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Stabilization over a non-simple directed cycle: application to opinion dynamics

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***Abstract*— This paper considers a variant of the classical cyclic pursuit problem where a multi-agent system (MAS) interacts over a directed cycle on n nodes. The edge weights of the directed cycle are considered to be positive, non-identical real numbers. However, instead of considering a simple directed cycle with these n edges alone, we admit the possibility of self-loops, which renders the resulting cycle digraph non-simple. Within this set-up, we show the effect of self-loops having positive and negative weights on the stability (or lack thereof) of the resulting system dynamics. Since this problem is relevant to the Taylor's model of opinion dynamics, where a positive self-loop weight corresponds to 'stubbornness', we draw an analogy between the two paradigms.Finally, we consider the unweighted version of the directed (non-simple) cycle and present an empirical study related to the effect of positive weighted self-loops in mitigating noise. Simulations are presented to support our results.**

I. INTRODUCTION

The cyclic pursuit problem has been widely investigated in the context of consensus of multi-agent systems. Under this paradigm, agent i pursues agent $i+1$ (modulo n), and owing to its simplicity, this problem was studied in the context of the 'n-bugs problem' [1] and other applications [2]. In several variants of cyclic pursuit [3], researchers have explored the possibility of heterogeneous edge weights with one of the weights being possibly negative. This antagonistic interaction was shown to be beneficial in expanding the set of points where agents may achieve consensus. Though the dynamics of agents under consideration ranged from single integrators to double integrators [4], all these models considered interactions over a simple directed cycle, with no self-loops.

There is a plethora of results available on the coordination of single and double integrators in continuous time, over more general undirected and directed graphs [5], [6], from various perspectives such as network synthesis [7], noise propagation [8] the effect of antagonistic interactions [9], [10], etc. However, one consistent feature of all these interaction topologies has been the fact that the underlying graph was always considered to be simple (with no self-loops). But graphs with self-loops do find their applications in natural sciences and engineering [11]–[14]. While several models of opinion dynamics exist (see [15]–[19]) to model interactions among individuals, the Taylor's model of opinion dynamics [20] has recently been investigated by some researchers from several perspectives [21], [22] and it turns out that the 'stubbornness' of an agent can be captured through a self-loop with positive edge weight. Though negative self-loop weights had not been considered in the classical Taylor's model, such a consideration may be justified to model self-doubting individuals.

In this context, the set-up considered in the present paper may be viewed as a way to study the possibility of consensus in the presence of stable and unstable first order linear agents or as a means to analyze the evolution of opinion dynamics according to Taylor's model in the presence of both stubborn and self-doubting individuals over a directed cycle. We consider a directed weighted cycle (with edges having positive real weights) that also has self loops whose weights can be both positive or negative. Under such conditions, we investigate how agreement or *consensus* can be achieved under varying levels of 'stubbornness' and 'self-doubt' in agents, captured through the magnitudes of positive and negative weights on self-loops. To be precise, we provide bounds on the self-loop edge weights (both negative and positive) for agreement/consensus to result. The specific topology considered here (directed cycle) may capture phenomenon such as propagation of information over the course of a telephone game. While negative edge weights have been previously considered for modelling antagonistic interactions within the consensus protocol [10], such results involved simple graphs with no self-loops, unlike this paper.

We study the effect of positive self-loops (stubborn agents) when there is noise in the network. For this problem, the edge weights of cyce graph as well as the self-loop weights are considered to be unity. Such investigations pertaining to noise have been carried out for the consensus protocol [8], [9] in the past, but in each of these cases, the graph under consideration was a simple graph, with no self-loops. Towards that end, this paper paves the way for answering important open questions pertaining to consensus over non-simple graphs.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Notations and definitions

$\|\cdot\|_k$ is the k- norm (e.g. $k=2,\ldots,\infty$), $\mathbb{R}$ is set of real numbers, and $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\text{Diag}(\gamma_i) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with γ_i being the i^{th} diagonal entry. For a matrix $A \in \mathbb{R}^{n\times n}$, $[A]_{ij}$ is the $(i,j)^{\text{th}}$ element and $\text{Det}(A)$ is the determinant, respectively. A weighted graph, $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{W})$, is a triplet, with $\mathscr{V}$ being the set of n nodes given by $\mathscr{V} = \{v_1, v_2, \ldots, v_n\}$, a

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set of edges $\mathscr{E} \in \mathscr{V} \times \mathscr{V}$ given by $\mathscr{E} = \{\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_{|\mathscr{E}|}\}$, and corresponding weights given by $\mathscr{W} : \mathscr{E} \to \mathbb{R}^{|\mathscr{E}|}$. With a slight abuse of notation, the edges and their weights may be also denoted by a double subscript to signify the indices of the nodes which they connect. Node v_q is a neighbour to node v_r if $\mathscr{E}_i \triangleq \mathscr{E}_{rq} = (v_r, v_q) \in \mathscr{E}$ for some i and $w_i \triangleq w_{rq}$ is the weight assigned to the adjacent of node v_q and node v_r, i.e., weight associated with edge $\mathscr{E}_i$. A graph is called simple if there are no self-loops, i.e., $(v_r, v_r) \notin \mathscr{E}$ for any r and no edge $\mathscr{E}_i$ appears multiple times in $\mathscr{E}$. $\mathscr{L} \in \mathbb{R}^{|\mathscr{V}|\times|\mathscr{V}|}$ is the graph Laplacian and the entries $[\mathscr{L}]_{ij} = l_{ij} = -w_{ij}$ and $[\mathscr{L}]_{ii} = l_{ii} = \sum_{i=1, i\neq j}^{n} w_{ij}$. The Laplacian, $\hat{\mathscr{L}}$, of the non-simple directed graph $\hat{\mathscr{G}}$, with self loops, changes to $\hat{\mathscr{L}} = \mathscr{L} + \Gamma$, where $\Gamma = \text{Diag}(\gamma_i)$ is a diagonal matrix and γ_i is the weight of self-loop on node v_i. We refer the reader to [8], [23] for further details on algebraic graph theory.

III. Stabilizability over directed cycle

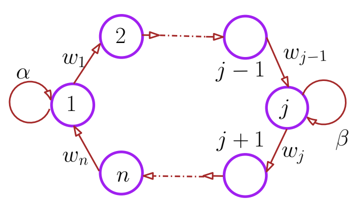


Fig. 1: Agents over a directed cycle with two self-loops (One with self-loop weight $\alpha \geq 0$ and other with self-loop weight $\beta \leq 0$)

Consider n agents with 1st order dynamics interacting over a directed graph $\mathscr{G}$, whose individual dynamics are given by

$$\dot{x}_i = -\gamma_i x_i + u_i, \tag{1}$$

such that $\gamma_i \in \mathbb{R}$, with the linear coordination law, u_i given by

$$u_i = \sum_{j=1}^{n} w_{ij}(x_j - x_i) \tag{2}$$

Therefore the linear dynamics of the system written in state-space form is

$$\dot{X} = -(\mathscr{L} + \Gamma)X = -\hat{\mathscr{L}}X, \tag{3}$$

where $X \in \mathbb{R}^n$ is the n-tuple representing states of all the agents, $\mathscr{L}$ is the Laplacian of underlying graph, $\mathscr{G}$, and $\Gamma = \text{Diag}(\gamma_i)$. The dynamics in (3) can be modelled as an agent interaction over a directed graph containing self-loops on each agent with weight γ_i. In such a setting, $\hat{\mathscr{L}}$ is the Laplacian of the non-simple graph $\hat{\mathscr{G}}$. In this paper, the interaction topology considered is a directed cycle with self-loops where $\gamma_1 = \alpha$, $\gamma_j = \beta$ with $\gamma_i = 0$ for any $i \notin \{1, j\}$, and resulting non-simple graph, $\hat{\mathscr{G}}$, with two self-loops, one at node 1 with $\alpha > 0$ and the other with $\beta < 0$ at node j is shown in Fig.(1). The Laplacian of the corresponding simple directed cycle is $\mathscr{L} = \text{Diag}(w_i) - \begin{bmatrix} 0_{n-1} & \text{Diag}(\bar{w}_i) \\ w_n & 0_{n-1}^{\top} \end{bmatrix}$, with $w_i > 0 \quad \forall i = 1:n$ being the weights associated with edges of the cycle digraph and $\bar{w}_i = w_i \forall i \in \{1, 2, \ldots, n-1\}$, while $\Gamma = \text{Diag}(\gamma_i)$, with the entries $\gamma_i = 0 \ \forall i \notin \{1, j\}$, $\gamma_1 = \alpha$, and $\gamma_j = \beta$. In other words, $\mathscr{L}$ is the Laplacian of the simple directed cycle (when considering $\alpha = \beta = 0$) and the characteristic polynomial of $-\mathscr{L}$ is hence given by

$$\phi_0(s) = \prod_{i=1}^{n}(s + w_i) - \prod_{i=1}^{n} w_i \tag{4}$$

Furthermore, for $w_i > 0 \forall i$, $\phi_0(s)$ has exactly one root at 0 with the remaining roots in the open left half of the complex plane [3]. In the presence of a negative self-loop, to analyse the stability of (3), we consider the characteristic polynomial $\phi(s) = \text{Det}(sI + \hat{\mathscr{L}})$ given by

$$\phi(s) = (s + w_1 + \alpha)(s + w_j + \beta) \prod_{i \neq 1, j}^{n}(s + w_i) - \prod_{i=1}^{n} w_i \tag{5}$$

The polynomial in (5) can be readily derived through Laplace's expansion of the determinant along the first column of $(sI + \hat{\mathscr{L}})$. Furthermore, $\phi(s)$ may be written as the sum of two polynomials, $\phi_1(s)$ and $\hat{\phi}_j(s)$, or $\phi_2(s)$ and $\varphi_j(s)$ as follows

$$\phi(s) = \phi_1(s) + \beta\hat{\phi}_j(s) = \phi_2(s) + \alpha\varphi_j(s) \tag{6}$$

$$\phi_1(s) = (s + w_1 + \alpha)\prod_{i \neq 1}^{n}(s + w_i) - \prod_{i=1}^{n} w_i \tag{7}$$

$$\hat{\phi}_j(s) = (s + w_1 + \alpha)\prod_{i \neq 1, j}^{n}(s + w_i) \tag{8}$$

$$\phi_2(s) = (s + w_j + \beta)\prod_{i \neq j}^{n}(s + w_i) - \prod_{i=1}^{n} w_i \tag{9}$$

$$\varphi_j(s) = (s + w_j + \beta)\prod_{i \neq 1, j}^{n}(s + w_i) \tag{10}$$

where $\phi_1(s)$ is the characteristic polynomial of the directed cycle with one self-loop ($\alpha > 0$, $\beta = 0$). Since the Laplacian of the graph with two self-loops is a rank- 1 update to the Laplacian of a graph with a single self-loop, equation (6) can be derived using the formula for rank-one update of determinants (see Lemma 1.1 in [24]). We now state the following result without proof in the interests of space.

Lemma 1. *Let $\hat{\mathscr{L}}$ be the Laplacian of a directed cycle with two arbitrary self-loops at nodes v_1 and v_j, having weights $\alpha \geq 0$ and $\beta \leq 0$, respectively. Then the following hold:*

1) *$-\hat{\mathscr{L}}$ is non Hurwitz if $\alpha = 0, \beta < 0$, and*
2) *$-\hat{\mathscr{L}}$ is Hurwitz if $\alpha > 0, \beta = 0$.*

Lemma 1 establishes that in the presence of an agent having a single self-loop with positive weight, the dynamics given in (3) will be stable, while in the presence of a single agent with negative self-loop weight, the dynamics will be unstable. Next, we will study the effects on the dynamics given in (3) in the presence of an agent of each type, that is one with a positive self-loop weight and another with a negative self-loop weight.

A. Destabilization problem:

This subsection will study how the presence of a self-loop with a negative weight (i.e., self-loop on node j with a negative weight $\beta < 0$) may cause (3) to lose its stability.

Theorem 1. *Suppose a group of agents evolve over a directed cycle according to* (3)*. Further, suppose there is one agent with positive self-weight with α (weight on self-loop), say agent 1, and an unstable agent, say agent j, with a self-loop of weight $\beta < 0$. The dynamics given in* (3) *will be stable if and only if $\beta > -\frac{\alpha w_j}{w_1+\alpha}$.*

Proof: The proof will involve the following steps. First, we shall evaluate the critical value of $\beta < 0$, given some fixed positive value of α, for which $\phi(s)$ in (6) has a root at 0. As a second step we shall show that for negative values of β *greater* than the value obtained in step 1, $\phi(s)$ cannot have any root in the open RHP. These two steps will then lead us to conclude that the value of β obtained in step 1 is indeed the critical value for stability.

Step 1: The determinant of the Laplacian is

$$\phi(0) = \mathrm{Det}(\hat{\mathscr{L}}) = (w_1+\alpha)(w_j+\beta)\prod_{i\neq 1,j}^{n} w_i - \prod_{i=1}^{n} w_i. \quad (11)$$

From (11) it follows that solving for $\mathrm{Det}(\hat{\mathscr{L}}) = 0$ leads to $\alpha\prod_{i\neq 1}^{n} w_i + \beta(\alpha+w_1)\prod_{i\neq 1,j}^{n} w_i = 0$, implying

$$\beta = -\frac{\alpha w_j}{\alpha+w_1} \quad (12)$$

For the value of β given in (12) of step 1, $-\hat{\mathscr{L}}$ is singular. Our interest is to determine the variation of the roots of the polynomial $\phi(s)$ for variation of parameter $\beta < 0$ for a fixed $\alpha > 0$. This problem is posed as a complementary root locus problem for the transfer function $\tilde{G}(s) := \frac{\hat{\phi}_j(s)}{\phi_1(s)}$, obtained by equation (6) to zero. Clearly, the n roots of $\phi_1(s)$ are open-loop poles of $\tilde{G}(s)$, and the $n-1$ roots of $\hat{\phi}_j(s)$ are open-loop zeros of $\tilde{G}(s)$ and they all belong to the LHP for $\alpha > 0$ (follows from Lemma 1). Consequently, the relative degree for $\tilde{G}(s)$ is 1, implying one branch of the complementary root locus will go to $+\infty$, passing through the origin, since the origin satisfies the angle criteria for the complementary root locus. In step 1, we have already evaluated this critical value of β at which this crossover happens.

Step 2: As the second step of this proof we shall now show that for $0 > \beta > -\frac{\alpha w_j}{\alpha+w_1}$, no other branch of the complementary root locus can exist on the imaginary axis. Equivalently, this would imply that crossing over of any branch to the RHP through any part of the imaginary axis, other than through the origin, is ruled out. This is important to verify since although only one branch of the complementary root locus can remain permanently in the RHP for $\beta < -\frac{\alpha w_j}{\alpha+w_1}$, there is a possibility that other branches of the complementary root locus might cross from LHP to RHP and subsequently return to LHP at some value of $\beta > -\frac{\alpha w_j}{\alpha+w_1}$. It stands to reason that if such an event does occur, then the critical value of β obtained in step 1, given by (12), will not be the only crossover value of β. Hence, we need to rule out this possibility. As a preliminary investigation, let us therefore probe the possibility of imaginary roots of $\phi(s)$ at $\beta = -\frac{\alpha w_j}{\alpha+w_1}$. Thereafter, we will endeavour to ascertain the possibility of such imaginary roots for $\beta > -\frac{\alpha w_j}{\alpha+w_1}$.

Assume $\phi(s)$ has some imaginary root when $\beta = -\frac{\alpha w_j}{\alpha+w_1}$ given by s_1. Thus, s_1 must satisfy equation (5). Now, $\beta + w_j = \frac{-\alpha w_j}{\alpha+w_j} + w_j = \frac{w_1 w_j}{\alpha+w_j}$ and

$$(s_1+w_1+\alpha)\left(s_1+\frac{w_1 w_j}{\alpha+w_j}\right)\prod_{i\neq 1,j}^{n}(s_1+w_i) - \prod_{i=1}^{n} w_i = 0 \quad (13)$$

$$\implies (s_1+w_1+\alpha)\left(s_1+\frac{w_1 w_j}{\alpha+w_j}\right)\prod_{i\neq 1,j}^{n}(s_1+w_i) = \prod_{i=1}^{n} w_i \quad (14)$$

Since the quantities on either side of equation (14) can be viewed as complex numbers, they must agree in both their magnitudes and arguments. Equating their magnitudes, we have

$$\left\|\frac{(s_1+w_1+\alpha)\left(s_1+\frac{w_1 w_j}{\alpha+w_j}\right)\prod_{i\neq 1,j}^{n}(s_1+w_i)}{\prod_{i=1}^{n} w_i}\right\| = 1 \quad (15)$$

$$\implies \underbrace{\left\|\frac{(s_1+w_1+\alpha)(s_1+\frac{w_1 w_j}{\alpha+w_j})}{w_1 w_j}\right\|}_{m_1} \cdot \underbrace{\left\|\prod_{i\neq 1,j}^{n}\frac{(s_1+w_i)}{w_i}\right\|}_{m_2} = 1 \quad (16)$$

Observe that (15) is written as the product $m_1 m_2$ in (16) and this is the magnitude criteria related to the root locus. Moreover, $s_1 + w_i$ represents a vector joining points s_1 on the imaginary axis and $-w_i$ on the negative real axis, while $|s_1 + w_i|$ is the length of this vector. Fig. (2) shows the case where the point $s_1 = j\delta$ is on the imaginary axis. From triangle inequality in Fig.(2) it follows that $\|w_i\| < \|j\delta + w_i\|$. Moreover, for point s_1 in RHP also, we will have $\|w_i\| < \|s_1 + w_i\|$. Considering $s_1 = j\delta$ in (16) we have

$$\underbrace{\left\|\frac{(j\delta+w_1+\alpha)(j\delta+\frac{w_1 w_j}{\alpha+w_j})}{w_1 w_j}\right\|}_{m_1} \cdot \underbrace{\left\|\prod_{i\neq 1,j}^{n}\frac{(j\delta+w_i)}{w_i}\right\|}_{m_2} = 1. \quad (17)$$

In (16) and (17), we clearly have $m_2 > 1$. Further, it follows

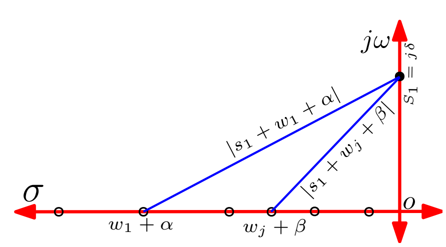


Fig. 2: Magnitude criteria in complex plane

that

$$m_1(j\delta) = \frac{(\delta^2 + (w_1+\alpha)^2)\left(\delta^2 + \left(\frac{w_1 w_j}{\alpha+w_j}\right)^2\right)}{w_1^2 w_j^2}$$
$$= 1 + \underbrace{\frac{\delta^4 + \delta^2(w_1+\alpha)^2 + \delta^2 \frac{w_1^2 w_j^2}{(w_1+\alpha)^2}}{w_1^2 w_j^2}}_{f(\delta^2)} > 1$$

Hence $m_1 > 1$ because $f(\delta^2) > 0, \forall \delta \in \mathbb{R}$. Since $m_2 > 1$, as established earlier, equality cannot hold for (16). Hence no root of $\phi(s)$ can exist in RHP or on the imaginary axis with β given in (12) and 0 is the only root of $\phi(s)$ at $\beta = -\frac{\alpha w_j}{\alpha+w_1}$. For any other $\beta > -\frac{\alpha w_j}{\alpha+w_1}$, the point $w_j+\beta$ will move further to the left on the real axis as shown in Fig.(2), which in turn will lead to $|s_1+w_j+\beta|$ and subsequently m_1 only increasing further, whereas m_2 remains unchanged. This means that no imaginary axis crossover can occur for $\beta > -\frac{\alpha w_j}{\alpha+w_1}$ either, as the magnitude criteria of root locus, given in (16), fails. Hence, $\beta = -\frac{\alpha w_j}{\alpha+w_1}$ is indeed the critical value for loss of stability of the dynamics given in (3). □

B. Stabilization problem

We now answer the following question: *Is it possible to stabilize the dynamics in* (3) *through introduction of a positively weighted self-loop in the group, given that the dynamics was initially unstable due to the presence of a negatively weighted self-loop?*

Theorem 2. *Suppose a group of agents evolve over a directed cycle. Further, suppose there is one unstable agent (self-loop with negative weight), say agent j, with an edge weight of $\beta < 0$ on its self-loop, and an agent, say agent* 1, *with positive self-loop weight $\alpha > 0$. The dynamics given in* (3) *can be stabilized if only if $\beta > -w_j$, and $\alpha > -\frac{\beta w_1}{w_j+\beta}$.*

IV. Application to Taylor's model for opinion dynamics

The evolution of i^{th} agent's opinion, according to Taylor's model [20], is given by

$$\dot{x}_i = \begin{cases} -\sum_j w_{ij}(x_i - x_j), & \text{if } i \notin \mathscr{V}_s \;, \\ -\sum_j w_{ij}(x_i - x_j) - \gamma_i(x_i - u_i), & \text{if } i \in \mathscr{V}_s. \end{cases} \tag{18}$$

where $x_i \in \mathbb{R}$ depicts opinion of agent i, γ_i is the confidence/stubbornness (for $\gamma_i > 0$) or lack of confidence/self-doubt of agent i (for $\gamma_i < 0$), $\mathscr{V}_s = \{v_i \in \mathscr{V} | \gamma_i \in \mathbb{R} \setminus \{0\}\}$ is the set of self-confident or self-doubting agents and w_{ij} represents the weight given by agent i to the opinion of agent j. Further, u_i is either the external influence, or $u_i = x_i(0)$, the initial opinion of i^{th} agent. For $u_i = x_i(0)$, $\gamma_i > 0$ represents the tendency of agent i to adhere to its initial opinion due to *confidence or stubbornness*, during the course of opinion evolution. Though Taylor's model conventionally does not consider $\gamma_i < 0$, one may consider a lack of self-confidence to be captured by negative values of γ_i, which subsequently drives an agent's opinion away from its own initial conviction. The evolution of opinions in (18) can be captured using a non-simple graph with self-loops having weight γ_i. Eqn. (18) can then be written in the following manner:

$$\dot{X} = -(\mathscr{L} + \Gamma)X + \Gamma U, \tag{19}$$

where $X \in \mathbb{R}^n$ is the vector of agents' opinions, $\mathscr{L}$ is the Laplacian of the digraph without self-loops, $U \in \mathbb{R}^n$, with $u_i = 0$ if $i \notin \mathscr{V}_s$ and equal to a constant otherwise, while Γ captures stubbornness/self-confidence of agents. Furthermore, if the underlying interaction topology is a directed cycle, it may be conceived of as exchange of opinion during the course of a telephone game with self loop weights representing stubborn ($\gamma_i > 0$) or self-doubting agents ($\gamma_i < 0$). In the presence of one self-doubting agent ($\gamma_i < 0$ for some i), and one stubborn agent, the stabilization of the opinion dynamics is possible under the conditions derived in the previous section.

V. Effectiveness under noisy environment

Multi-agent systems are susceptible to the effects of undesirable noises, and a way to characterize a network's resilience to such noises is through the H_2 norm. In [8] such analyses have been carried out for undirected graphs, where closed form solutions were obtained for relevant Riccati equations by exploiting the symmetric nature of interactions in undirected graphs. An empirical study is presented here, which culminates in a few conjectures, arrived at through extensive simulations. Here, we consider an unweighted directed cycle graph, i.e., all the edge weights, including self-loops, are considered unity. The input has been set to zero ($U = 0$) in (19), and the only external signal present is in the form of noises at the nodes. Thus, the resulting dynamics turns out to be

$$\dot{X} = -(\mathscr{L} + \Gamma)X + IV \qquad \text{and} \qquad Y = E^\top X, \tag{20}$$

with $V \in \mathbb{R}^n$ being the noise vector, where the entry V_i is the noise on node i. The noise is considered to be 0 mean Gaussian, satisfying $\mathbf{E}(VV^\top) = \sigma_v^2 I_n$. E is the incidence matrix. The norm $\|Y\|_2$ is used as a measure to study the effect of noise. The computation of $\|Y\|_2$ [25] for the system $\dot{X} = AX + BU$ and $Y = CX$ requires the solution of the Lyapunov equation $AP + PA^\top = -BB^\top$ or $AQ + QA^\top = -C^\top C$ and $\|Y\|_2 = \text{Trace}(BPB^\top) = \text{Trace}(C^\top QC)$. For the system in (20), the associated Lyapunov equation is thus:

$$-(\mathscr{L} + \Gamma)P - P(\mathscr{L} + \Gamma)^\top = -I_n. \tag{21}$$

The effect of noise on opinions in (20) are characterized by

$$||Y||_2^2 = \text{Trace}(E^\top PE) = \text{Trace}(\mathscr{L}P). \tag{22}$$

We proceed to solve (22) numerically. If there are multiple self-loops, the locations of these self-loops also affect the performance against noise. This problem is equivalent to finding a $0,1$ diagonal matrix Γ with a given number of entries being 1 so as to minimize $\|Y\|_2$. Note that the matrix $\mathscr{L} + \Gamma$ will be singular if $\Gamma = 0$. We use an edge perspective, as proposed for general digraphs in [26], to evaluate the

H_2 performance of the simple graph so as to compare it with the case when one or more confident agents, modelled by self-loops, are present.The performance with and without any self-loop is tabulated in the Table I for a directed cycle graph with the number of nodes being chosen as $n = 7$, 10, 51, 100, 501, 1000. One may observe that without any self-loop, $\|Y\|_2 = n-1$ in each case.

(n)	7	10	51	100	501	1000
$\lvert\mathscr{V}_s\rvert = 1$	5.725	8.587	48.366	96.452	493.503	989.559
$\lvert\mathscr{V}_s\rvert = 0$	6	9	50	99	500	999

TABLE I: $\|Y\|_2$ without and with one positive self-loop agent

From Table (I), it follows that the addition of a self-loop mitigates the effect of noise.

Conjecture 1. *Interactive dynamics within a group of agents in cyclic pursuit is more resilient to noise when at least one self-loop is present, compared with the situation when no self-loop is present.*

Similarly, the effect on the minimum and maximum values of $\|Y\|_2$, while two self-loops are placed at different locations in the directed cycle are tabulated in Table II.

(n)	7	10	51	100	501	1000
Min.	5.154	7.914	47.081	94.780	490.185	985.020
Max.	5.227	8.036	47.564	95.517	492.016	987.665

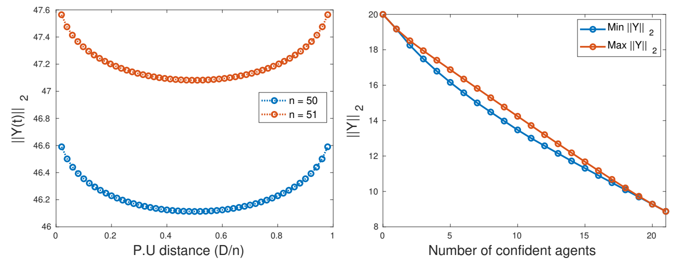
TABLE II: Range of $\|Y\|_2$ with two positive self-loops

From Table (II), we notice that addition of a 2^{nd} self-loop ensures a further reduction of $\|Y\|_2$ when compared with the case of a single self-loop, leading to another conjecture.

Conjecture 2. *The more the number of self-loops in a group of agents following cyclic pursuit, the more resilient is the overall dynamics to noise.*

Fig. 3a shows the variation of $\|Y\|_2$ norm with the per unit distance $\frac{D}{n}$, where D is the Edge distance between self loops in the directed cycle. Cycle digraphs with 51 and 52 nodes are considered for the simulation. We note that the best performance is achieved at $D/n = 0.5$, implying the two self loops are at a maximum distance along the graph. Performance thus improves with an increase in distance between the confident agents. Furthermore, while the number of nodes is odd, i.e. in the case of 51, one of the stubborn agents fixed at node 1, the second one at 25 or 26 will have the same minimum $\|Y\|_2$. With $n = 50$, the minimum $\|Y\|_2$ was achieved when the second self-loop was on node 26 (distance between self-loops was 25). This trend continued in general across several simulations.

Next, we considered a directed cycle over $n = 21$ nodes and varied the number of self-loops. For each different choice of a number of self-loops, we calculated the maximum and minimum values of $\|Y\|_2$ across different placements of self-loops. Fig 3b reveals that for a given number of self-loops, a minimization of $\|Y\|_2$ resulted from placing the self-loops in an evenly spaced fashion over the cycle while the



(a) (b)

Fig. 3: Performance against noise a) with varying distance between self-loops and b) varying number of positive self-loops.

'worst'/maximum value of $\|Y\|_2$ resulted from placing self-loops adjacent to one another.

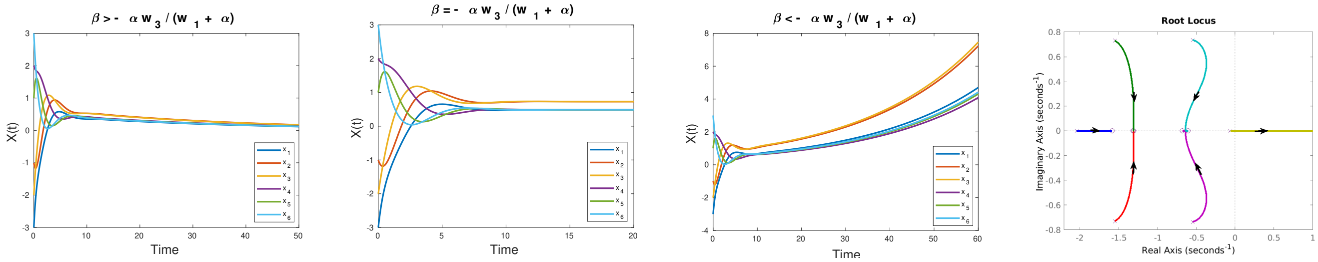
Conjecture 3. *For a group of n agents in cyclic pursuit, for a fixed number of self-loops, the maximum resilience of the collective dynamics against noise results from evenly distributing the self-loops across the directed cycle..*

VI. Simulations

A directed cycle graph on 6 nodes and edge weights $w_1 = 1$, $w_2 = 1.2$, $w_3 = 0.8$, $w_4 = 0.7$, $w_5 = 1.5$, $w_6 = 0.6$ is considered with initial conditions $X(0) = [-3, -1, -2, 2, 1, 3]^T$. Fig.4 shows the responses with different values of negative self-loop edge-weight on node 3, (say β) while the weight on the self-loop at position one $\alpha = 0.5$. As our calculations indicate, the system is on the brink of instability at the critical value of self-loop weight equal to -0.2667. On the other hand, Fig.5 depicts the situation with a negative edge weight at node 3, given by $\beta = -0.5$, and a self-confident agent 1 with self-loop weight $\alpha > 0$. The simulations shows that the dynamics are stable only if $\alpha > 1.667$, as expected from our analyses. Moreover, the complementary root locus for the variation in $\beta < 0$ with $\alpha = 0.5$, and the root locus for varying $\alpha > 0$ with $\beta = -0.5$ validate these bounds.

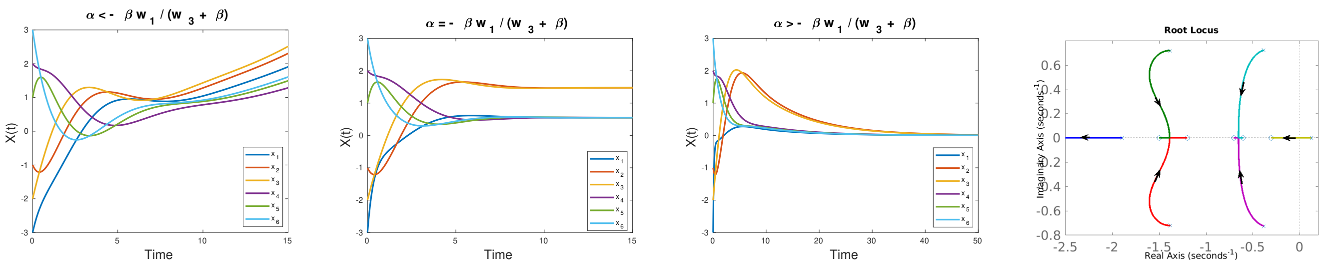
VII. Conclusions

In this paper, a cyclic pursuit problem over a non-simple directed cycle, containing self-loops, was considered. The effects of positive and negative edge weights on these self loops were investigated with respect to the stability of the resulting multi-agent system dynamics. The negative self-loops captured models of unstable agents, whereas the positive self-loops modelled a stable agent. We presented conditions for stability of the dynamics, depending on the values of self-loop weights and edge weights. As an application to opinion dynamics, in situations similar to a telephone game, it has also been shown that Taylor's opinion model can achieve consensus in the presence of a single stubborn agent. Finally, a numerical study has been presented to investigate the effectiveness of self-loops and their locations in the directed cycle with all the weights being unity, in mitigating noise.



a) $\beta = -0.15$ b) $\beta = -0.2667$ c) $\beta = -0.4$ d) complementary root locus vs β

Fig. 4: Dynamics with $\alpha = 0.5$ and different choices of β



a) $\alpha = 0.1$ b) $\alpha = 1.667$ c) $\alpha = 6$ d) root locus vs α

Fig. 5: Dynamics with $\beta = -0.5$ and different choices of α

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