchronizing DMPC (S-DMPC) scheme for constrained MASs with synchronization and recursive feasibility guarantees. Firstly, we consider the local dynamics of each agent in closed-loop with an unconstrained synchronizing control law constructed using standard results [6]. This leads to dynamically coupled local agent dynamics because the local control laws depend on the outputs of neighbouring agents. To guarantee feasibility, we then use the concept of an invariant family of sets introduced in [12] to construct a terminal set for the closed-loop dynamics of each agent. LMI computation of an invariant family of sets was proposed in [13], but their solution is infeasible for the synchronizing closed-loop dynamics considered in our paper. Hence, to reduce conservatism, we exploit the reducibility properties of the Laplacian matrix to arrive at simpler set dynamics that render the LMIs for computing the invariant family of sets feasible. Secondly, we use a cost function as in [10], i.e., which only penalizes the difference between the MPC control input and the unconstrained synchronizing controller, and a shrinking horizon to steer the output trajectories of the agents to the family of terminal sets in finite-time. The developed S-DMPC algorithm for achieving constrained synchronization is tested in different benchmark examples, namely, balancing capacitor voltages for MMCs and harmonic oscillators, yielding faster synchronization.

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Basic Notation: The identity matrix is denoted as $\mathbf{I}_n$. A block diagonal matrix with matrices A_1 to A_m on the diagonal is denoted by $\mathrm{diag}\,(A_1, \ldots, A_m)$.

We write $A \succ 0 (A \succcurlyeq 0)$ for a symmetric, positive (semi)definite matrix $A = A^\top \in \mathbb{R}^{n\times n}$. The closed Euclidean unit ball is given by $\mathcal{B}^n := \left\{x \in \mathbb{R}^n : \sqrt{x^T x} \leq 1\right\}$. The Minkowski set addition for two sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^n$, is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$. The cardinality of vector is obtained by $|\cdot|$, suhc that $|\cdot| : \mathbb{R}^{n\times 1} \to n \in \mathbb{N}_+$. The operation $\|\cdot\|_A : \mathbb{R}^n \to \mathbb{R}$ such that $\|x\|_A = x^\top Ax$. The i eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_i(A)$ and the spectral radius of A is $\rho(A) := \max\{|\lambda_1(A)|, ..., |\lambda_n(A)|\}$.

A communication network comprising of systems called agents is defined by directed graph, i.e., as $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} \in \mathbb{N}^n$ is the set of agents and $\mathcal{E} \in \mathbb{Z}^{n\times n}$ is the edge matrix describing the connection between agents. For the i^{th} agent of the communication network, their set of neighbours, which share information with, is denoted by $\mathcal{N}_i$. The matrix representations of $\mathcal{G}$, such as the adjacency $\mathcal{A} \in \mathbb{Z}^{n\times n}$, degree $\mathcal{D} \in \mathbb{Z}^{n\times n}$ and Laplacian $\mathcal{L} \in \mathbb{Z}^{n\times n}$ matrices, are obtained using the methodology from [6].

II. Preliminaries and Problem Description

In this paper, we consider a homogeneous multi-agent system (MAS) with a leader-follower topology comprising of N_a follower agents and one leader agent, i.e., Σ_L. The follower agents are labelled by 1, 2, ..., N_a and present linear dynamics defined as

$$\Sigma_i \begin{cases} x_i(k{+}1) = A_i x_i(k) {+} B_i u_i(k), \\ y_i(k) = C_i x_i(k), \end{cases} \tag{1}$$

where, for all time instant $k \in \mathbb{N}$ $y_i \in \mathbb{Y}_i \subset \mathbb{R}$, $x_i \in \mathbb{X}_i \subset \mathbb{R}^{n_i}$ and $u_i \in \mathbb{U}_i \subset \mathbb{R}^{m_i}$ are the i^{th} follower output, state and control input, respectively. The leader agent labelled by L, i.e., $L = N_a{+}1$, has linear dynamics, such that

$$\Sigma_L \begin{cases} x_L(k{+}1) = A_L x_L(k) {+} B_L u_L(k), \\ y_L(k) = C_L x_L(k). \end{cases} \tag{2}$$

Additionally, the set containing all the agents of the MAS ($\mathcal{V}$) is defined as $\mathcal{V} = \{1, .., N_a, L\}$. In this case, we consider a communication network with a fixed topology. Finally, an overall system is defined as

$$\Sigma_d := \begin{cases} \boldsymbol{x}(k{+}1) = \boldsymbol{A}\boldsymbol{x}(k) {+} \boldsymbol{B}\boldsymbol{u}(k), \\ \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k), \end{cases} \tag{3}$$

with

$$\boldsymbol{A} = \mathrm{diag}\,(A_1, ..., A_L)\,,\ \boldsymbol{B} = \mathrm{diag}\,(B_1, ..., B_L)\,,$$
$$\boldsymbol{C} = \mathrm{diag}\,(C_1, ..., C_L)\,,$$

where the state, control input and output vectors are

$$\boldsymbol{x}(k) := [x_1(k), \ldots, x_L(k)]^\top \in \mathbb{X} \subset \mathbb{R}^{n(N_a+1)},$$
$$\boldsymbol{u}(k) := [u_1(k), \ldots, u_L(k)]^\top \in \mathbb{U} \subset \mathbb{R}^{m(N_a+1)} \text{ and}$$
$$\boldsymbol{y}(k) := [y_1(k), \ldots, y_L(k)]^\top \in \mathbb{Y} \subset \mathbb{R}^{N_a+1}.$$

A. Synchronization: Problem Description

We say that the MAS as in (3) achieves synchronization if for all the initial states $x_i(0)$, the agent outputs yield

$$\lim_{k\to\infty} (y_i(k) - y_s(k)) = 0,\ \forall i \in \mathcal{V}, \tag{4}$$

where $y_s(k)$ is the synchronous trajectory. In [6, Chapter 4], a solution for the synchronization problem is proposed using local output feedback controllers, i.e.,

$$\bar{u}_i(x_{\mathcal{N}_i}(k)) = -k_i^f \sum_{j=1}^{L} a_{ij}(y_i(k) - y_j(k)), \tag{5}$$

where a_{ij} are the coefficients of the adjacency matrix, i.e., $\mathcal{A}$. Note that, there exists a $k_i^f \in \mathbb{R}$ such that (4) holds if the following assumptions hold.

Assumption II.1 The pairs (A_i, B_i) and (A_i, C_i) in (1) are stabilisable and detectable for all $i \in \{1, \ldots, L\}$.

Assumption II.2 As in Definition 4.3 from [6, page 168], we assume that for the set $\{\Sigma_1, \ldots, \Sigma_{N_a}, \Sigma_L\}$, there exists a system intersection, i.e., $\Sigma_s := \bigcap_{i=1}^{L} \Sigma_i$, such that for every initial states ($x_s(0) \in \mathbb{R}^{n_s}$) there exist initial states $x_1(0) \in \mathbb{R}^{n_1}, \ldots,\ x_L(0) \in \mathbb{R}^{n_L}$ for which the system Σ_d behaves synchronously, i.e., $y_1(k) = ... = y_L(k)$, $k \geq 0$.

The adjacency matrix of a communication network with a leader yields $a_{Lj}{=}0$ for all $j{\in}\mathcal{V}$. Hence, the leader closed-loop dynamics is independent of the followers' dynamics, yet the leader's output defines the synchronisation trajectory, i.e., $y_L(k){=}y_s(k)$.

Theorem II.3 (Unconstrained synchronization [6]) *Consider a MAS as (3) with identical follower agents, i.e., $(A_i, B_i, C_i){=}(A, B, C)$ for all $i{\in}\{1, ..., N_a\}$, a controller as (5), and suppose that Assumption II.1 and II.2 hold, then if*

$$\exists k_i^f \in \mathbb{R} : \rho\left(A_i - k_i^f B_i C|\mathcal{N}_i|\right) < 1,\ \forall i{\in}\{1, ..., N_a\}, \tag{6}$$

the closed-loop system composed by (3) and (5) reaches synchronization, i.e., $\lim_{k\to\infty} \boldsymbol{y}(k){=}\mathbf{1}_L\bar{y}$, where $\bar{y} = y_L{\in}\mathbb{R}$ is the synchronizing trajectory, i.e., the leader output.

III. Main Results

To guarantee stability and recursive feasibility in S-DMPC, we first construct an invariant family of sets considering synchronized MASs closed-loop dynamics.

A. Invariant Family of Sets for MAS

Definition III.1 (Positively Invariant Set) A set $\mathcal{S} \subseteq \mathcal{X}$ with $\mathcal{X}$ a proper compact set, is said to be positively invariant for the system $x(k{+}1) = Ax(k)$ if it holds $A\mathcal{S} \subseteq \mathcal{S}$.

Definition III.2 (Invariant family of sets [13]) Given a set of parameters $\Theta{\subseteq}\mathbb{R}_+^n$ and a collection of sets $\mathcal{S}{=}\{\mathcal{S}_i \subseteq \mathbb{X}_i : i \in \mathcal{N}_i\}$, where $\mathbb{X}_i \subset \mathbb{R}^{n_i}$. Then, a parameterized family of sets, i.e., $\mathbb{S}(\mathcal{S}, \Theta) := \{(\theta_1\mathcal{S}_1, \theta_2\mathcal{S}_2, \cdots, \theta_n\mathcal{S}_n) : \boldsymbol{\theta} = [\theta_1, \theta_2, \cdot\cdot\cdot, \theta_n]^\top {\in}\Theta\}$, is a positively invariant and constraint admissible family of sets for the agents (1) in closed-loop with (5) expressed as

$$\Sigma_i : x_i(k+1) = Ax_i(k) + Bu_i(k) + \sum_{j\in\mathcal{N}_i} \phi_{i,j}(x_j(k)), \quad (7)$$

if for all $\boldsymbol{\theta}(k) \in \Theta$ and $i \in \mathcal{N}_i$ there exists $\boldsymbol{\theta}(k+1) \in \Theta$, such that $x_i(k) \in \theta_i(k)\mathcal{S}_i \subseteq \mathbb{X}_i$ implies $u_i(k) \in \mathbb{U}_i$ and $x_i(k+1) = f(x_i(k), u_i(k)) \in \theta_i(k+1)\mathcal{S}_i \subseteq \mathbb{X}_i$.

The closed-loop dynamics of (7) follows the formulation from [13]. In our case, due to the designed output feedback controller (5), the agent dynamics as in (1) are rewritten in the form of (7) with the interconnection function, i.e., $\phi_{i,j}(\cdot) : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$, defined as

$$\phi_{i,j}(x_j(k)) = k_i^f B_i C_i x_j, \quad (8)$$

and the local control input is reformulated as

$$u_i(k) = K_i x_i(k), \quad (9)$$

where $K_i{=}-k_i^f|\mathcal{N}_i|C_i$. Note that $\phi_{i,j}(\cdot)$ is a linear function, hence, for all $j \in \mathcal{N}_i$ and $\beta{\in}\mathbb{R}$, $\phi_{i,j}(\beta x_j) = \beta\phi_{i,j}(x_j)$.

As in [13], we reduce the computational complexity by assuming the neighbour's behaviour is bounded over the scaled Euclidean ball with a factor $\eta_{i,j}$.

Assumption III.3 (Bounded Neighbour's Dynamics) For all $i{\in}\mathcal{N}_i$, $\exists\eta_{i,j}$ such that $\phi_{i,j}(\mathbb{X}_j){\subseteq}\eta_{i,j}\mathcal{B}^{n_i}$ for all $j{\in}\mathcal{N}_i$.

For a MAS as in (3), the factor $\eta_{i,j}$ is formulated as

$$\eta_{i,j} = \|k_i^f B_i C_i x_j^{\max}\|_\infty, \quad (10)$$

where $x_j^{\max}$ is defined by the constraint of $\mathbb{X}_j$. This choice is based on the results from [6], where it is known that if Assumption II.1 and II.2 hold, there exists a k_i^f, such that

$$\bar{A}_i = A_i - \lambda_i\{\mathcal{L}\}k_i^f B_i C_i, \quad \forall i = \{1, 2, ..., N_a\} \quad (11)$$

with $\rho(\bar{A}_i) \leq 1$, for all $i = \{1, 2, \cdots, N_a\}$.

Remark III.4 In Definition III.2, finding the invariant family of sets depends on the existence of a vector of scaling factors, i.e., $\boldsymbol{\theta}$, that evolves in time such that $\theta_i(k)\mathcal{S}_i \subseteq \mathbb{X}_i$ and $\theta_i(k+1)\mathcal{S}_i \subseteq \mathbb{X}_i$. The dynamics of $\boldsymbol{\theta}$ can be expressed in an autonomous linear fashion, i.e.,

$$\boldsymbol{\theta}(k+1){=}M\boldsymbol{\theta}(k), \text{ with } M{=}\begin{bmatrix} M_{11} & \dots & M_{1N_a} \\ \vdots & \ddots & \vdots \\ M_{L1} & \dots & M_{LL} \end{bmatrix}, \quad (12)$$

where $M_{ij} \in \mathbb{R}_+$ for all $(i,j) \in \{1, ..., L\}$. Note, $\rho(M) \leq 1$ ensures that θ_i are contracting scalar factors compatible with Definition III.2. In this case, we can exploit the relationship between the Laplacian matrix and M to reduce the conservatism of the LMIs, i.e., [13, (2h), (5h)].

Lemma III.5 *Let us define the state constraints and the control input constraints as*

$$\mathbb{X}_i := \{x_i \in \mathbb{R}^{n_i} : g_{i,t_x}^\top x_i \leq 1, \forall t_x \in \{1, 2, ..., s_i\}\}, \quad (13a)$$
$$\mathbb{U}_i := \{u_i \in \mathbb{R}^{m_i} : h_{i,t_u}^\top u_i \leq 1, \forall t_u \in \{1, 2, ..., l_i\}\}, \quad (13b)$$

where $s_i, l_i \in \mathbb{N}$, and suppose the Assumption II.1 and II.2 are satisfied. Consider a MAS as in (3), output feedback controller as (5) such that $K_i = -k_i^f|\mathcal{N}_i|C_i \in \mathbb{R}^{m_i\times n_i}$, $\eta_{ij} \in \mathbb{R}^+$ as in (10) and $0{<}\xi_{i,i}{<}1$. If there exists Q_i, Z_i, $\alpha_i{>}0$ and $\xi_{i,j}$ for all $i \neq j$ and $(i,j){\in}\mathcal{V}$ such that:

$$\begin{bmatrix} \xi_{i,i}Z_i & (A_iZ_i{+}B_iQ_i)^\top \\ A_iZ_i{+}BQ_i & Z_i \end{bmatrix} \succcurlyeq 0, \quad (14a)$$
$$\forall j{\in}\mathcal{N}_i, \quad Z_i \succcurlyeq \xi_{i,j}\eta_{i,j}^2 I_{n_i}, \quad (14b)$$
$$\forall t_x{\in}\{1, ..., s_i\}, \quad \begin{bmatrix} Z_i & Z_i g_{it_x} \\ (Z_i g_{it_x})^\top & \alpha_i \end{bmatrix} \succcurlyeq 0, \quad (14c)$$
$$\forall t_u{\in}\{1, ..., l_i\}, \quad \begin{bmatrix} Z_i & Q_i h_{it_u} \\ h_{it_u}^\top Q_i{}^\top & \alpha_i \end{bmatrix} \succcurlyeq 0, \quad (14d)$$
$$0 < \alpha_i \leq 1, \quad (14e)$$
$$\xi_{i,j} \geq 0, \quad (14f)$$

where $Q_i{=}K_iZ_i$, then $\mathcal{S} := \{\mathcal{S}_i \subseteq \mathbb{X}_i \ : \ i \in \mathcal{V}\}$ with

$$\mathcal{S}_i := \{x_i \in \mathbb{R}^{n_i} \ : \ x_i^\top P_i x_i \leq 1\}, \text{ and } P_i = Z_i^{-1} \quad (15)$$

define an invariant family of sets for all agents $i \in \mathcal{V}$ connected via a communication network with a reducible Laplacian matrix.

Proof: The LMIs (14) can be rewritten as:

$$(A_i + B_iK_i)^\top P_i(A_i + B_iK_i) \preceq \xi_{i,i}P_i \quad (16a)$$
$$\forall j{\in}\mathcal{N}_i, \quad \eta_{ij}^2 I_{d_i} \preceq \xi_{i,j}P_i^{-1} \quad (16b)$$
$$\forall t_x{\in}\{1, 2, ..., s_i\}, \quad \frac{1}{\sqrt{\alpha_i}}\sqrt{g_{it_x}^\top P_i^{-1} g_{it_x}} \leq 1 \quad (16c)$$
$$\forall t_u{\in}\{1, 2, ..., l_i\}, \quad \frac{1}{\sqrt{\alpha_i}}\sqrt{h_{it_u}^\top K_i P_i^{-1} K_i^\top h_{it_u}} \leq 1 \quad (16d)$$
$$0 < \alpha_i \leq 1 \quad (16e)$$
$$\xi_{i,j} \geq 0 \quad (16f)$$

where (16a)-(16d) are obtained by applying Schur complement, respectively. If (16) is feasible for a given $0 < \xi_{i,i} < 1$, $A_i \in \mathbb{R}^{n_i\times n_i}$, $B \in \mathbb{R}^{n_i\times m_i}$, $\eta_{i,j} \in \mathbb{R}^+$, g_{it_x} and h_{it_u} of compatible dimension for all $j \in \mathcal{N}_i$, $t_x{\in}\{1, 2, ..., s_i\}$ and $t_u{\in}\{1, 2, ..., l_i\}$, then i) $(A_i + B_iK_i)\mathcal{S}_i \subseteq \sqrt{\xi_{i,i}}\mathcal{S}_i$, ii) $\eta_{i,j}\mathcal{B}^{n_i} \subseteq \xi_{i,j}^{-0.5}\mathcal{S}_i$, iii) $\frac{1}{\sqrt{\alpha_i}}\mathcal{S}_i \subseteq \mathbb{X}_i$, iv) $\mathcal{S}_i \subseteq \mathbb{X}_i$, v) $\frac{1}{\sqrt{\alpha_i}}K_i\mathcal{S}_i \subseteq \mathbb{U}_i$.

Since $\phi_{i,j}(\mathcal{S}_j) \subseteq \phi_{i,j}(\mathbb{X}_j) \subseteq \eta_{i,j}\mathcal{B}^{n_j} \subseteq \xi_{i,j}^{-0.5}\mathcal{S}_i$ and $\theta_j\phi_{i,j}(\mathcal{S}_j) = \phi_{i,j}(\theta_j\mathcal{S}_j)$, we have

$$\phi_{i,j}(\theta_j\mathcal{S}_j) \subseteq \xi_{i,j}^{-0.5}\theta_j\mathcal{S}_i, \quad (17)$$

and, as in (12), the scaling factors dynamics yields

$$\boldsymbol{\theta}(k+1) = M\boldsymbol{\theta}(k), \quad (18)$$

where

$$M_{ij} = \begin{cases} \xi_{i,i}^{0.5}, & \text{if} \quad i = j \\ \xi_{i,j}^{-0.5}, & \text{if} \quad j \in \mathcal{N}_i \\ 0, & \text{otherwise} \end{cases}.$$

Note that having a reducible $\mathcal{L}$ implies that M is a lower triangular matrix. Hence, the spectral radius of M is defined by the given values of $\xi_{i,i}$. Together with $(A_i + B_iK_i)\mathcal{S}_i \subseteq \sqrt{\xi_{i,i}}\mathcal{S}_i$, we can have

$$(A_i + B_iK_i)\theta_i S_i \bigoplus_{j\in\mathcal{N}_i} \phi_{ij}(\theta_j \mathcal{S}_j) \subseteq \xi_i^{0.5}\theta_i\mathcal{S}_i \bigoplus_{j\in\mathcal{N}_i} \theta_j\mathcal{S}_i \quad (19)$$

with

$$\xi_i^{0.5}\theta_i\mathcal{S}_i \bigoplus_{j\in\mathcal{N}_i} \theta_j\mathcal{S}_i = \theta_i(k+1)\mathcal{S}_i, \quad (20)$$

which means whenever $x_i(k) \in \theta_i\mathcal{S}_i$, we have $x_i(k+1) \in \theta_i(k+1)\mathcal{S}_i$, for all $i\in\mathcal{V}$. Hence, we have constructed an admissible invariant family of sets. ■

The main difference concerning the conditions given in [13] is the constraints of $\xi_{i,j}$ in [13, (5h)]. Most of the arguments in the proof [13] still apply; nevertheless, for completeness, we presented the full derivation of the results.

Remark III.6 (16c), (16d) yields that $\frac{1}{\sqrt{\alpha_i}}\mathcal{S}_i \subseteq \mathbb{X}_i$. Hence, an admissible initial value for θ_i is $\theta_i(0)=\frac{1}{\sqrt{\alpha_i}}$ and then $\boldsymbol{\theta}(k+1) = M^k\boldsymbol{\theta}(0)$ for all $k\in\mathbb{N}_+$. Nonetheless, $\theta_i(0)$ is not an upper limit.

B. Synchronizing Distributed Model Predictive Control

Next, for implementing the distributed model predictive control (DMPC) to synchronize the MAS (3), we define the prototype local MPC problem for each follower agent.

Problem III.7 (Local MPC problem) Consider a MAS as (3), then for each agent $i \in \{1, ..., N_a\}$,

$$\min_{U_i(k)} J_i(U_i(k), x_{\mathcal{N}_i}(k)) \quad (21a)$$

$$\text{s.t. } x_i(h+1|k) = A_ix_i(h|k)+B_iu_i(h|k), \quad (21b)$$

$$y_i(h|k) = C_ix_i(h|k),\ \forall h \in \{0, ..., N(k)\}, \quad (21c)$$

$$y_i(h|k) \in \mathbb{Y}_i,\ \forall h \in \{1, ..., N(k)\}, \quad (21d)$$

$$x_i(h|k) \in \mathbb{X}_i,\ \forall h \in \{1, ..., N(k)\}, \quad (21e)$$

$$u_i(h|k) \in \mathbb{U}_i,\ \forall h \in \{0, ..., N(k)-1\}, \quad (21f)$$

$$x_i(0|k) = x_i(k), \quad (21g)$$

$$x_i(N(k)|k) \in \mathbb{X}_i^T, \quad (21h)$$

where $J_i(U_i(k), x_{\mathcal{N}_i}(k))$:$\mathbb{R}^{N(k)m_i}\times\mathbb{R}^{|\mathcal{N}_i|n_i} \to \mathbb{R}$ is the local cost function; $N(k)\in\mathbb{N}_+$ is the prediction horizon; the sets $\mathbb{X}_i$, $\mathbb{Y}_i$, $\mathbb{U}_i$ and $\mathbb{X}_i^T$ are the constraints of the state, output, control input and terminal states respectively.

Theorem III.8 *Consider Problem III.7 with $\mathbb{X}_i^T=\theta_i(0)\mathcal{S}_i$ as Lemma III.5 and local cost function as*

$$J_i\left(U_i(k), \bar{u}_i(k)\right) = \sum_{h=0}^{N(k)} \|u_i(h|k) - \bar{u}_i(x_{\mathcal{N}_i}(k))\|_{R_i}^2, \quad (22)$$

where $N(k)$ is a shrinking prediction horizon such that

$$N(k) = \begin{cases} \bar{N} \in \mathbb{N}_+, & \textit{if} \quad k = 0 \\ \bar{N}(k-1) - 1, & \textit{if} \quad 1 \le k \le \bar{N} - 1 \\ 1, & \textit{if} \quad k \ge \bar{N} \end{cases}. \quad (23)$$

Then the MAS system (3) obtained by clustering the dynamics of all agents in closed-loop with the corresponding DMPC controller as in Problem III.7 achieve synchronization. Moreover, Problem III.7 is recursively feasible for all agents and the closed-loop trajectories of all agents reach the invariant family of terminal sets, i.e., $\mathbb{X}_i^T$, in $\bar{N}$ steps or less.

Proof: Let us assume that $x_i(0)$ belongs to a constraints admissible feasible set, i.e., $x_i(0) \in \mathbb{X}_i^F \subseteq \mathbb{X}_i$, such that

$$\exists\, \bar{N} \in \mathbb{N}_+ : x_i(h|0) \in \mathbb{X}_i^T,\ \forall h \ge \bar{N}, \quad (24)$$

where $\mathbb{X}_i^T=\theta_i(0)\mathcal{S}_i$ with $\theta_i(0)$ defined as in Remark III.6. Consequently, there exists an optimal control input, i.e., $U_i^*(k) = \left[u^*(0|k)\, u^*(1|k) \ldots u^*(N(k)|k)\right]^\top$. By shifting $U_i^*(k)$, the optimal control input at the next time step, i.e., $k+1$, yields

$$U_i^*(k+1) = \left[u_i^*(1|k)\, u_i^*(2|k) \ldots u_i^*(N(k)|k)\right]^\top. \quad (25)$$

Consequently, $x_i(N(k+1)|k+1) \in \mathbb{X}_i^T$ with $N(k+1) = N(k)-1$, meaning that the new appended local control law, i.e., $U_i^*(k+1)$, is feasible too. Thanks to the feasibility of $U_i^*(k+1)$, the prediction horizon keeps decreasing, i.e.,

$$N(k+1) < N(k). \quad (26)$$

Hence, due to the shrinking horizon policy, there exists a finite time k^* such that $N(k^*)$=1 and $x(1|k^*) \in \mathbb{X}_i^T$. For all $k \ge k^*+1$, $N(k)$=1 and the optimal control input sequence, i.e., $U_i^*(k)$ reduces to the unconstrained synchronizing control law, which is feasible in terminal set, i.e.,

$$u_i^*(0|k) = \bar{u}_i(k),\ \forall k \ge k^* + 1. \quad (27)$$

Then synchronization follows from Theorem II.3. ■

Remark III.9 The results of Lemma III.5 can also be used to guarantee recursive feasibility for the formulation of Problem III.7 where the horizon N is kept constant (i.e., non-shrinking), as in [10], or for any synchronizing cost function that calculates the references using the unconstrained synchronization protocol, e.g., as in [7]–[9]. However, it is generally beneficial to reach the family of terminal sets in finite time, as this can speed up synchronization. Moreover, Theorem III.8 can be extended to achieve finite-time synchronization, since there exists a finite-time feedback synchronization, see [14, Theorem 1]. Hence, combining it with a shrinking-horizon policy, synchronization can be reached in $k_{FT}^*=\bar{N}+2n_i$.

IV. Illustrative Examples

In this section, we present two illustrative examples: (*i*) MMCs and (*ii*) harmonic oscillators. For assessing the synchronization level, we use the consensus index, i.e.,

$$J_c(k) := \frac{1}{L}\sum_{i=1}^{L}\left\| y_i(k) - \frac{1}{L}\sum_{j=1}^{L} y_j(k) \right\|. \tag{28}$$

Hence, in the following section, we compare S-DMPC with terminal constraints (labelled S-DMPC[X]) and without (labelled S-DMPC).

A. Modular Multilevel Converters (MMCs)

Distributed balancing control of the capacitor voltages of the MMCs have been discussed in [10]. In this case, we synchronize the capacitor voltages using an arc path network that has a reducible Laplacian matrix. The module dynamics are expressed as

$$\Sigma_i : x_i(k+1) = x_i(k) - \frac{N_a T_s i^r}{C_y} u_i(k), \tag{29}$$

for all $i \in \mathcal{V}$, where $N_a=21$, $x_i \in \mathbb{R}$ is the state describing the capacitor voltages (v_{C_i}), $u_i \in \mathbb{R}$ is the control input describing the insertion indeces, $k \in \mathbb{N}$, $T_s=0.2$ms is the sampling period, $C_y=0.4$mF and $i^r=10$A. The constraints are defined as $8.7485\text{kV} \leq x_i \leq 8.7505\text{kV}$ and $-1 \leq u_i \leq 1$. The S-DMPC[X] is comprised of $R_i=1$, $\xi_{i,i}=0.25$, $k_i^f=1$ and $\bar{N}=15$. The values P_i, and α_i are common to all agents due to identical agents and the communication network structure.

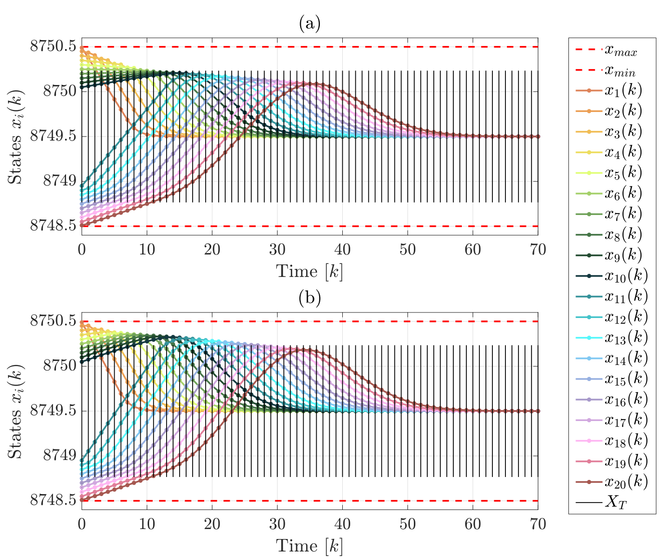

Fig. 1: Comparison of the MMC capacitor voltage dynamics with respect to $\mathbb{X}_i^T(0)$ from different DMPCs: (a) S-DMPC[X] as (21) and (b) S-DMPC without terminal constraints.

By applying Lemma III.5, we obtain a $P_i = 2.9380$, $\xi_{i,j} = 1.5097$ and $\alpha_i = 0.7439$. As in Remark III.6, $\mathbb{X}_i^T(0) = \theta_i(0)\mathcal{S}_i$ with $\theta_i(0) = \frac{1}{\sqrt{\alpha_i}} = 1.1594$ and $\mathcal{S}_i = \{x \in \mathbb{R} : x^\top P_i x \leq 1\}$. In Figure 1, we compare the performance of S-DMPC[X] and the S-DMPC by analysing the capacitor voltage dynamics. Both solutions synchronize the MMC, yet it seems like S-DMPC[X] has a better performance as the agents reach the initial terminal set, i.e., $\mathbb{X}_i^T(0)=\theta_i(0)\mathcal{S}_i$, in less than 15 discrete-time steps. Figure 2 shows the $J_c(k)$ dynamics corresponding to both DMPC schemes. The S-DMPC[X] achieves the fastest synchronization of the MMC. Note that at time step 30, we observe a 20% improvement of the S-DMPC[X]. On average, the S-DMPC[X] consensus index improves by 5.12%.

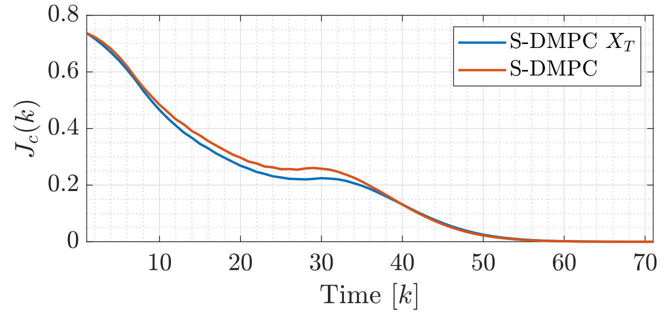

Fig. 2: Comparison of $J_c(k)$ from different DMPCs: (-) S-DMPC[X] as (21) and (-) S-DMPC.

B. Harmonic oscillators

As in [6], we examine two AC generators, i.e., harmonic oscillators, connected by an arc path network and modelled as

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} x_i(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i(t) \\ y_i(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x_i(t) \end{cases} \tag{30}$$

where $\omega=5$Hz, $-1 \leq u_i \leq 1$ and $-20\mathbf{1}_{n_i} \leq x_i \leq 20\mathbf{1}_{n_i}$. System (30) is discretized using the zero-order-hold method with sampling time $T_s=2$ms. The S-DMPC[X] is comprised of $R_i=\mathbf{I}_{m_i}$, $\xi_{i,i}=0.995$, $k_i^f=0.5$ and $\bar{N}=200$. Similar to the MMC, we used an arc path that communicates with the leader AC generator and the other two followers' AC generators. By applying Lemma III.5, we find $P_i = \begin{bmatrix} 19.622 & 0.2005 \\ 0.2005 & 0.7892 \end{bmatrix}$ and $\alpha_i = 0.62$. The initial terminal set ($\mathbb{X}_i^T(0)$) and initial scaling factor ($\theta_i(0)$) are defined as in III.6 like in section IV-A.

Figure 3 and 4 show the state trajectories for the followers AC generators and the leader AC generator, i.e., the black dashed line. In this case, Figure 3 shows that having a terminal set increases the convergence speed when the S-DMPC[X] is used. Both AC generators' initial conditions are outside the terminal set, i.e., $x_i(0) \notin \mathbb{X}_i^T(0), \forall i \in \{1,2\}$.

The resulting autonomous auxiliary system, i.e., $\theta(t+1)=M\theta(t)$ for all $t=k-k^* \in \mathbb{N}$, yields $\rho(M) \leq 1$, where

$$M = \begin{bmatrix} 0.9995 & 0 & 0 \\ 0.1274 & 0.9995 & 0 \\ 0 & 0.1274 & 0.9995 \end{bmatrix}. \tag{31}$$

Note that even when P_i, and α_i are common to all agents due to the identical agents and the communication network

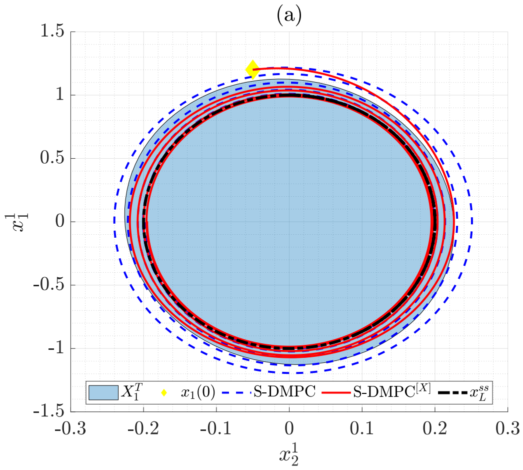

Fig. 3: Comparison of state trajectories of Agent 1: (- -) leader trajectories,i.e., x_L; (-) S-DMPC$^{[X]}$ as (21) and (- -) S-DMPC with $x_1(0)$ denoted by (♦) and $\mathbb{X}_i^T(0)$ as (●).

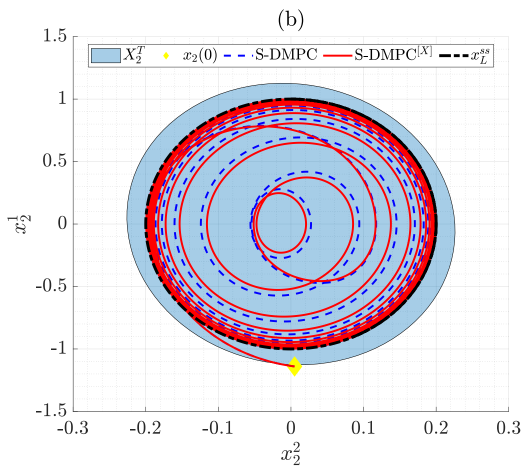

Fig. 4: Comparison of state trajectories of Agent 2: (- -) leader trajectories,i.e., x_L; (-) S-DMPC$^{[X]}$ as (21) and (- -) S-DMPC with $x_1(0)$ denoted by (♦) and $\mathbb{X}_i^T(0)$ as (●).

structure, the trajectories of θ_i are not identical, see Figure 5. There, we can observe the stable trajectory of $\boldsymbol{\theta}$ and confirm the validity of invariant family of sets since, $\theta_2(543)\mathcal{S}_2 \in \mathbb{X}_2$, i.e., the maximal parameterized set is within constraints.

V. Conclusions

In this paper, we proposed a S-DMPC with stability and recursive feasibility guarantees, denoted as S-DMPC$^{[X]}$. In this case, by exploiting the characteristics of the Laplacian matrix, we relaxed the LMI formulation of [13] to compute an invariant famility of terminal sets for the MASs dynamics in closed-loop with an unconstrained synchronizing control law. The proposed method is limited to MAS which has a communication network with a spanning tree and reducible Laplacian matrix. Nonetheless, communication networks with those properties are commonly found in many applications since all the strongly connected graphs have that property. The experiment results showed that besides

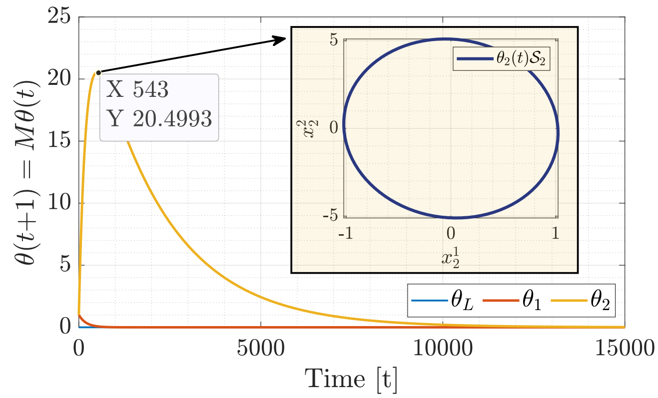

Fig. 5: Trajectory of $\theta_i(t)$ and corresponding parameterized invariant set of the AC generator 2 at iteration $t = 543$, i.e., $\theta_2(t)\mathcal{S}_2$.

recursive feasibility, faster synchronization is obtained using the proposed method.

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