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Dissipativity-based scalable $\mathcal{L}_2$-gain analysis for nonlinear networked systems

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***Abstract*— By making use of dissipativity theory we can provide an easily verifiable condition which ensures a *scalable* $\mathcal{L}_2$-gain in a network of nonlinear systems, i.e. a gain independent on the number of systems. However, ensuring bounded $\mathcal{L}_2$-gains may become insufficient, since the energy of the input may grow unbounded. Therefore, we can also give a proof that the same condition may be used to bound the energy of the local systems. Such a bound ensures that the effects of increasing network size do not accumulate in any of the systems. We end the paper with an example in which we demonstrate that scaling the network size does not lead to an accumulation in any part of the network.**

I. Introduction

Current advances in technology and ongoing research efforts are leading to a growing implementation of Multi-Agent Systems, or interconnected systems, which can be found for instance in the management of electrical grids, [1], [2] and the coordination of vehicle platoons [3], [4], [5]. It is expected that the use of Multi-Agent Systems will continue to expand in the future, with network structure becoming dynamic, changing as new agents join or leave the network, or as interconnections are reconfigured. This dynamic nature of network structure and size introduces new challenges in terms of stability analysis. For instance, as network size N grows, consensus protocols may become unstable if the number of connections to each system do not grow with N, [6].

As a result, requirements on local systems ensuring stability of dynamic networks are needed. These conditions should rely solely on local information and should be independent of the number of systems (also referred to as subsystems or agents) or structure of the network. Such conditions would allow for a scalable performance, which has been described in [7]. In our previous research we have focused on linear systems, see [8] and [9], and provided locally verifiable conditions to ensure a bounded scalable $\mathcal{L}_\infty$-gain. These results, however, do not extend naturally for nonlinear systems. Other approaches for nonlinear systems, such as [10] and later works [11], provide conditions on the gradient of the local dynamics, which ensure a scalable $\mathcal{L}_\infty$-gain.

In this paper we do not consider the $\mathcal{L}_\infty$-gain, but instead we turn to dissipativity theory in order to guarantee upper bounds on the energy associated to the local systems. This leads to an abstraction of the system dynamics and leads to conditions, which are easily interpreted and locally verifiable.

The seminal works on dissipativity of [12], [13], [14] provided a generalisation of passivity and Lyapunov theory. The concept of dissipativity provides an attractive abstraction of the local dynamics for interconnected systems. Instead of considering the dynamics directly, features of the systems can be used to guarantee network properties, such as stability.

Dissipativity is indeed a useful concept to guarantee stability of interconnected systems, as one may use properties of the individual systems to verify if the interconnection will be stable. In [15], a method using Sum of Squares (SOS) was presented to find invariant safe sets, such that cascaded failures are avoided. However, finding these sets using the SOS-method may need non-local information.

In this paper, we introduce and analyze the notion of a scalable $\mathcal{L}_2$-gain, i.e., an $\mathcal{L}_2$-gain on a network that is uniform of the network size. Here, we rely on results on dissipativity theory for networked systems [16]. We also show that this scalable $\mathcal{L}_2$-gain does not necessarily prevent network disturbances to accumulate at a single subsystem, which might be undesirable. To address this, we introduce an alternative signal norm, which we refer to as the $\mathcal{L}_2(\infty)$-norm. With this, we can use standard definitions of dissipativity to find requirements on the local systems which ensure that the effects of increasing network size do not accumulate at any of the individual systems. Continuing, such a bound on the energy of any system is found via a supply rate using the ∞-norm, which during time intervals may be formulated in a quadratic form. The supply rate over a larger time interval can then be found as a summation of the quadratic supply rates. In [17], a similar supply rate, i.e. a summation of supply rates active on diffierent time intervals, was used to show dissipativity in switching systems. Using such a supply rate allows us to evaluate the problem as a set of LMIs. However, due to the structure of the LMIs we will see that the conditions may be easily verified. To the best of our knowledge we have not found similar approaches to use dissipativity for *scalable* networks. For a recent review of the developments in dissipativity theory see [18].

Outline

In section II we will give a thorough problem definition, including system definition and assumptions. We follow this with our results regarding a scalable $\mathcal{L}_2$-gain in section III before we extend this in section IV and show that the same assumptions and conditions result in a scalable $\mathcal{L}_2(\infty)$-gain.

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We end the paper with two examples regarding networks with scalable $\mathcal{L}_2(\infty)$-gains. Further, our contributions are

- a gain-constraint on the local systems ensuring a scalable $\mathcal{L}_2$-gain of the network, independent of size and structure;
- a definition of the $\mathcal{L}_2(\infty)$ signal norm;
- a gain-constraint on the local systems which ensure a bounded scalable $\mathcal{L}_2(\infty)$-gain of the network, independent of size or structure.

Notation: Let $\mathbb{R}$ denote the real numbers. We denote the $\mathcal{L}_2$-norm as $\|\cdot\|_{\mathcal{L}_2}$ and use $\|x(t)\|_{\mathcal{L}_2}^2 = \int_0^t |x(\tau)|_2^2 d\tau$, where $|\cdot|_2$ denotes the Euclidean norm. w_i denotes the ith cartesian unit vector and $\mathbb{I}$ denotes the identity matrix with suitable dimension. Additionally, with $v \in \mathbb{R}^n$ let $\operatorname{diag}(v)$ denote a diagonal matrix with the elements of v on its diagonal. Let $\rho(A)$ denote the spectral radius of A, i.e. $\rho(A) = \max_i |\lambda_i(A)|$, $\lambda_i(A)$ an eigenvalue of the matrix A. Lastly, we use $\|\cdot\|_\infty$ to denote the induced ∞-norm.

II. Problem statement

A. System definition

Consider a network of N systems of the following form

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \\ y_i(t) = h_i(x_i(t)), \end{cases} \tag{1}$$

with state $x_i(t) \in \mathbb{R}^n$, input $u_i(t) \in \mathbb{R}^m$ and output $y_i(t) \in \mathbb{R}^m$ for all $i \in \{1, 2, \ldots, N\}$. We will drop the dependency on t and assume $f(0,0) = 0$, $h(0) = 0$ in the remainder.

Definition II.1. Let a supply rate $s : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be given. A system of the form (1) is said to be dissipative with respect to supply rate $s(u_i, y_i)$ if there exists a continuously differentiable function $V_i : \mathbb{R}^n \to \mathbb{R}$ with $V_i(0) = 0$ satisfying $V_i(x_i) \geq 0$, for all $x_i \in \mathbb{R}^n$, such that

$$\nabla V_i(x_i)^{\mathrm{T}} f_i(x_i, u_i) \leq s(u_i, y_i), \tag{2}$$

In particular, we make the following assumptions on the systems defined in (1):

Assumption 1. All systems i, defined by (1), are dissipative with respect to

$$s(u_i, y_i) = \begin{bmatrix} u_i \\ y_i \end{bmatrix}^{\mathrm{T}} X \begin{bmatrix} u_i \\ y_i \end{bmatrix}, \; X = \begin{bmatrix} \gamma^2 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

Assumption 1 ensures that all subsystems have a bounded $\mathcal{L}_2$-gain, i.e. $\|y_i\|_{\mathcal{L}_2} \leq \gamma \|u_i\|_{\mathcal{L}_2}$ when the trajectories satisfy $x_i(0) = 0$. If, in addition, $V_i(x_i) > 0$ holds for all $x_i \neq 0$, then $V_i(x_i)$ acts as a Lyapunov function for the system Σ_i, as can be concluded from (2) and (3) by setting $u_i = 0$.

We can now focus on the interconnections in the network. Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ describe the directed graph formed by the network, with $\mathcal{V} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, with $(i,j) \in \mathcal{E}$ if system i has an incoming connection from system j. Further, we define the adjacency matrix $A(\mathcal{G})$ as

$$[A(\mathcal{G})]_{ij} = \begin{cases} 1 \text{ if } (i,j) \in \mathcal{E} \\ 0 \text{ otherwise.} \end{cases} \tag{4}$$

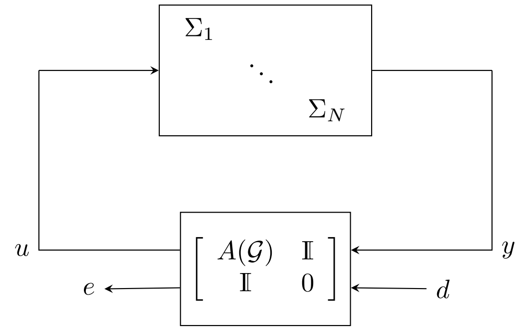

Fig. 1. Visualisation of the network and the interconnections.

Let $u = \left[u_1^{\mathrm{T}}, \ldots, u_N^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{mN}$, $y = \left[y_1^{\mathrm{T}}, \ldots, y_N^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{mN}$ and add disturbances $d = [d_1^{\mathrm{T}}, \ldots, d_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mN}$, where d_i acts on system i. We also add performance outputs $e = [e_1^{\mathrm{T}}, \ldots, e_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mN}$. The relationship between u, y, d and e can be described by

$$\begin{bmatrix} u \\ y \\ d \\ e \end{bmatrix} = M \begin{bmatrix} y \\ d \end{bmatrix}, \; M = \begin{bmatrix} A(\mathcal{G}) \otimes \mathbb{I}_m & \mathbb{I} \\ \mathbb{I} & 0 \\ 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}. \tag{5}$$

We define networks by interconnecting the systems Σ_i through the matrix M and graph $\mathcal{G}$ and we will use the notation $\Sigma(\mathcal{G}, \{\Sigma_i\})$ to consider a single network. In the remainder we will be interested in families of $\Sigma(\mathcal{G}, \{\Sigma_i\})$ which we obtain by instead considering classes of graphs and a family of systems $\{\Sigma_i\}$ satisfying some common properties. Given a class of graphs $\{\mathcal{G}\}$ and family $\{\Sigma_i\}$ we denote the corresponding family of networks as $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$. Further, we consider $\{\mathcal{G}\}$ with arbitrary number of nodes, such as the class $\{\mathcal{G}\}$ with bounded in-degree. A block diagram of a network with N systems can be seen in Fig. 1.

Remark II.2. The interconnection in (5) is defined for any $m \geq 1$, however, in the remainder we consider $m = 1$.

B. Problem statement

Broadly speaking, we are concerned with characterizing the gain from disturbance d to performance output e and identifying locally verifiable conditions on systems, which ensure the existence of a bounded gain in families $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$. Importantly, we are interested in ensuring that the effects of disturbances d do not grow or accumulate in any one system, even if the number of systems, N, within a network increases. To accurately specify our problem, we need the following definitions.

Definition II.3. Let $\{\mathcal{G}\}$ be some class of graphs. Then, the family of networks $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$ is said to have a *scalable $\mathcal{L}_2$-gain* if there exists a $\beta > 0$ such that, for any graph $\mathcal{G} \in \{\mathcal{G}\}$ and any finite number of nodes N, each network $\Sigma(\mathcal{G}, \{\Sigma_i\})$ satisfies

$$\|e\|_{\mathcal{L}_2} \leq \beta \|d\|_{\mathcal{L}_2}, \tag{6}$$

for any trajectory satisfying $x(0) = 0$.

Thus, a family of networks $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ with a scalable $\mathcal{L}_2$-gain has a uniform gain β for all graphs in $\{\mathcal{G}\}$ with an arbitrary number of nodes N.

Definition II.3, however, does not prohibit the accumulation of energy in some specific system in a network. This may lead to a situation in which all energy in some network in $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ is accumulated in a single system i, i.e.

$$\|y_i\|_{\mathcal{L}_2} = \beta\|d\|_{\mathcal{L}_2},\ \|y_j\|_{\mathcal{L}_2} = 0, \forall j \neq i.$$

To avoid this, we seek to limit the energy of the individual systems i with respect to the energy of the disturbance.

Definition II.4. Let $\{\mathcal{G}\}$ be some class of graphs. Then, the family of networks $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ has *bounded accumulating effects* if there exists a $\beta > 0$ such that, for any $\mathcal{G} \in \{\mathcal{G}\}$ with any finite number of nodes N, every system Σ_i in $\Sigma(\mathcal{G},\{\Sigma_i\})$ fulfills

$$\|y_i\|_{\mathcal{L}_2} \leq \beta\|d_{\max}\|_{\mathcal{L}_2}, \tag{7}$$

for trajectories satisfying $x_i(0) = 0$ and with $d_{\max}$ a signal satisfying $|d_i(t)| \leq d_{\max}(t)$ for all $t \geq 0$ and all i.

To finalize our problem statement, we are interested in finding conditions on the systems Σ_i, such that for some class of graphs $\{\mathcal{G}\}$ the family $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ has bounded accumulating effects and every Σ_i fulfills (7).

III. $\mathcal{L}_2$-SCALABILITY

Before we can state the main results on scalability, we consider a single network $\Sigma(\mathcal{G},\{\Sigma_i\})$ by fixing some graph $\mathcal{G}$ with nodes $\mathcal{V} = \{\Sigma_1,\dots,\Sigma_N\}$, for some arbitrary N, and interconnection structure as in (5). Attempting to analyze its properties, we define the function $V(x)$ as

$$V(x) = \sum_{i=1}^{N} p_i V_i(x_i). \tag{8}$$

with $p_i > 0$ for all i. Then, using (2), it is clear that

$$\dot{V}(x) = \sum_{i=1}^{N} p_i \nabla V_i(x_i)^{\mathrm{T}} f_i(x_i,u_i) \leq \sum_{i=1}^{N} p_i s_i(u_i,y_i). \tag{9}$$

Hence, if

$$\sum_{i=1}^{N} p_i s_i(u_i,y_i) \leq \bar{s}(d,e) \tag{10}$$

for some network supply rate $\bar{s}(d,e)$, we have that the network $\Sigma(\mathcal{G},\{\Sigma_i\})$ is dissipative with respect to $\bar{s}(d,e)$. Here, we recall that the relationship in (5) holds. Motivated by Definition II.3, we choose $\bar{s}(d,e)$ as

$$\bar{s}(d,e) = \begin{bmatrix} d \\ e \end{bmatrix}^{\mathrm{T}} W \begin{bmatrix} d \\ e \end{bmatrix},\ W = \begin{bmatrix} \beta^2\mathbb{I} & \\ & -\mathbb{I} \end{bmatrix}. \tag{11}$$

Now, let $P = \mathrm{diag}(p_1,\dots,p_N)$, and define

$$\mathrm{X} = \begin{bmatrix} \gamma^2 P & \\ & -P \end{bmatrix}. \tag{12}$$

We state the following result from [16], which essentially is a translation of (10) as a Linear Matrix Inequality (LMI).

Lemma III.1. Let $\mathcal{G}$ be a directed graph. A network $\Sigma(\mathcal{G},\{\Sigma_i\})$, with each Σ_i defined as in (1) and fulfilling Assumption 1 is dissipative with respect to (11) if there exists $p_i > 0$ for $i \in \{1,\dots,N\}$ such that

$$\begin{bmatrix} A(\mathcal{G}) & \mathbb{I} \\ \mathbb{I} & 0 \\ 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{X} & \\ & -W \end{bmatrix} \begin{bmatrix} A(\mathcal{G}) & \mathbb{I} \\ \mathbb{I} & 0 \\ 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \leq 0, \tag{13}$$

with X as in (12). Furthermore, if $\Sigma(\mathcal{G},\{\Sigma_i\})$ is dissipative with respect to (11) it has a bounded $\mathcal{L}_2$-gain with bound β, i.e. (6) holds.

We can now state the following theorem.

Theorem III.2. Consider a family of networks $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$, with $\{\mathcal{G}\}$ defined as the class of graphs with maximum in-degree less than or equal to $\mathcal{N}$. Further, let each system Σ_i, $i = 1,\dots,N$, be defined as in (1) and fulfill Assumption 1. Then, if

$$\gamma^2\mathcal{N}^2 < 1,$$

the family $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ has a scalable $\mathcal{L}_2$-gain. In particular, β can be found by solving

$$(1-\gamma^2\mathcal{N}^2)\alpha \geq 1 + \frac{\alpha^2\gamma^4\mathcal{N}^2}{\beta^2 - \alpha\gamma^2} \tag{14}$$

for $\beta^2 > \alpha\gamma^2$ and $\alpha > 0$. In this case, for any network $\Sigma(\mathcal{G},\{\Sigma_i\})$ in the family, the choice $p_i = \alpha$ results in a storage function (8) for dissipativity with respect to (11).

Proof: Let Σ be an arbitrary network drawn from $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$. Let $p_i = \alpha$ for all i, such that $P = \alpha\mathbb{I}$. Then, by performing the matrix multiplications in (13) and reversing the inequality we get the following

$$\begin{bmatrix} P - \gamma^2 A(\mathcal{G})^{\mathrm{T}} P A(\mathcal{G}) - \mathbb{I} & -\gamma^2 A(\mathcal{G})^{\mathrm{T}} P \\ -\gamma^2 P A(\mathcal{G}) & \beta^2\mathbb{I} - \gamma^2 P \end{bmatrix} \geq 0. \tag{15}$$

After substituting $P = \alpha\mathbb{I}$, by the Schur complement we know that if $\beta^2 > \alpha\gamma^2$ then (15) is satisfied if and only if

$$\alpha\mathbb{I} - \alpha\gamma^2 A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G}) - \mathbb{I} - \frac{\alpha^2\gamma^4}{\beta^2 - \alpha\gamma^2} A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G}) \geq 0. \tag{16}$$

Since we assume that each system has a maximum of $\mathcal{N}$ neighbours, we have $\|A(\mathcal{G})\|_\infty \leq \mathcal{N}$. Further, as $A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})$ is positive semi-definite, all eigenvalues of $A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})$ are real and non-negative and the maximum eigenvalue can be bounded by $\lambda_{\max}(A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})) = \rho(A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})) \leq \|A(\mathcal{G})\|_\infty^2$. This leads to the bound

$$\lambda_{\max}\left(\alpha\gamma^2 A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})\right) \leq \alpha\gamma^2\mathcal{N}^2.$$

Similarly as above,

$$\lambda_{\max}\left(\frac{\alpha^2\gamma^4}{\beta^2 - \alpha\gamma^2} A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})\right) \leq \frac{\alpha^2\gamma^4\mathcal{N}^2}{\beta^2 - \alpha\gamma^2}. \tag{17}$$

As (16) is equivalent to

$$\alpha\mathbb{I} \geq \alpha\gamma^2 A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G}) + \mathbb{I} + \frac{\alpha^2\gamma^4}{\beta^2 - \alpha\gamma^2} A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})$$

we use [19, Cor. 7.7.4], which gives that (16) is satisfied if

$$\alpha \geq \lambda_{\max}\Big(\alpha\gamma^2 A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G}) + \mathbb{I} + \frac{\alpha^2\gamma^4}{\beta^2 - \alpha\gamma^2} A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})\Big).$$

From Weyl's inequality [19, Thm. 4.3.1] the maximum eigenvalue can be bounded from above according to

$$\lambda_{\max}\Big(\alpha\gamma^2 A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G}) + \mathbb{I} + \frac{\alpha^2\gamma^4}{\beta^2 - \alpha\gamma^2} A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})\Big) \leq \alpha\gamma^2\mathcal{N}^2 + \frac{\alpha^2\gamma^4\mathcal{N}^2}{\beta^2 - \alpha\gamma^2} + 1. \quad (18)$$

The combination of the above results shows that satisfying

$$\alpha \geq \alpha\gamma^2\mathcal{N}^2 + \frac{\alpha^2\gamma^4\mathcal{N}^2}{\beta^2 - \alpha\gamma^2} + 1,$$

will ensure that (15) is fulfilled. This can be rewritten as

$$(1 - \gamma^2\mathcal{N}^2)\alpha \geq 1 + \frac{\alpha^2\gamma^4\mathcal{N}^2}{\beta^2 - \alpha\gamma^2}.$$

Which is equal to (14). If $\gamma^2\mathcal{N}^2 < 1$ as assumed in the statement of the theorem, then $(1 - \gamma^2\mathcal{N}^2)$ will be positive and there exists $\alpha > 0$ and $\beta > 0$ such that (15) is satisfied and Σ is dissipative with respect to (11). In order to show that this holds for any network in the family $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$ we emphasize that we have only made use of the property of a maximum in-degree less than or equal to $\mathcal{N}$ combined with each system Σ_i being dissipative with respect to (3). Consequently, the family $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$ has a bounded scalable $\mathcal{L}_2$-gain according to Definition II.3. □

Though the choice $P = \alpha\mathbb{I}$ may lead to conservative results it is out of the scope of this paper to find other choices of P. From Theorem III.2 we can also deduce that the same condition on the local systems ensures that the network has a scalable $\mathcal{L}_2$-gain when there is only one disturbance acting on the network and only one output is considered, as this will not affect the eigenvalues of (15). This will be useful in the continuation of the paper. To do this we define a new interconnection structure as

$$\tilde{M} = \begin{bmatrix} A(\mathcal{G}) & w_i \\ \mathbb{I} & 0 \\ 0 & 1 \\ w_j^{\mathrm{T}} & 0 \end{bmatrix}. \quad (19)$$

Corollary III.3. Consider a family of networks $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$ and let each Σ_i be defined as in (1) fulfilling Assumption 1 and connected through $\tilde{M}$ in (19). Further, consider $\{\mathcal{G}\}$ as the class with maximum in-degree less than or equal to $\mathcal{N}$. Then, $\Sigma(\{\mathcal{G}\}, \{\Sigma_i\})$ has a scalable $\mathcal{L}_2$-gain β if $\gamma^2\mathcal{N}^2 < 1$. The gain β can be found by solving (14) for $\beta^2 > \alpha\gamma^2$ and $\alpha > 0$.

Proof: The proof follows from the proof of Theorem III.2 using that $\lambda_{\max}(w_j^{\mathrm{T}} w_j) = 1$ followed with $\mathrm{rank}(A(\mathcal{G})w_i) = 1$, which gives

$$\lambda_{\max}\left((A(\mathcal{G})w_i)^{\mathrm{T}} A(\mathcal{G})w_i\right) = \mathrm{tr}\left((A(\mathcal{G})w_i)^{\mathrm{T}} A(\mathcal{G})w_i\right) \leq \mathcal{N}. \quad \square$$

Remark III.4. In Thm. III.2 and Cor. III.3 we considered the class of graphs with a maximum in-degree in order to limit the eigenvalues of $A(\mathcal{G})$. However, other classes of graphs and their adjacency matrices may give tighter bounds on the scalable $\mathcal{L}_2$-gain. This can be seen in Theorem III.2 and (14), in which the factor $\mathcal{N}^2$ is used as a bound on the maximum eigenvalue $\lambda_{\max}\left(A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})\right)$. Thus, for some classes of matrices, e.g. classes of undirected graphs, it may be possible to find a smaller bound $r < \mathcal{N}^2$, such that for any graph $\mathcal{G}$ of the class $\{\mathcal{G}\}$ $\lambda_{\max}\left(A(\mathcal{G})^{\mathrm{T}} A(\mathcal{G})\right) \leq r$.

IV. Mixed norm

As the number of systems increases and each system is subject to some bounded disturbance d_i, the overall energy of the network increases as well. An issue with $\mathcal{L}_2$-scalability, as discussed in Sec. II-B, is that there is no bound on the energy of the individual outputs, other than the total amount of energy in the network. Thus, an increase of energy in a network can be accumulated in a single system. In particular, Theorem III.2 together with Corollary III.3 guarantees

$$\|y_i(t)\|_{\mathcal{L}_2} \leq \beta\|d_j(t)\|_{\mathcal{L}_2}, \ \forall i, j \quad (20)$$

$$\|y(t)\|_{\mathcal{L}_2} \leq \beta\|d(t)\|_{\mathcal{L}_2}. \quad (21)$$

However, if multiple subsystems have disturbance inputs, (20) and (21) will only guarantee $\|y_i(t)\|_{\mathcal{L}_2} \leq \beta\|d(t)\|_{\mathcal{L}_2}$. In order to guarantee bounded accumulating effects, defined as in Definition II.4, we define the following norm.

Definition IV.1. For a signal partitioned as $y(t) = [y_1(t)^{\mathrm{T}}, \ldots, y_N(t)^{\mathrm{T}}]^{\mathrm{T}}$, with $y_i(t) \in \mathbb{R}^m$, its $\mathcal{L}_2(\infty)$-norm is defined as

$$\|y\|^2_{\mathcal{L}_2(\infty)} = \int_0^\infty \max_i |y_i(t)|_2^2 dt. \quad (22)$$

We immediately have the following results.

Lemma IV.2. Let y be a signal partitioned as $y = [y_1^{\mathrm{T}}, \ldots, y_N^{\mathrm{T}}]^{\mathrm{T}}$. Then, for all $i \in \{1, 2, \ldots, N\}$,

$$\|y_i\|_{\mathcal{L}_2} \leq \|y\|_{\mathcal{L}_2(\infty)}. \quad (23)$$

Proof: This follows immediately from the definition as

$$\int_0^\infty |y_i(t)|_2^2 dt \leq \max_i \int_0^\infty |y_i(t)|_2^2 dt \leq \int_0^\infty \max_i |y_i(t)|_2^2 dt,$$

proving (23). □

Hence, the $\mathcal{L}_2(\infty)$-norm gives an upper bound on the $\mathcal{L}_2$-norms of the subsignals. By regarding $\|y_i\|_{\mathcal{L}_2}$ as the energy associated to system i in a network, we can use Lemma IV.2 to analyze how to provide bounded accumulated effects as in Definition II.4. We start by defining a supply rate

$$s(d, e) = \max_i \beta^2 |d_i|_2^2 - \max_j |e_j|_2^2 \quad (24)$$

and see that if a network is dissipative with respect to (24) it has a bounded $\mathcal{L}_2(\infty)$-gain, as stated next.

Proposition IV.3. Suppose a network $\Sigma(\mathcal{G},\{\Sigma_i\})$ is dissipative with respect to (24) for some $\beta > 0$. Then, $\Sigma(\mathcal{G},\{\Sigma_i\})$ has a bounded $\mathcal{L}_2(\infty)$-gain with bound β, i.e.

$$\|e\|_{\mathcal{L}_2(\infty)} \leq \beta \|d\|_{\mathcal{L}_2(\infty)},$$

for any trajectory satisfying $x(0) = 0$. Furthermore, if $\Sigma(\mathcal{G},\{\Sigma_i\})$ has a bounded $\mathcal{L}_2(\infty)$-gain it will have bounded accumulating effects.

Proof: By Definition II.1, there exists a function $V(x)$ such that $\dot{V}(x) \leq s(d,e)$, which can be integrated over the time interval $[0,T]$ to obtain

$$V(x(T)) - V(x(0)) \leq \int_0^T \beta^2 \tilde{d}^2(t) - \tilde{e}^2(t)dt, \qquad (25)$$

where we have defined $\tilde{d}(t) = \max_i |d_i(t)|_2$ and $\tilde{e}(t) = \max_i |e_i(t)|_2$. Noting that $V(x(0)) = 0$ and $V(x(T)) \geq 0$, rearranging terms in (25) gives

$$\int_0^T \tilde{e}^2(t)dt \leq \beta^2 \int_0^T \tilde{d}^2(t)dt, \qquad (26)$$

after which taking the limit for $T \to \infty$ gives the desired results. Further, by setting $d_{\max}(t) = \tilde{d}(t)$ and recalling the relationship in (5) we see that

$$\|y_i\|_{\mathcal{L}_2} \leq \beta \|d_{\max}\|_{\mathcal{L}_2},$$

such that $\Sigma(\mathcal{G},\{\Sigma_i\})$ has bounded accumulating effects, according to Definition II.4. □

We will now define matrices W_{ij} in order to rewrite the supply rate in (24) in a quadratic form. Let w_i be a unit vector with a one on index $i = \arg\max_i |d_i(t)|_2^2$, and similarly w_j with $j = \arg\max_i |y_i(t)|_2^2 = \arg\max_i |e_i(t)|_2^2$. [1] We assume that during an interval $[t_k, t_{k+1})$ the indices i and j do not change. Now, let W_{ij} be a matrix defined as

$$W_{ij} = \begin{bmatrix} \beta^2 \mathrm{diag}(w_i) & 0 \\ 0 & -\mathrm{diag}(w_j) \end{bmatrix}. \qquad (27)$$

Then, for $t \in [t_k, t_{k+1})$ (24) can be rewritten as

$$s(d(t),e(t)) = \begin{bmatrix} d(t) \\ e(t) \end{bmatrix}^{\mathrm{T}} W_{ij} \begin{bmatrix} d(t) \\ e(t) \end{bmatrix}.$$

With the supply rate in (24) on this form, we can give the following results regarding dissipativity with respect to (24).

Lemma IV.4. Let $\Sigma(\mathcal{G},\{\Sigma_i\})$ be a network, for some graph $\mathcal{G}$, with each Σ_i of the form of (1) and fulfilling Assumption 1. Then, if

$$\begin{bmatrix} A(\mathcal{G}) & \mathbb{I} \\ \mathbb{I} & 0 \\ 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{X} & \\ & -W_{ij} \end{bmatrix} \begin{bmatrix} A(\mathcal{G}) & \mathbb{I} \\ \mathbb{I} & 0 \\ 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \leq 0 \qquad (28)$$

is satisfied for all possible combinations of i and j, $\Sigma(\mathcal{G},\{\Sigma_i\})$ will have a $\mathcal{L}_2(\infty)$-gain with bound β.

[1] If there are more than one maximum, i and j may be chosen arbitrarily between those indices where the maximum is attained.

Proof: From [16] we know that if the inequality (28) is satisfied with $p_i > 0$ for some fixed i and j the network $\Sigma(\mathcal{G},\{\Sigma_i\})$ is dissipative with respect to the supply rate

$$\begin{bmatrix} d \\ e \end{bmatrix}^{\mathrm{T}} W_{ij} \begin{bmatrix} d \\ e \end{bmatrix},$$

with storage function $V(x)$ as in (8). If W_{ij} are selected as by (27), we can rewrite the dissipation inequality as

$$\begin{aligned} V(x(T)) &\leq V(x_0) \\ &\quad + \sum_k \int_{t_k}^{t_{k+1}} \begin{bmatrix} d(t) \\ e(t) \end{bmatrix}^{\mathrm{T}} W_{ij} \begin{bmatrix} d(t) \\ e(t) \end{bmatrix} dt \\ &= V(x_0) + \int_0^T s(d(t),e(t))dt, \end{aligned} \qquad (29)$$

for $T = t_{k+1}$ and $s(d(t),e(t))$ equal to (24).

With Lemma IV.4 in place we can state the following theorem, in which we show how to ensure bounded accumulating effects. This will also conclude our main results.

Theorem IV.5. Let the family of networks $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ be defined over the class of graphs $\{\mathcal{G}\}$ with maximum in-degree less than or equal to $\mathcal{N}$. Further, let each Σ_i be of the form of (1) and fulfill Assumption 1. Then, if

$$\gamma^2 \mathcal{N}^2 < 1$$

there exists a $\beta > 0$ for which $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ has a $\mathcal{L}_2(\infty)$-gain with bound β. Further, this β can be found by solving (14) for $\beta^2 > \alpha\gamma^2$ and $\alpha > 0$. Thus, the choice $p_i = \alpha$ will, for any network in the family $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$, result in a storage function as in (8). Such that, any network in $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ is dissipative with respect to (24) and has bounded accumulating effects.

Proof: Consider Σ to be a network drawn from the family $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$. From Thm. III.2 and Cor. III.3 we know that if $\gamma^2\mathcal{N}^2 < 1$ there exists an $\alpha > 0$, such that with $p_i = \alpha$, Σ will fulfill (28) for any disturbance input j and performance output i. Furthermore, with $\beta^2 > \alpha\gamma^2$ fulfilling (14) and Lemma IV.4 this ensures that the network Σ is dissipative with respect to the supply rate in (24). Lastly, by Proposition IV.3 a network dissipative with respect to (24) has a bounded $\mathcal{L}_2(\infty)$-gain.

Consequently, as we have only made use of the class $\{\mathcal{G}\}$ having in-degree less than or equal to $\mathcal{N}$ combined with each Σ_i being dissipative with respect to (3), we can conclude that the family $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ has a bounded $\mathcal{L}_2(\infty)$-gain less than or equal to β. Thereby, $\Sigma(\{\mathcal{G}\},\{\Sigma_i\})$ will also have bounded accumulating effects. □

V. Example

A. Contour of feasible α and β

To visualise the feasible set of α and β for a 2-regular network of fictive systems with $\mathcal{L}_2$-gain less than or equal to $\gamma = 1/2.1$ the plot in Fig. 2 was created. The feasible set was found via a grid search over solutions for (14) and the lowest β was found as $\beta = 10.02$, the α for this β was

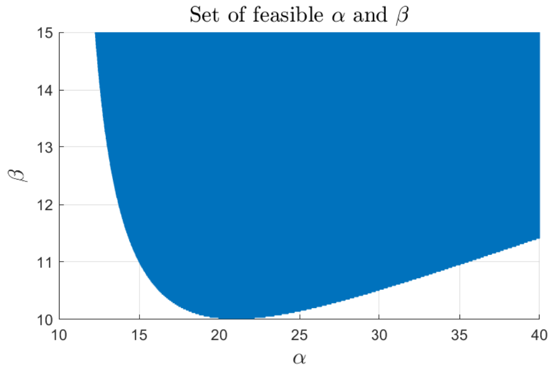

Fig. 2. Contour of α and β fulfilling ensuring a scalable $\mathcal{L}_2(\infty)$-gain. In this example $\gamma = 1/2.1$ and $\mathcal{N} = 2$.

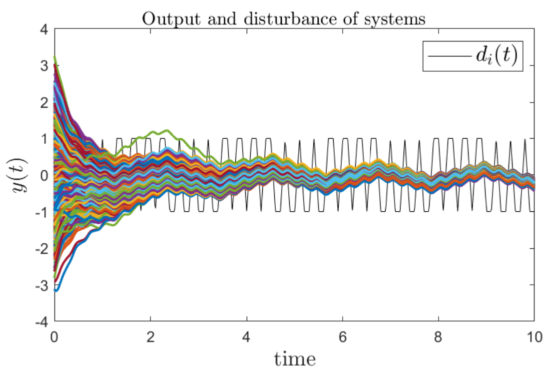

Fig. 3. Outputs of the systems. The disturbance is the same for each system and can be seen in black.

found to be $\alpha = 20.98$. It should be noted that the lower contour of the set is not linear, but rather satisfies $\beta > \sqrt{\alpha}\gamma$.

B. Simulation results

To simulate a network with bounded $\mathcal{L}_2(\infty)$-gain we created a network connected in a circular graph with disturbances acting on each subsystem, defined as below

$$\Sigma_i : \begin{cases} \dot{x}_{i_1} = x_{i_2}^2 \\ \dot{x}_{i_2} = -0.1x_{i_1}^3 x_{i_2} - 2.1x_{i_2} + y_{i-1} + y_{i+1} + d_i, \\ y_i = x_{i_2}. \end{cases}$$

Thus, all systems fulfill $\gamma^2 \mathcal{N}^2 < 1$ and according to Thm. IV.5 there exists a $\mathcal{L}_2(\infty)$-gain. From the simulations, seen in Fig. 3-4, we can see that in the sequence of increasing N there is no system with a monolithic increase in energy.

VI. Conclusion

In this paper we have presented locally verifiable conditions on the $\mathcal{L}_2$-gain of systems which, if fulfilled, guarantee a bounded scalable $\mathcal{L}_2$-gain of the network. Further, by using the mixed $\mathcal{L}_2(\infty)$-norm we have shown that the same conditions may be used to ensure that an upper bound of the energy of the local systems exists. As the same condition on the local systems results in a scalable $\mathcal{L}_2$-gain as well as limiting the energy of local systems we assume that the conditions are conservative. Future research would

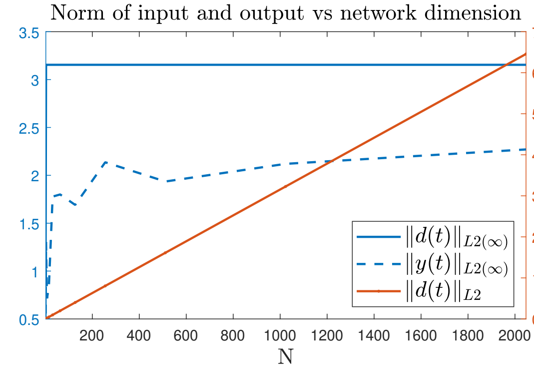

Fig. 4. Norm of the system outputs and the disturbance. As expected, $\|d(t)\|_{\mathcal{L}_2}$ grows with N, whereas $\|y(t)\|_{\mathcal{L}_{2(\infty)}}$ does not.

entail describing and reducing the conservativeness of these results.

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