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Decentralized Strong Stabilizability of Time-delay Systems

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Abstract— **The decentralized strong stabilization problem, i.e., the problem of designing a decentralized stable stabilizing controller, is considered for linear time-invariant (LTI) multi-input multi-output time-delay systems. A characterization of decentralized blocking zeros is given and it is shown that the parity interlacing property, where decentralized blocking zeros, instead of centralized ones, should be used, is also a necessary condition for decentralized strong stabilizability of LTI time-delay systems. A numerical example is also presented to demonstrate a possible application of the theoretical results.**

I. INTRODUCTION

The necessity of modeling and control of large-scale systems is increasing as a result of the magnitude and complexity of real-world industrial operations. For many large-scale systems, gathering all the information in one location, processing it there, and distributing the control commands from there may be prohibitively expensive or perhaps impossible [1]. Practical examples of large-scale systems are electric power networks, transportation and traffic systems, water distribution systems, inventory management systems, and the global economic system (see [2], [3], [4] and references therein). The idea of information flow structure usually describes the fundamental difference between the feedback control of small-scale and large-scale systems. Unlike small-scale systems, the overall plant is not controlled by a single controller but by several independent local controllers each of which can operate only on certain input-output channels of the system. This control strategy is known as *decentralized control*, and those independent local controllers all together represent the overall *decentralized controller* [1]. Given the numerous input-output channels and high level of complexity of large-scale systems, time delays are an unavoidable part of such systems. As a result, it is essential to take time delays into account in the design of decentralized controllers for such systems [5], [6], [3].

Even though unstable controllers can attain closed-loop stability, they may have some drawbacks in some real-world applications [7]. Because of that, designing a stable stabilizing controller (known as *strong stabilization problem*) has become an important topic of control theory and applications. Even though the centralized strong stabilization problem has been well studied in the past four decades (e.g., [8]–[13]), the literature is not that rich for the decentralized counterpart of this problem. To the authors' best knowledge, the decentralized strong stabilization problem has been explicitly studied only in [14]–[18]. In those studies, the *decentralized blocking zeros* have been characterized for finite-dimensional systems. Then, it has been shown that the *parity interlacing property*[1], where decentralized blocking zeros are considered instead of centralized blocking zeros, is a necessary condition for decentralized strong stabilizability of finite-dimensional linear time-invariant (LTI) systems, and also a sufficient condition under some assumptions. A stable controller design method, which is based on the sequential design of stable local controllers, has also been proposed for finite-dimensional systems in the same works. However, to the authors' best knowledge, apart from a recent dissertation [20], no works on decentralized strong stabilization of time-delay systems have appeared to date in the literature. Therefore, in the present study, we consider this problem for a broad class of LTI time-delay systems. We show that the parity interlacing property, where decentralized blocking zeros are considered instead of centralized blocking zeros, is a necessary condition for decentralized strong stabilization of LTI time-delay systems.

In the next section, the description of multi-agent time-delay systems is given along with an outline of spectral properties of such systems. Then, in Section III, decentralized time-delay controllers are introduced. In Section IV, the main result of the paper is given. A numerical example is presented in Section V. Finally, in Section VI, some concluding remarks are made.

Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ denote the sets of real numbers, complex numbers, and integers, respectively. Furthermore, $\mathbb{C}^e$ denotes the extended complex plane, i.e., $\mathbb{C}^e := \mathbb{C} \cup \{\infty\}$. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ denotes the real part of s. For $\varepsilon \in \mathbb{R}$, $\mathbb{C}_\varepsilon^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \varepsilon\}$, $\mathbb{C}_\varepsilon^{+e} := \mathbb{C}_\varepsilon^+ \cup \{\infty\}$, $\mathbb{R}_\varepsilon^+ := \{s \in \mathbb{R} \mid s \geq \varepsilon\}$, and $\mathbb{R}_\varepsilon^{+e} := \mathbb{R}_\varepsilon^+ \cup \{\infty\}$. $\mathscr{H}_{\mathbb{R}}^\infty$ denotes the Hardy space of real functions which are bounded and analytic in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$. For non-negative integers k and l, $\mathbb{R}^k$ and $\mathbb{R}^{k\times l}$ respectively denote the spaces of k-dimensional real vectors and $k \times l$-dimensional real matrices. I and 0 denote the identity and the zero matrices of appropriate dimensions, respectively. j denotes the imaginary unit; $\det(\cdot)$, $\mathrm{rank}(\cdot)$, $\mathrm{nrank}(\cdot)$, $\rho(\cdot)$, and $(\cdot)^T$ denote the determinant, the rank, the normal rank, the spectral radius, and the transpose of $(\cdot)$, respectively. Finally, $\mathrm{bdiag}(\dots)$ denotes a block diagonal matrix with $(\dots)$ on the main diagonal.

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[1]A system is said to satisfy the parity interlacing property if the number of poles (counted according to their McMillan degrees) between any pair of blocking zeros on the extended positive real-axis is even [19].

II. Multi-agent Time-delay Systems

We consider an LTI time-delay system with ν control agents, described as

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=0}^{\sigma}\Big(A_i x(t-h_i) + \sum_{k=1}^{\nu} B_{i,k} u_k(t-h_i)\Big) \\ y_k(t) &= \sum_{i=0}^{\sigma}\Big(C_{i,k} x(t-h_i) + \sum_{l=1}^{\nu} D_{i,k,l} u_l(t-h_i)\Big)\,, \end{aligned} \tag{1}$$

$k = 1,\ldots,\nu$, where $x(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^{m_k}$, and $y_k(t) \in \mathbb{R}^{q_k}$ are, respectively, the state, the input, and the output vectors at time t. Here, σ indicates the number of distinct time delays involved in (1) and $h_i > 0$, $i = 1,\ldots,\sigma$, are the time delays. We use $h_0 := 0$ for notational convenience, thus $i = 0$ corresponds to the delay-free part of the system. All the matrices involved are constant real matrices. The system may have an arbitrary number of time delays both in the autonomous part and the input-output channels. Although we represent the time delays compactly here, the state, the input, and the output time delays may indeed be different, in which case certain matrices in (1) would be zero. We note that a system in which the delayed derivatives of the state appear explicitly can be brought into the form (1) by an extension of the state vector (see [21]). Thus, description (1) is quite general and covers all LTI proper systems with constant pointwise time delays.

In the case $\mathrm{rank}(E) < n$, let the columns of U (respectively V) be a minimal basis for the left (respectively right) null space of E. Then, $U \in \mathbb{R}^{n\times\tilde{n}}$ and $V \in \mathbb{R}^{n\times\tilde{n}}$, where

$$\tilde{n} := n - \mathrm{rank}(E)\,, \tag{2}$$

are such that

$$U^T E = 0 \quad \text{and} \quad EV = 0\,. \tag{3}$$

In order to ensure the solvability of (1), it is assumed that either $\mathrm{rank}(E) = n$ or $U^T A_0 V$ is nonsingular [22], [23].

In the remainder of this section, we will outline the spectral properties of (1) to establish a background for the following sections. For the details and further properties, the reader may refer to [20].

Definition 1: For any given $\varepsilon \in \mathbb{R}$, the *set of ε-modes* of (1) is defined as

$$\Omega_\varepsilon = \left\{ s \in \mathbb{C}_\varepsilon^+ \mid \det(\phi(s)) = 0 \right\}\,, \tag{4}$$

where

$$\phi(s) := sE - A(s) \tag{5}$$

is the *characteristic matrix* of the system, where

$$A(s) := \sum_{i=0}^{\sigma} A_i e^{-sh_i}. \tag{6}$$

Furthermore, any $s \in \Omega_0$ is said to be an *unstable mode*.

When $\tilde{n} = 0$, where $\tilde{n}$ is the rank deficiency of E as defined in (2), or $U^T A_i V = 0$, for $i = 1,\ldots,\sigma$, (1) has only delay-differential equations; hence it describes a *retarded* system. For retarded systems, Ω_ε is a finite set for any $\varepsilon \in \mathbb{R}$ [24]. In other words, even though the time-delay system has infinitely many modes, there are always finitely many of them on the right-hand side of any vertical line on the complex plane. As a result, a retarded system always has a finite number of unstable modes.

On the other hand, in the case $1 \le \tilde{n} < n$, i.e., when E is a rank deficient matrix, (1) describes a *neutral* system if $U^T A_i V \neq 0$, for at least one $i \in \{1,\ldots,\sigma\}$. In this case, the associated delay-difference equations of (1) can be described as

$$\sum_{i=0}^{\sigma} \hat{A}_i \hat{x}(t-h_i) = 0\,, \tag{7}$$

where $\hat{A}_i := U^T A_i V$, $i = 0,\ldots,\sigma$, and U and V are as in (3). Here, $\hat{x}(\cdot) \in \mathbb{R}^{\tilde{n}}$ is a dummy state vector.

The system (1) is exponentially stable if there exist a $\xi > 0$, such that $\Omega_{-\xi} = \emptyset$ [24].

Definition 2: (1) is said to be *insensitively stable* [2] if it is stable for nominal values of the time delays and remains stable when infinitesimal perturbations occur in the time delays.

Definition 3: The *insensitive bound*, indicated by C_D, is the least upper bound of the real part of the modes of (7) under all infinitesimal perturbations in the time delays.

The insensitive bound, C_D, is equal to the unique root of

$$g(\zeta) - 1 = 0\,,$$

where, for $\sigma = 1$,

$$g(\zeta) := \rho\left(\hat{A}_0^{-1}\hat{A}_1 e^{-\zeta h_1}\right)\,,$$

and, for $\sigma \ge 2$,

$$g(\zeta) := \max\, \rho\left(\hat{A}_0^{-1}\left[\hat{A}_1 e^{-\zeta h_1} + \sum_{k=2}^{\sigma}\hat{A}_k e^{-\zeta h_k} e^{\mathrm{j}\theta_k}\right]\right)\,,$$

where the max is taken over $\{\theta_2,\ldots,\theta_\sigma\} \in [0,2\pi]^{\sigma-1}$ [22].

Note that the set of ε-modes is finite for all $\varepsilon > C_D$ and remains finite even when infinitesimal perturbations in the time delays occur. Furthermore, for any $\varepsilon > C_D$, all the ε-modes of (1) can be calculated by the spectral method of [26].

Definition 4: For a given $\varepsilon \in \mathbb{R}$, the *set of ε-decentralized fixed modes (ε-DFMs)* of (1) is defined as

$$\Xi_\varepsilon = \{ s \in \mathbb{C}_\varepsilon^+ \mid \det(\phi_{\Sigma,\mathscr{K}}(s)) = 0,\ \forall \mathscr{K} \in \mathbf{K} \}\,, \tag{8}$$

where $\phi_{\Sigma,\mathscr{K}}(s)$ is the characteristic matrix of the closed-loop system obtained by applying controller $\mathscr{K}$ to (1), and $\mathbf{K}$ is the class of all decentralized LTI feedback controllers under which the closed-loop system remains proper (see [27] for details). Furthermore, any $s \in \Xi_0$ is said to be an *unstable DFM*.

Note that any ε-DFM is a ε-mode, since the class $\mathbf{K}$ includes the *null controller* (i.e., the controller which applies

[2] We should note that, first in [25], then in some other studies (see [24] and references therein) the term *strong stability* was used instead of *insensitive stability*. However, *strong stability* is a term well-established to mean *closed-loop stability achieved by a stable controller* (see [7], [11] and references therein). Therefore, in order to avoid this ambiguity, here we use the term *insensitive stability* as in [21] and [13].

no feedback). Once the ε-modes of (1) are calculated for any $\varepsilon > C_D$, the ε-DFMs can be determined using the *rank test* given in [27] or the *numerical test* given in [20].

Definition 5: The *transfer function matrix (TFM)* of (1) is given as

$$\mathscr{T}(s) := C(s)\,(sE - A(s))^{-1} B(s) + D(s)\;, \tag{9}$$

where $A(s)$ is as defined in (6),

$$B(s) := \sum_{i=0}^{\sigma} B_i e^{-sh_i}\;,\;\; C(s) := \sum_{i=0}^{\sigma} C_i e^{-sh_i}\;, \tag{10}$$

and

$$D(s) := \sum_{i=0}^{\sigma} D_i e^{-sh_i}\;, \tag{11}$$

where $B_i := [B_{i,1} \quad \cdots \quad B_{i,\nu}]$, $C_i := [C_{i,1}^T \quad \cdots \quad C_{i,\nu}^T]^T$, and

$$D_i := \begin{bmatrix} D_{i,1,1} & \cdots & D_{i,1,\nu} \\ \vdots & & \vdots \\ D_{i,\nu,1} & \cdots & D_{i,\nu,\nu} \end{bmatrix}. \tag{12}$$

Furthermore, $z \in \mathbb{C}^e$ is said to be a (centralized) *blocking zero* of (1) if $\mathscr{T}(z) = 0$.

III. Decentralized Controllers

For a large-scale time-delay system with ν control agents, described as (1), we have ν *local controllers* which all together represent the decentralized controller. The k^{th} local controller, $k = 1, \dots, \nu$, can access only $\{u_k, y_k\}$ pairs of the system given in (1). Here, we consider the most general form of decentralized LTI output feedback controllers with pointwise time delays, where the k^{th} local controller is described as

$$\begin{aligned} L_k \dot{z}_k(t) &= \sum_{i=0}^{\hat{\sigma}_k} \Big(F_{i,k} z_k(t - \hat{h}_{i,k}) + G_{i,k} y_k(t - \hat{h}_{i,k}) \Big) \\ u_k(t) &= \sum_{i=0}^{\hat{\sigma}_k} \Big(H_{i,k} z_k(t - \hat{h}_{i,k}) + K_{i,k} y_k(t - \hat{h}_{i,k}) \Big)\;, \end{aligned} \tag{13}$$

where $z_k(t) \in \mathbb{R}^{l_k}$ is the state vector at time t.[3] Here, $\hat{h}_{i,k} > 0$, $i = 1, \dots, \hat{\sigma}_k$, are the time delays, where $\hat{\sigma}_k$ indicates the number of distinct time delays of the k^{th} local controller (here σ_k may be zero, i.e., some or all of the local controllers may be finite-dimensional), and $\hat{h}_{0,k} := 0$ is used for notational convenience. All the matrices involved are constant real matrices. It is assumed that either $\operatorname{rank}(L_k) = l_k$ or $U_k^T F_{0,k} V_k$ is nonsingular, where the columns of U_k (respectively V_k) form a minimal basis for the left (respectively right) null space of L_k. Furthermore, it is assumed that the matrices in (13) are such that the overall closed-loop system is proper, i.e., these matrices satisfy

$$\det(I - (K_0 - \mathscr{K})(D_0 - \mathscr{D})) \neq 0\;, \tag{14}$$

[3]Here, some or all of the local controllers may involve no state vector, in which case one will have $l_k = 0$ and the controller will be defined simply as $u_k(t) = \sum_{i=0}^{\hat{\sigma}_k} K_{i,k} y_k(t - \hat{h}_{i,k})$.

where $K_0 := \operatorname{bdiag}\{K_{0,1}, \dots, K_{0,\nu}\}$, $\mathscr{K} := \operatorname{bdiag}\{\mathscr{K}_1, \dots, \mathscr{K}_\nu\}$, where

$$\mathscr{K}_k := \begin{cases} 0\,, & \text{if}\operatorname{rank}(L_k) = l_k \\ H_{0,k} V_k (U_k^T F_{0,k} V_k)^{-1} U_k^T G_{0,k}\,, & \text{otherwise} \end{cases}\;,$$

D_0 is as defined in (12) (with $i = 0$), and

$$\mathscr{D} := \begin{cases} 0\,, & \text{if}\operatorname{rank}(E) = n \\ C_0 V (U^T A_0 V)^{-1} U^T B_0\,, & \text{otherwise} \end{cases}\;,$$

where B_0 and C_0 are as defined following (11) (with $i = 0$). Note that (14) reduces to $\det(I - K_0 D_0) \neq 0$ when $\operatorname{rank}(E) = n$ and $\operatorname{rank}(L_k) = l_k$, $k = 1, \dots, \nu$.

IV. Decentralized Strong Stabilizability

It has been shown in [28] that it is not possible to assign infinitely many unstable modes of a LTI time-delay system to the stable region in the complex plane by using a proper LTI controller. According to this result, to be able to design a stabilizing proper LTI decentralized controller, it must be ensured that the open-loop system has finitely many unstable modes. Therefore, $C_D < 0$ is a necessary condition for stabilizability of (1) since only then it is guaranteed that the open-loop system has finitely many unstable modes. To this end, we make the following assumption before going into the decentralized strong stabilizability of (1).

Assumption 1: (1) satisfies $C_D < 0$.

It has been shown in [27] that, given $C_D < 0$, (1) is stablizable by a decentralized LTI controller whose k^{th} local controller is in the form of (13) if and only if it does not have any unstable DFMs. For this reason, we make the following assumption in addition to Assumption 1.

Assumption 2: (1) does not have any unstable DFMs.

Under Assumptions 1 and 2, all the unstable modes of (1) can be assigned to the left-hand side of the complex plane by a decentralized LTI controller [27].

A necessary condition for decentralized strong stabilizability was derived in [18] for finite-dimensional systems, more specifically, for systems in the form of (1) with $\sigma = 0$ and $E = I$. Here, we extend this result to time-delay systems described by (1), by following the line of [18], and using the results of [11] which considers the centralized strong stabilizability problem for a broad class of LTI infinite-dimensional systems, including time-delay systems. In the remainder of this section, we extensively use bicoprime factorization for the system and left/right coprime factorization for the controller. We note that there are several advantages of using both of these two types of factorization approaches (see [29] and [30]).

Let us denote the TFM of (1) as $\mathscr{T} = [\mathscr{T}_{i,j}]$, where $\mathscr{T}_{i,j}$, $i, j = 1, \dots, \nu$, are the $q_i \times m_j$ dimensional TFMs corresponding to the channel with input u_j and the output y_i. Also, let us denote the $m_k \times q_k$ dimensional TFM of the k^{th} local controller (13) as $\mathscr{C}_k$, then denote the TFM of the overall decentralized controller as $\mathscr{C} = \operatorname{bdiag}\{\mathscr{C}_1, \dots, \mathscr{C}_\nu\}$. The decentralized strong stabilization problem can be defined as designing a decentralized stable controller $\mathscr{C}$, which is composed of stable local controllers $\mathscr{C}_k$, $k=1,\ldots,\nu$, for $\mathscr{T}$ such that the overall closed-loop system of $(\mathscr{T},\mathscr{C})$ is stable.

It is known that the ordered triple (P,Q,R) is bicoprime in $\mathscr{H}^\infty_\mathbb{R}$ if the pairs (P,Q) and (Q,R) are right and left coprime over $\mathscr{H}^\infty_\mathbb{R}$, respectively [30]. Furthermore the ordered triple (P,Q,R) is a bicoprime factorization of $\mathscr{T}$ over $\mathscr{H}^\infty_\mathbb{R}$ if (P,Q,R) is bicoprime in $\mathscr{H}^\infty_\mathbb{R}$, Q is a square matrix with $\det(Q(\infty))\neq 0$, and $\mathscr{T}=PQ^{-1}R$, where P, Q and R are $q\times\xi$, $\xi\times\xi$ and $\xi\times m$ dimensional matrices in $\mathscr{H}^\infty_\mathbb{R}$, respectively. Here, ξ must be such that $\operatorname{nrank}(\mathscr{T})\leq\xi$ [30]. Note that, for $\mathscr{T}$, one can always find these pairs (P_r,Q_r) and (Q_l,R_l) which are right and left coprime in $\mathscr{H}^\infty_\mathbb{R}$, respectively [31]. Then a bicoprime factorization can be found via right or left coprime factorization. For example, once a right coprime factorization of $\mathscr{T}=P_rQ_r^{-1}$ over $\mathscr{H}^\infty_\mathbb{R}$ is found, then the ordered triple (P,Q,R) can be determined as a bicoprime factorization of $\mathscr{T}$ over $\mathscr{H}^\infty_\mathbb{R}$ by choosing $P=P_r$, $Q=Q_r$, and $R=I$. A dual procedure can also be applied by using the left coprime factorization. Furthermore, a bicoprime factorization can be directly derived from (1) if and only if (1) is both spectrally stabilizable and spectrally detectable [30].

If $\mathscr{T}=PQ^{-1}R$ is a bicoprime (also if $\mathscr{T}=PQ^{-1}$ is a right coprime or $\mathscr{T}=Q^{-1}R$ is a left coprime) factorization, where $\mathscr{T}$ is the TFM of a stabilizable and detectable system (note that under Assumption 2, the system (1) is stabilizable and detectable [27]), then the $\mathbb{C}_0^+$ modes of the system are precisely the $\mathbb{C}_0^+$ roots of $\det(Q)(s)=0$.

A decentralized controller $\mathscr{C}$, whose each local controller is in the form of (13), has always a *right coprime factorization* $\mathscr{C}=P_cQ_c^{-1}$ over $\mathscr{H}^\infty_\mathbb{R}$ such that P_c and Q_c are right coprime over $\mathscr{H}^\infty_\mathbb{R}$ where P_c and Q_c are $m\times q$ and $q\times q$ dimensional matrices in $\mathscr{H}^\infty_\mathbb{R}$ [31].

Now, let us introduce the notion of *decentralized blocking zeros* which will play a central role in the remainder of this section.

Definition 6: $z_0\in\mathbb{C}^e$ is a *decentralized blocking zero* of $\mathscr{T}=[\mathscr{T}_{i,j}]$, $i,j=1,\ldots,\nu$, if, when evaluated at z_0, all the main block diagonal entries and the entries below the main diagonal blocks of the system TFM become zero after a suitable symmetric permutation of the block rows and columns. In other words, z_0 is a decentralized blocking zero of $\mathscr{T}=[\mathscr{T}_{i,j}]$ if for some permutation $\{i_1,\ldots,i_\nu\}$, $\mathscr{T}_{i_k,i_l}(z_0)=0$, for $k=1,\ldots\nu$ and $l=1,\ldots,k$. Thus, the set of decentralized blocking zeros can be described as

$$\Lambda_\mathscr{T}:=\{z_0\in\mathbb{C}^e\mid\exists\, a\ permutation\ \{i_1,\ldots,i_\nu\}\ of\ \{1,\ldots,\nu\}$$

such that

$$\begin{bmatrix}\mathscr{T}_{i_1,i_1} & 0 & 0 & \ldots & 0\\ \mathscr{T}_{i_2,i_1} & \mathscr{T}_{i_2,i_2} & 0 & \ldots & 0\\ \mathscr{T}_{i_3,i_1} & \mathscr{T}_{i_3,i_2} & \mathscr{T}_{i_3,i_3} & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ \mathscr{T}_{i_\nu,i_1} & \mathscr{T}_{i_\nu,i_2} & \mathscr{T}_{i_\nu,i_3} & \ldots & \mathscr{T}_{i_\nu,i_\nu}\end{bmatrix}(z_0)=0\}. \qquad (15)$$

For instance, when $\nu=2$, the set of decentralized blocking zeros of $\mathscr{T}$ can be defined as

$$\Lambda_\mathscr{T}=\{z_0\in\mathbb{C}^e\mid\begin{bmatrix}\mathscr{T}_{1,1} & 0\\ \mathscr{T}_{2,1} & \mathscr{T}_{2,2}\end{bmatrix}(z_0)=0\ or$$

$$\begin{bmatrix}\mathscr{T}_{1,1} & \mathscr{T}_{1,2}\\ 0 & \mathscr{T}_{2,2}\end{bmatrix}(z_0)=0\}\ .$$

Some other equivalent descriptions of the decentralized blocking zeros can be found in [18]. We note that, for a single-channel system, i.e, when $\nu=1$, the decentralized blocking zeros are equal to the centralized blocking zeros. By definition, while a decentralized blocking zero may not be a centralized blocking zero, any centralized blocking zero is a decentralized blocking zero. Consequently, the set of decentralized blocking zeros can be a much larger set than the set of centralized blocking zeros.

Obviously, to determine the decentralized blocking zeros of (1), roots of certain quasi-polynomials must be calculated. Such a calculation can be done by using the *quasi-polynomial mapping based root-finder* (QPmR) of [32].[4]

Let us denote the right-half plane decentralized blocking zeros of the system as

$$\Lambda^+_\mathscr{T}=\Lambda_\mathscr{T}\cap\mathbb{C}_0^{+e}.$$

The following two lemmas, which are borrowed from [14], are needed to prove our main result.

Lemma 1: Let (1) has no unstable DFMs. Then the set of modes of (1) and $\Lambda^+_\mathscr{T}$ are disjoint.

Lemma 2: Let (1) has no unstable DFMs and $\mathscr{T}=PQ^{-1}R$ be a bicoprime factorization over $\mathscr{H}^\infty_\mathbb{R}$, where $P=[P_1^T\ldots P_\nu^T]^T$ and $R=[R_1\ldots R_\nu]$. Let $\mathscr{C}_\nu=P_{c\nu}Q_{c\nu}^{-1}$ be the local controller applied to the ν^{th} channel of $\mathscr{T}$, where $(P_{c\nu},Q_{c\nu})$ is a pair of right coprime matrices over $\mathscr{H}^\infty_\mathbb{R}$ with appropriate dimensions. Then, the resulting $\nu-1$ channel system has a bicoprime factorization over $\mathscr{H}^\infty_\mathbb{R}$ as

$$\mathscr{T}^{\nu-1}:=\begin{bmatrix}P_1 & 0\\ \vdots & \vdots\\ P_{\nu-1} & 0\end{bmatrix}\begin{bmatrix}Q & R_\nu P_{c\nu}\\ -P_\nu & Q_{c\nu}\end{bmatrix}^{-1}\begin{bmatrix}R_1 & \cdots & R_{\nu-1}\\ 0 & \cdots & 0\end{bmatrix} \qquad (16)$$

and $\Lambda^+_\mathscr{T}\subset\Lambda^+_{\mathscr{T}^{\nu-1}}$.

Now we present our main result.

Theorem 1: Let Assumptions 1 and 2 hold. Let $z_1,z_2,\ldots,z_t$ denote the elements of $\Lambda_\mathscr{T}\cap\mathbb{R}_0^{+e}$ arranged in an ascending order. Also, let η_i denote the number of $\mathbb{R}_0^+$ modes of (1) counted with multiplicities in the interval (z_i,z_{i+1}), $i=1,2,\ldots,t-1$. Define η be the number of odd integers in the set $\{\eta_1,\ldots,\eta_{t-1}\}$. Then, every decentralized stabilizing

[4]Although QPmR can find the roots in a finite region of the complex plane, to check the condition of Theorem 1, we only need to calculate the $\mathbb{R}_0^+$ decentralized blocking zeros of (1) on the finite interval $[0,\sigma_{\max}]$, where $\sigma_{\max}$ is the right-most $\mathbb{R}_0^+$ mode of (1) and check whether there is also at least one real decentralized blocking zero to the right of $\sigma_{\max}$ (which may be at ∞; see [20]).

controller $\mathscr{C} = \text{bdiag}\{\mathscr{C}_1, \ldots, \mathscr{C}_\nu\}$ has at least η modes in $\mathbb{R}_0^+$ (counting multiplicities).[5]

Proof: Let $\mathscr{T} = PQ^{-1}R$ be a bicoprime factorization over $\mathscr{H}_{\mathbb{R}}^\infty$, where $P = [P_1^T \ldots P_\nu^T]^T$ and $R = [R_1 \ldots R_\nu]$. Also let $\mathscr{C}_j = P_{cj}Q_{cj}^{-1}$, $j = 1, \ldots, \nu$, be a right coprime factorization over $\mathscr{H}_{\mathbb{R}}^\infty$. Without loss of generality, assume that $i_j = j$, $j = 1, \ldots, \nu$, where $\{i_1, \ldots, i_\nu\}$ is such that (15) holds. By applying (16) $\nu - 1$ times we obtain:

$$\mathscr{T}^1 := \begin{bmatrix} P_1 & 0 & \cdots & 0 \end{bmatrix} \tilde{Q}^{-1} \begin{bmatrix} R_1^T & 0 & \cdots & 0 \end{bmatrix}^T, \tag{17}$$

which, by Lemma 2, is a bicoprime factorization over $\mathscr{H}_{\mathbb{R}}^\infty$, where

$$\tilde{Q} := \begin{bmatrix} Q & R_\nu P_{c\nu} & \cdots & R_2 P_{c2} \\ -P_\nu & Q_{c\nu} & & 0 \\ \vdots & & \ddots & \\ -P_2 & 0 & & Q_{c2} \end{bmatrix}.$$

Note that

$$\det(\tilde{Q}) = \det(Q)\det(\tilde{Q}_c - \tilde{P}Q^{-1}\tilde{R}), \tag{18}$$

where $\tilde{Q}_c := \text{bdiag}\{Q_{c\nu}, \ldots, Q_{c2}\}$, $\tilde{P} := [-P_\nu^T \quad \cdots \quad -P_2^T]^T$, and $\tilde{R} := [R_\nu P_{c\nu} \quad \cdots \quad R_2 P_{c2}]$. Note that, by (15), $P_k Q^{-1} R_l(z_i) = \mathscr{T}_{kl}(z_i) = 0$, for $k = 1, \ldots, \nu$, $l = 1, \ldots, k$, $i = 1, \ldots, t$. Thus,

$$\det(\tilde{Q}_c - \tilde{P}Q^{-1}\tilde{R}) = \det(\tilde{Q}_c) = \det(Q_{c\nu}) \cdots \det(Q_{c2})$$

at any z_i, $i = 1, \ldots, t$. Therefore,

$$\det(\tilde{Q}) = \det(Q)\det(Q_{c\nu}) \cdots \det(Q_{c2}) \tag{19}$$

at any z_i, $i = 1, \ldots, t$. For $j = 1, \ldots, \nu$, let n_j denote the number of $\mathbb{R}_0^+$ modes (counting multiplicities) of $\mathscr{C}_j$. Since $\mathscr{C}$ is assumed to be stabilizing, $\mathscr{T}^1$ can not have any unstable fixed modes. Then, by Theorem 1 of [11], $\mathscr{C}_1$ stabilizes $\mathscr{T}^1$ only if the number of sign changes of $\det(\tilde{Q})$ at the $\mathbb{R}_0^{+e}$ blocking zeros of $\mathscr{T}^1$ is not greater than n_1 (note that since $\mathscr{T}^1$ can not have any unstable fixed modes, $\det(\tilde{Q})$ is non-zero at the $\mathbb{C}_0^{+e}$ blocking zeros of $\mathscr{T}^1$). Let $\tilde{\eta}$ denote the number of sign changes of $\det(\tilde{Q})$ at the sequence $z_1, z_2, \ldots, z_t$. Then, since by Lemma 2 the sequence $z_1, z_2, \ldots, z_t$ is a subset of the blocking zeros of $\mathscr{T}^1$, $\mathscr{C}_1$ stabilizes $\mathscr{T}^1$ only if $\tilde{\eta} \leq n_1$. However, since by the assumption of the theorem, $\mathscr{C}$ is a stabilizing decentralized controller, $\mathscr{C}_1$ must stabilize $\mathscr{T}^1$. Thus, we must have $\tilde{\eta} \leq n_1$. On the other hand, by the assumption of the theorem, the number of sign changes of $\det(Q)$ at the sequence $z_1, z_2, \ldots, z_t$ is equal to η [11] (note that, by Assumption 2 and Lemma 1, $\det(Q)(z_i) \neq 0$, $i = 1, \ldots, t$). Then, by (19), $\tilde{\eta} \geq \eta - n_\nu - \ldots - n_2$. Thus, $\eta - n_\nu - \ldots - n_2 \leq n_1$, or $\eta \leq n_1 + \ldots + n_\nu$, which proves the theorem. ■

[5] We note that $\mathscr{T}$ may have infinitely many $\mathbb{R}_0^{+e}$ decentralized blocking zeros (see [11] for the centralized case, which may be extended to the decentralized case), in which case we replace the finite sequence $z_1, z_2, \ldots, z_t$ by the infinite sequence $z_1, z_2, \ldots$, which diverges to $+\infty$. Furthermore, even the sequence $z_1, z_2, \ldots, z_t$ is finite, we may have $z_t = \infty$ [11], which must also be taken into account. However, there can be only finitely many such blocking zeros in any finite interval [11]. Furthermore, by Assumption 1, there can be only finitely many unstable modes of (1), thus η is always finite.

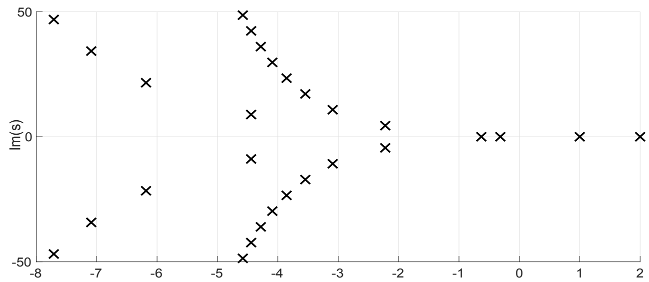

Fig. 1. The -8-modes of the system with imaginary parts between -50 and 50.

Remark 1: Theorem 1 is a generalization of part (i) of Theorem 2 in [18] to the time-delay case. However, its proof is different than the proof given in [18]. Firstly, rather than using an induction argument as in [18], we directly proved the result for a ν-channel system. More importantly, here we used the results of [11], which were given for a broad class of infinite-dimensional systems, instead of those of [7], which were restricted to finite-dimensional systems.

The following result is now apparent from Theorem 1.

Corollary 1: Under Assumptions 1 and 2, (1) is decentralized strongly stabilizable only if $\eta = 0$.

We note that Corollary 1 is a generalization of the necessity part of Theorem 1 in [11] to the decentralized case.

V. Numerical Example

We consider an LTI time-delay system with two control agents described as

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t-1)$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t-0.5) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1(t-h)$$
$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2(t) + \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} u_2(t-0.3) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} u_2(t-h)$$
$$y_1(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} x(t)$$
$$y_2(t) = \begin{bmatrix} 1 & 0 & -1 & 1 \end{bmatrix} x(t),$$

with four distinct pointwise time delays in which $h > 0$ is left as a parameter. Note that the above description is in the form of (1) where $E = I$. Therefore the system is a retarded type time-delay system since $\tilde{n} = 0$. Accordingly, one can immediately claim that $C_D = -\infty$, thus Assumption 1 is satisfied. Relying on that, we can calculate the $\mathbb{C}_\varepsilon^+$ modes, for $\varepsilon = -1$, as $\Omega_\varepsilon = \{-0.6298, -0.3149, 1, 2\}$ using the spectral method of [26]. Also, the modes of the system in a prescribed region, calculated using QPmR of [32], are shown in Fig. 1. Obviously, the system is unstable due to two $\mathbb{C}_0^+$ modes $s_1 = 1$ and $s_2 = 2$. We then check whether s_1 and/or s_2 is a

DFM, using the rank test of [27]. We find that s_1 is a DFM when $h=\log(2)$ and s_2 is a DFM when $h=\frac{\log(2)}{2}$.[6] However, for any other $h>0$, the system is free of any unstable DFMs, thus Assumption 2 is satisfied for $h \notin \{\log(2), \frac{\log(2)}{2}\}$.

In order to calculate the decentralized blocking zeros, we first obtain the TFM as

$$\mathscr{T}(s) = \begin{bmatrix} \frac{1-2e^{-hs}}{s+1-0.5e^{-s}} & \frac{4e^{-0.3s}}{s-2} \\ \frac{1-2e^{-hs}}{s-1} & \frac{1-2e^{-hs}}{s+2-e^{-0.5s}} \end{bmatrix}.$$

According to Definition 6, the set of $\mathbb{C}_0^{+e}$ decentralized blocking zeros are given as

$$\Lambda_{\mathscr{T}} = \left\{ \frac{1}{h}\log(2) + \mathrm{j}\frac{2l}{h}\pi \right\}_{l\in\mathbb{Z}} \cup \{\infty\}.$$

Therefore, the $\mathbb{R}_0^{+e}$ decentralized blocking zeros are $z_1 = \frac{\log(2)}{h}$ and $z_2 = \infty$.[7] Thus, $\eta = 0$ when $h < \frac{\log(2)}{2}$ or $h > \log(2)$ and $\eta = 1$ when $\frac{\log(2)}{2} < h < \log(2)$. Therefore, according to Corollary 1, the system is not decentralized strongly stabilizable when $\frac{\log(2)}{2} < h < \log(2)$.

VI. Conclusion

It has been shown that, under Assumptions 1 and 2, a LTI time-delay system is strongly stabilizable by a decentralized LTI output feedback time-delay controller only if the parity interlacing property (where decentralized blocking zeros are considered instead of centralized blocking zeros) is satisfied. Whenever the system is strongly stabilizable by decentralized LTI feedback, a time-delay or finite-dimensional decentralized stable LTI controller can be designed by extending the constrained optimization-based approach of [13] to the decentralized case.

References

[1] D. D. Šiljak, *Decentralized Control of Complex Systems*. San Diego, USA: Academic Press, 1991.

[2] J. Lunze, *Feedback Control of Large Scale Systems*. New York, USA: Prentice Hall, 1992.

[3] L. Bakule, "Decentralized control: Status and outlook," *Annual Reviews in Control*, vol. 38, pp. 71–80, 2014.

[4] J. Guanetti, Y. Kim, and F. Borrelli, "Control of connected and automated vehicles: State of the art and future challenges," *Annual Reviews in Control*, vol. 45, pp. 18–40, 2018.

[5] L. Bakule, "Decentralized control: An overview," *Annual Reviews in Control*, vol. 32, pp. 87–98, 2008.

[6] ——, "Decentralized control and communication," *Annual Reviews in Control*, vol. 36, pp. 1–10, 2012.

[7] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, USA: M.I.T. Press, 1985.

[8] M. Zeren and H. Özbay, "On the strong stabilization and stable $\mathscr{H}^\infty$ controller design problems for MIMO systems," *Automatica*, vol. 36, pp. 1675–1684, 2000.

[6] s_1 is in fact a centralized fixed mode (an uncontrollable mode) when $h=\log(2)$, i.e., it is not possible to move s_1 by centralized or decentralized LTI feedback when $h=\log(2)$; however, s_2 is a DFM when $h=\frac{\log(2)}{2}$, although it is not a centralized fixed mode, i.e., it is possible to move s_2 by centralized LTI feedback, but not by decentralized LTI feedback when $h=\frac{\log(2)}{2}$.

[7] We note that the only centralized blocking zero is $z=\infty$. Thus, according to the results of [11], the system is strongly stabilizable by LTI centralized feedback for any positive $h \neq \log(2)$ (recall that $s_1 = 1$ is a centralized fixed mode when $h=\log(2)$, thus the system is not stabilizable when $h=\log(2)$).

[9] A. N. Gündeş and H. Özbay, "Strong stabilization of a class of MIMO systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 431–435, 2011.

[10] H. U. Ünal and A. İftar, "Stable $\mathscr{H}^\infty$ controller design for systems with multiple input/output time-delays," *Automatica*, vol. 48, pp. 563–568, 2012.

[11] ——, "On strong stabilizability of MIMO infinite-dimensional systems," *Automatica*, vol. 121, no. 109178, 2020.

[12] H. U. Ünal, "On stable $\mathscr{H}^\infty$ controller design for plants with infinitely many unstable zeros and poles," *Automatica*, vol. 138, no. 110036, 2021.

[13] S. M. Özer and A. İftar, "Delay-based stabilisation and strong stabilisation of LTI systems by nonsmooth constrained optimisation," *International Journal of Control*, vol. 96, no. 8, pp. 2044–2057, 2023.

[14] K. A. Ünyelioğlu, "Decentralized blocking zeros in the control of large scale systems," Ph.D. dissertation, Bilkent University, Ankara, Turkey, July 1992.

[15] A. B. Özgüler and K. A. Ünyelioğlu, "Decentralized strong stabilization problem," in *Proceedings of the American Control Conference*, Chicago, IL, June 1992, pp. 3294–3298.

[16] K. A. Ünyelioğlu and A. B. Özgüler, "Decentralized blocking zeros-Part I: Decentralized strong stabilization problem," in *Proceedings of the IEEE Conference on Decision and Control*, Tucson, Arizona, December 1992, pp. 1340–1345.

[17] ——, "Decentralized blocking zeros-Part II: A unifiying approach to decentralized synthesis problem," in *Proceedings of the American Control Conference*, San Francisco, California, June 1993, pp. 2745–2749.

[18] K. A. Ünyelioğlu, A. B. Özgüler, and Ü. Özgüner, "Decentralized blocking zeros and the decentralized strong stabilization problem," *IEEE Transactions on Automatic Control*, vol. 40, no. 11, pp. 1905–1918, 1995.

[19] D. C. Youla, J. J. Bongiorno, and C. N. Lu, "Single-loop feedback stabilization of linear multivariable dynamical plants," *Automatica*, vol. 10, p. 159173, 1974.

[20] S. M. Özer, "Stabilization and strong stabilization of LTI systems by structured time-delay controllers," Ph.D. dissertation, Eskişehir Technical University, Eskişehir, Turkey, 2022.

[21] S. M. Özer and A. İftar, "Eigenvalue optimisation-based centralised and decentralised stabilisation of time-delay systems," *International Journal of Control*, vol. 95, no. 8, pp. 2245–2266, 2022.

[22] W. Michiels, "Spectrum based stability analysis and stabilisation of systems described by delay differential algebraic equations," *IET Control Theory and Applications*, vol. 5, pp. 1829–1842, 2011.

[23] E. Fridman and U. Shaked, "$\mathscr{H}_\infty$-control of linear state-delay descriptor systems: an LMI approach," *Linear Algebra and its Applications*, vol. 351-352, pp. 271–302, 2002.

[24] W. Michiels and S.-I. Niculescu, *Stability, Control, and Computation for Time-Delay Systems: An Eigenvalue-Based Approach*. Philadelphia, USA: SIAM, 2014.

[25] J. Hale and S. Verduyn-Lunel, *Introduction to functional differential equations*. New York, USA: Springer, 1993.

[26] Z. Wu and W. Michiels, "Reliably computing all characteristic roots of delay differential equations in a given right half plane using a spectral method," *Journal of Computational and Applied Mathematics*, vol. 236, pp. 2499–2514, 2012.

[27] H. E. Erol and A. İftar, "Stabilization of decentralized descriptor-type neutral time-delay systems by time-delay controllers," *Automatica*, vol. 64, pp. 262–269, 2016.

[28] J. J. Loiseau, M. Cardelli, and X. Dusser, "Neutral-type time-delay systems that are not formally stable are not BIBO stabilizable," *IMA Journal of Mathematical Control and Information*, vol. 19, pp. 217–227, 2002.

[29] C. A. Desoer and A. N. Gündeş, "Bicoprime factorizations of the plant and their relation to right and left-coprime factorizations," *IEEE Transactions on Automatic Control*, vol. 33, pp. 672–676, 1988.

[30] M. Tsiakkas and A. Lanzon, "Foundations of a bicoprime factorization theory," *IEEE Transactions on Automatic Control*, vol. 62, pp. 4598–4603, 2017.

[31] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York, USA: Springer-Verlag, 1995.

[32] T. Vyhlidal and P. Zitek, "QPmR - Quasi-polynomial root-finder: Algorithm update and examples," in *Delay Systems: From Theory to Numerics and Applications*. New York: Springer, 2014, ch. 10, pp. 299–312.