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Uniform Ultimate Boundedness Analysis for Linear Systems with Asymmetric Input Backlash and Dead-zone: A Piecewise Quadratic Lyapunov Function Approach

A. Pierron[1,2] J. Kreiss[1] M. Jungers[1] G. Millérioux[1] J. Dupont[2] M. Martig[2]

Abstract— **This paper deals with the interconnection between a linear system and a nonlinear operator consisting of asymmetric input backlash and asymmetric dead-zone. The uniform ultimate boundedness of the system is studied. A piecewise quadratic Lyapunov function, suitable with the polyhedral description of the nonlinear operator is proposed. The conservatism of existing results is therefore reduced. The effectiveness and improvement of the results are assessed using a numerical example.**

Index Terms— **Backlash, dead-zone, piecewise quadratic Lyapunov function, uniform ultimate boundedness.**

I. Introduction

The system investigated in this paper is a linear plant whose input acts via a nonlinear operator exhibiting asymmetric backlash and asymmetric deadzone. The presence of a backlash operator (labelled also as a play operator [1]) is motivated by a large range of nonlinear system applications (see for instance [2]) and in particular when a cylinder is used. Its intrinsic presence in the well known Hysteresis phenomena and its relation to nonlinear memory effects (see Prandtl-Ishlinkskii models [3], [4] for instance) makes it essential in many applications. The deadzone is widespread in mechanics to model solid frictions for instance. These two types of nonlinearities are nevertheless rarely studied together, even if they can be coupled in real systems. A main difficulty is that the backlash operator is only piecewise differentiable. In addition and because of the memory effect, such a phenomena must be described by its time derivative, when it exists. Various solutions propose to compensate the nonlinear operator by an inversion as for instance in [4], [5]. However, this requires an exact knowledge of the nonlinearity.

In this paper, we propose a different approach by analysing the properties of the state trajectories. Due to the structure of the studied system which may fail to be stable, we are interested in providing guarantees of the Uniform Ultimately Boundedness (UUB) property, by following [6]. The analysis is performed thanks to a UUB-Lyapunov function which presents the following properties: (i) outside the unit level set (i.e. when its value is greater than 1), it decreases along the trajectory of the system and (ii) after reaching this level-set, the trajectory remains inside it. The aim, here, is to provide tractable sufficient conditions as parameterized Linear Matrix Inequalities (LMIs) and an optimization problem, as a semidefinite program, that minimizes the size of the unit level-set of this UUB-Lyapunov function.

In order to reduce the conservatism of considering only common quadratic UUB-Lyapunov function to ensure the UUB property [7], we propose here a piecewise quadratic Lyapunov function. This kind of Lyapunov functions has been introduced in the middle of the 90's to deal with piecewise affine systems, which are defined as affine dynamics over regions corresponding to a polyhedral partition of the state-space (see for instance [8] and references therein). Such a tool has been used successfully for specific hybrid systems [9] or systems with saturation or deadzone [10], [11], and even more complex nonlinearities [12]. To the best of our knowledge, such a tool has not been considered for complex nonlinear operator with backlash, as the one investigated here.

The paper is organized as follows. Section II introduces the problem by describing the nonlinear operator and recalling the definition of Uniform Ultimate Boundedness. Section III details the preliminaries dealing with the structure of the piecewise quadratic UUB-Lyapunov functions and the characterization of the nonlinear operator. Section IV provides the main result, that are parameterized LMIs ensuring the UUB property. An optimization problem to minimize the resulting UUB-set is given. Section V demonstrates on an example the effectiveness of our method and the providing improvement with respect to the literature. We finally conclude in Section VI.

Notation: I_n stands for the identity matrix of dimension n and $0_{n\times m}$ stands for the null matrix of dimensions $n\times m$, with $0_n = 0_{n\times n}$. For a matrix $M \in \mathbb{R}^{n\times m}$, $M^\top$ denotes its transpose. For a square matrix $M \in \mathbb{R}^{n\times n}$, $\mathrm{He}(M) = M + M^\top$ and $M \succ 0_n$ ($M \succeq 0_n$) means that M is positive (semi-)definite. For two symmetric matrices M_1, M_2 of same dimension, $M_1 \succ M_2$ ($M_1 \succeq M_2$) means $M_1 - M_2 \succ 0$ ($M_1 - M_2 \succeq 0$). The same symbol is used for vectors as follows: for a vector $z \in \mathbb{R}^n$, $z \succeq 0$ means that all entries of the vector z are non-negative. For a matrix M, $M_{(j)}$ and $M_{(i,j)}$ denote respectively its j-th row and its (i,j)-th entry. The symbol $\star$ denotes a symmetric block in symmetric matrices. For square matrices W and Z, $\mathrm{diag}(W,Z)$ corresponds to the block-diagonal matrix.

[1] Aurélien Pierron, Jérémie Kreiss, Marc Jungers and Gilles Millérioux are with Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France. {jeremie.kreiss, marc.jungers, gilles.millerioux}@univ-lorraine.fr

[2] Aurélien Pierron, Jeremy Dupont and Martial Martig are with SPIE Industrie, F-57000 Metz, France. {aurelien.pierron, jeremy.dupont, m.martig}@spie.com

II. PROBLEM FORMULATION

A. System description

We consider in this paper the continuous-time system:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B\Phi[f](t), \\ f(t) & = & Kx(t), \end{array} \tag{1}$$

where we denote the state $x(t) \in \mathbb{R}^n$, with initial state $x(0) = x_0 \in \mathbb{R}^n$, and the input $f(t) \in \mathbb{R}$ which is scalar for the sake of simplicity. However, the same developments may be applied to extend the study to the multiple input case. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $K \in \mathbb{R}^{1\times n}$ are constant and given. The nonlinear operator Φ gathers asymmetric backlash and asymmetric dead-zone. It is the main source of difficulty and its definition requires a special attention.

The nonlinear operator $\Phi[f](\cdot)$ exhibits an hysteresis memory and is defined for continuous and piecewise differentiable function $f \in \mathcal{C}^1_{\text{pw}}([0,+\infty);\mathbb{R})$. If the unbounded sequence of times $\{t_j\}_{j\in\mathbb{N}}$ is such that f is differentiable over $]t_j, t_{j+1}[$, then we have the evolution of the nonlinearity output given for all time $t \in]t_j, t_{j+1}[$, $\forall j \in \mathbb{N}$:

$$\dot{\overbrace{\Phi[f](t)}} = \begin{cases} l_a\dot{f}(t) \quad \text{if } \Phi[f](t) \geq 0 \text{ and } \Big(\big(\dot{f} \geq 0 \text{ and} \\ \quad \Phi[f](t) = l_a(f(t) - \rho_a - h)\big) \text{ or} \\ \quad \big(\dot{f}(t) \leq 0 \text{ and } \Phi[f](t) = l_a(f(t) - \rho_a)\big)\Big) \\ l_b\dot{f}(t) \quad \text{if } \Phi[f](t) \leq 0 \text{ and } \Big(\big(\dot{f} \leq 0 \text{ and} \\ \quad \Phi[f](t) = l_b(f(t) + \rho_b + h)\big) \text{ or} \\ \quad \big(\dot{f}(t) \geq 0 \text{ and } \Phi[f](t) = l_b(f(t) + \rho_b)\big)\Big) \\ 0 \quad \text{otherwise.} \end{cases} \tag{2}$$

The parameters of the nonlinearity Φ are $h, l_a, l_b, \rho_a, \rho_b \in \mathbb{R}_{>0}$ and are respectively called the backlash width, the inclination and the threshold when f is positive and negative. The characteristic of the nonlinear operator Φ is depicted on Figure 1.

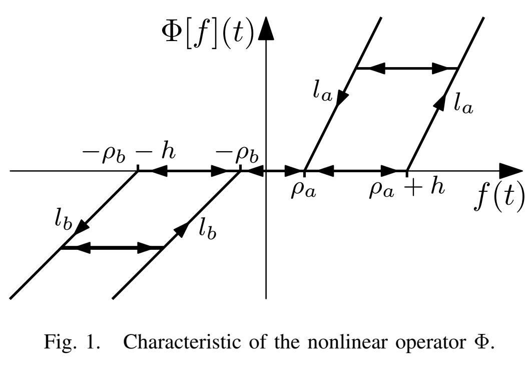

Fig. 1. Characteristic of the nonlinear operator Φ.

The nonlinear operator having affine branchs with respect to $f(t) = Kx(t)$, we introduce the next assumption.

Assumption 1: The triplet (A, B, K) is such that the matrices $(A - Bl_kK)$, $k \in \{a, b\}$, are Hurwitz. □

Assumption 1 allows to cope with possibly non-Hurwitz matrix A and will be a necessary condition for the feasability of the LMIs in the main result (see Theorem 1 and also [6]).

B. Activation of the non-linearity

The function $t \mapsto \Phi[f](t)$ is continuous. The evolution of the output $\Phi[f](t)$ is well defined by relation (2). Nevertheless, the initial condition $\Phi[f](0)$ is not uniquely defined by the knowledge of the function $f \in \mathcal{C}^1_{\text{pw}}([0,+\infty);\mathbb{R})$ but belongs to an interval depending on the initial value $f(0)$:

$$\Phi[f](0) \in \begin{cases} [l_b(f(0)+\rho_b); \min(0, l_b(f(0)+\rho_b+h)], \\ \qquad \text{if } f(0) \leq -\rho_b; \\ \{0\}, \quad \text{if } -\rho_b \leq f(0) \leq \rho_a; \\ [\max(0, l_a(f(0)-\rho_a-h)); l_a(f(0)-\rho_a)], \\ \qquad \text{if } \rho_a \leq f(0). \end{cases} \tag{3}$$

It can be interpreted as a memory effect.

The following sets Σ_i, $i \in \mathcal{I} = \{1, \cdots, 4\}$, are introduced for the formal developments and proofs given in the sequel.

$$\begin{aligned} \Sigma_1 = &\{(x,\phi) \in \mathbb{R}^{n+1}, \phi \geq 0, \\ &\quad \phi + l_a\rho_a \leq l_aKx \leq \phi + l_a(\rho_a + h)\}, \\ \Sigma_2 = &\{(x,\phi) \in \mathbb{R}^{n+1}, 0 \leq Kx \leq \rho_a,\ \phi = 0\}, \\ \Sigma_3 = &\{(x,\phi) \in \mathbb{R}^{n+1}, -\rho_b \leq Kx \leq 0,\ \phi = 0\}, \\ \Sigma_4 = &\{(x,\phi) \in \mathbb{R}^{n+1}, \phi \leq 0, \\ &\quad \phi - l_b(\rho_b + h) \leq l_bKx \leq \phi - l_b\rho_b\}, \\ \Sigma = &\cup_{i\in\mathcal{I}}\Sigma_i. \end{aligned}$$

We have the following lemma, whose proof is developed in [7] and inspired by [13].

Lemma 1: For any function $f \in \mathcal{C}^1_{\text{pw}}([0,+\infty);\mathbb{R})$, let consider an initial condition for the output $\Phi[f](0)$ such that $(f(0), \Phi[f](0)) \in \Sigma$. Then, the function $t \mapsto \Phi[f](t)$ is uniquely defined and $(f(t), \Phi[f](t)) \in \Sigma$, $\forall t \in \mathbb{R}_{\geq 0}$.

Hereafter, we will consider the following assumption.

Assumption 2: The initial condition $\Phi[Kx](0)$ is chosen such that $(Kx(0), \Phi[Kx](0)) \in \Sigma$. □

C. Uniform Ultimate Boundedness property

The major concern of this study is in the neighborhood of the origin. Indeed, because of the dead zone behavior of Φ around zero, one can easily see that $\Phi[Kx](t) = 0$ and System (1) is in open-loop. The open-loop system might present unstability (since no assumptions are made on matrix A) for the origin, which prevents us to study stability of System (1). Moreover, (1) may exhibit infinite number of equilibrium points (this is the case when zero is an eigenvalue of A). For all these reasons, we are interested in a weaker notion than stability which is known as Uniform Ultimate Boundedness (UUB) for system (1), whose definition is as follows.

Definition 1 (UUB): [14, Definition 4.6] The trajectory of system (1) is *uniformly ultimately bounded* with ultimate bound b if there exist positive constants b and c, independent of $t_0 > 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that $\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b$, $\forall t \geq t_0 + T$. A set of the state-space implying that $\|x\| \leq b$ is called a UUB-set. □

Based on Definition 1, we study the following problem.

Problem 1 (Uniform Ultimate Boundedness analysis): Given the system (1) and a gain K such that Assumptions 1 and 2 hold, determine the UUB-set of the system (1), as small as possible and associated with the UUB property. □

III. PROBLEM SETTING AND PRELIMINARIES

Our approach to solve Problem 1 is based on the idea presented in [14, Theorem 4.18] using a Lyapunov-like function (called in the following UUB-Lyapunov function) that decreases along the trajectory outside its unit level set and remains less than 1 inside.

Since the system is defined over a polyhedral partition (due to the non-linear operator), and in order to fit as much as possible the UUB-set, we will consider piecewise Lyapunov functions over this partition. This is the main improvement with respect to the result in [7]. In this case, we need to ensure continuous property of the overall UUB-Lyapunov function on the borders of each region. In the following, we will ensure, over each intersection between a region of the partition and the complementary of the unit level set of the UUB-Lyapunov function, that the regional Lyapunov function is decreasing. The associated conditions will involve the state x, the nonlinearity Φ and its time-derivative $\dot{\Phi}$.

Therefore, we are looking in this paper at *continuous piecewise quadratic UUB-Lyapunov function*. This section will provide the tools and notations, before presenting the main result in Section IV. Subsection III-A treats the structure of continuous piecewise quadratic UUB-Lyapunov function. Subsection III-B offers a background for the nonlinear operator.

A. Structure of a continuous piecewise quadratic UUB-Lyapunov function

From the definitions of Σ_i, $i \in \mathcal{I}$, we can divide the state space $\mathbb{R}^n$ in four regions bounded by the three parallel hyperplans $f(t) = Kx(t) \in \{-\rho_b; 0; \rho_a\}$. Figure 1 shows the relevance of such a decomposition. In order to highlight the polyhedral partition, we introduce the notation $\bar{x} = \begin{pmatrix} x^\top & 1 \end{pmatrix}^\top \in \mathbb{R}^{n+1}$ and the following polyhedral partition (see [8, Definition A.8]) of the state-space $\mathbb{R}^n$ related to the definition of sets Σ_i, $i \in \mathcal{I}$. Let us define $\mathcal{X}_i = \{x \in \mathbb{R}^n,\ X_i\bar{x} \succeq 0\}$, $i \in \mathcal{I}$, with $X_i \in \mathbb{R}^{n_i \times (n+1)}$,

$$X_1 = \begin{bmatrix} K & -\rho_a \end{bmatrix}, \qquad X_4 = \begin{bmatrix} -K & -\rho_b \end{bmatrix},$$
$$X_2 = \begin{bmatrix} -K & \rho_a \\ K & 0 \end{bmatrix}, \qquad X_3 = \begin{bmatrix} K & \rho_b \\ -K & 0 \end{bmatrix}.$$

We can check that $\cup_{i\in\mathcal{I}}\mathcal{X}_i = \mathbb{R}^n$ and that only the intersections between consecutive sets are not empty, such that only $\mathcal{X}_i \cap \mathcal{X}_{i+1} \neq \emptyset$, $i \in \{1, 2, 3\}$. In addition, the sets $\mathcal{X}_i$ are closed and the intersections of their interiors are empty. The origin belongs only to $\mathcal{X}_2$ and $\mathcal{X}_3$. We introduce $\mathcal{I}_0 = \{2, 3\} \subset \mathcal{I}$, the set of indices i such that $0 \in \mathcal{X}_i$. The projection of Σ_i on its n-first components belongs to $\mathcal{X}_i$, $i \in \mathcal{I}$.

By following the method detailed in [8, Appendix A], we can build the continuity matrices $\bar{F}_i \in \mathbb{R}^{(n+3)\times(n+1)}$, $i \in \mathcal{I}$, based on the three halfplanes $\{x \in \mathbb{R}^n,\ Kx - \rho_a \succeq 0\}$, $\{x \in \mathbb{R}^n,\ Kx + 0 \succeq 0\}$ and $\{x \in \mathbb{R}^n,\ -Kx - \rho_b \succeq 0\}$ as

$$\bar{F}_1 = \begin{bmatrix} K & -\rho_a \\ K & 0 \\ 0_{1\times n} & 0 \\ I_n & 0_{n\times 1} \end{bmatrix}, \quad \bar{F}_2 = \begin{bmatrix} 0_{1\times n} & 0 \\ K & 0 \\ 0_{1\times n} & 0 \\ I_n & 0_{n\times 1} \end{bmatrix},$$
$$\bar{F}_3 = \begin{bmatrix} 0_{1\times n} & 0 \\ 0_{1\times n} & 0 \\ 0_{1\times n} & 0 \\ I_n & 0_{n\times 1} \end{bmatrix}, \quad \bar{F}_4 = \begin{bmatrix} 0_{1\times n} & 0 \\ 0_{1\times n} & 0 \\ -K & -\rho_b \\ I_n & 0_{n\times 1} \end{bmatrix}.$$

By construction, the origin belonging in $\mathcal{X}_2$ and $\mathcal{X}_3$, the last column of $\bar{F}_2$ and $\bar{F}_3$ are trivial. This choice allows to impose

$$\bar{F}_i\bar{x} = \bar{F}_{i+1}\bar{x},\ \forall x \in \mathcal{X}_i \cap \mathcal{X}_{i+1},\ i \in \{1, 2, 3\}. \tag{4}$$

The last block of rows in $\bar{F}_i$, i.e. $\begin{bmatrix} I_n & 0_{n\times 1} \end{bmatrix}$, translates the continuity of the state in $\mathbb{R}^n$.

This polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ of $\mathbb{R}^n$, compatible with the sets Σ_i, $i \in \mathcal{I}$, depends only on the state x. For this reason, we focus on regional UUB-Lyapunov functions that are quadratic forms with respect to the state. As a consequence, we can look for a UUB-Lyapunov function in the class of continuous piecewise quadratic functions, defined by

$$V(x) = V_i(x), \quad x \in \mathcal{X}_i, \tag{5}$$

where the regional quadratic functions $V_i(x)$ are given by

$$V_i(x) = \begin{cases} \bar{x}^\top \bar{P}_i \bar{x} = \bar{x}^\top \bar{F}_i^\top T \bar{F}_i \bar{x},\ \forall i \in \mathcal{I}/\mathcal{I}_0, \\ \bar{x}^\top \bar{P}_i \bar{x} = x^\top P_i x = x^\top F_i^\top T F_i x,\ \forall i \in \mathcal{I}_0, \end{cases} \tag{6}$$

with $T = T^\top \in \mathbb{R}^{(n+3)\times(n+3)}$, a weighting matrix and where for $i \in \mathcal{I}_0$, we denote $F_i \in \mathbb{R}^{(n+3)\times n}$, the extraction of the first n columns of $\bar{F}_i$.

Lemma 2 ([9]): The function $V(\cdot)$ defined by (5) and (6) is continuous over $\mathbb{R}^n$, irrespectively to the matrix T. □

Proof: The regional quadratic function (6) are continuous. To prove the continuity property, we have to check the continuity of V on the non-empty intersections of the regions $\mathcal{X}_i$. Thanks to the relation (4), we have $\forall x \in \mathcal{X}_i \cap \mathcal{X}_{i+1}$, $i \in \{1, \cdots, 3\}$

$$V_i(x) = \bar{x}^\top \bar{F}_i^\top T \bar{F}_i \bar{x} = \bar{x}^\top \bar{F}_{i+1}^\top T \bar{F}_{i+1} \bar{x} = V_{i+1}(x).$$

The piecewise quadratic function V is thus continuous over $\mathbb{R}^n$, without constraint about matrix T. ■

In order to obtain other required properties of V via Lemmas 3 and 4, let us describe the building of the polyhedral cell bounding related to the polyhedral partition of $\mathbb{R}^n$ [8, Algorithm A.1]:

- If $i \in \mathcal{I}_0$, $\bar{E}_i$ is obtained by deleting all rows of X_i whose last entry is non-zero. E_i is then obtained by extracting the n first columns of the resulting $\bar{E}_i$.
- If $i \in \mathcal{I} \setminus \mathcal{I}_0$, $\mathcal{X}_i$ is unbounded and $\bar{E}_i$ is obtained by augmenting X_i with the row $\begin{bmatrix} 0_{1\times n} & 1 \end{bmatrix}$.

This procedure leads to

$$\bar{E}_1 = \begin{bmatrix} K & -\rho_a \\ 0_{1\times n} & 1 \end{bmatrix},\ \bar{E}_4 = \begin{bmatrix} -K & -\rho_b \\ 0_{1\times n} & 1 \end{bmatrix},\ E_2 = -E_3 = K,$$

and ensures the implications

$$\forall i \in \mathcal{I}_0, \qquad x \in \mathcal{X}_i \Rightarrow E_i x \succeq 0, \tag{7}$$

$$\forall i \in \mathcal{I} \setminus \mathcal{I}_0, \qquad x \in \mathcal{X}_i \Rightarrow \bar{E}_i \bar{x} \succeq 0. \tag{8}$$

Lemma 3: Assume that V obeys Equations (6), then there exists a scalar $\beta > 0$ such that $V(x) \leq \beta \|x\|^2$, $\forall x \in \mathbb{R}^n$. □

Proof: The idea of the proof can be found in the appendix of [9]. The main argument is that in an open neighborhood of the origin, V is piecewise quadratic with respect to the state x (instead of $\bar{x}$). For $i \in \mathcal{I}_0$, there exists $\beta_i > 0$ such that $V_i(x) \leq \beta_i \|x\|^2$. By construction, there exists $\epsilon > 0$ small enough such that $\|x\|^2 \leq \epsilon$ implies that $x \in \cup_{i \in \mathcal{I}_0} \mathcal{X}_i$. Contraposing this statement implies, $x \in \cup_{i \in \mathcal{I} \setminus \mathcal{I}_0}$ induces $\|x\|^2/\epsilon \geq 1$. For each $i \in \mathcal{I} \setminus \mathcal{I}_0$, there exists $\tilde{\beta}_i > 0$ such that $V_i(x) \leq \tilde{\beta}_i \|\bar{x}\|^2 \leq \beta_i \|x\|^2$, with $\beta_i = \tilde{\beta}_i(1 + 1/\epsilon)$. Selecting $\beta = \max_{i \in \mathcal{I}} \beta_i$ ends the proof. ■

Lemma 4: If there exist a symmetric matrix $T \in \mathbb{R}^{(n+3)\times(n+3)}$ and matrices U_i, $i \in \mathcal{I}$ of adequate dimensions with nonnegative entries, such that the LMIs

$$F_i^\top T F_i - E_i^\top U_i E_i \succ 0_n, \; i \in \mathcal{I}_0 \tag{9}$$

$$\bar{F}_i^\top T \bar{F}_i - \bar{E}_i^\top U_i \bar{E}_i \succ 0_{n+1}, \; i \in \mathcal{I} \setminus \mathcal{I}_0, \tag{10}$$

hold, then there exists $\alpha > 0$ such that $V(x) \geq \alpha \|x\|^2$. □

Proof: The strict inequalities (9) allow the existence of $\alpha_i > 0$ small enough such that 0_n can be replaced in (9) by $\alpha_i I_n$. Multiplying left and right the resulting inequalities by $x^\top$ and x respectively leads to $i \in \mathcal{I}_0$: $V_i(x) - \alpha_i \|x\|^2 \geq \sum_{j,k} U_{i,(j,k)} (E_{i,(j)} x)(E_{i,(k)} x) \geq 0$. The latter inequality comes from implication (7). We proceed in the same way for inequalities (10) by replacing 0_{n+1} by $\mathrm{diag}(\alpha_i I_n, 0)$ and multiplying the result by $\bar{x}^\top$ and $\bar{x}$, $i \in \mathcal{I} \setminus \mathcal{I}_0$: $V_i(x) - \alpha_i \|x\|^2 \geq \sum_{j,k} U_{i,(j,k)} (\bar{E}_{i,(j)} \bar{x})(\bar{E}_{i,(k)} \bar{x}) \geq 0$. The latter inequality is a consequence of implication (8). Selecting $\alpha = \min_{i \in \mathcal{I}} \alpha_i$ completes the proof. ■

Due to Lemmas 2, 3 and 4, V can act as a candidate continuous UUB-Lyapunov function.

B. Characteristics of the nonlinear operator

This subsection aims at providing sufficient conditions ensuring that the UUB-Lyapunov function V decreases outside its unit level set, (that is on $\{x \in \mathbb{R}^n, \; V(x) \geq 1\}$). To use Assumption 1, we consider an adaptation of the dual nonlinear operator Ψ related to Φ, (see [6] and [7]) by:

$$\Psi[Kx](t) = \Phi[Kx](t) - l_i K x(t), \; \forall x \in \mathcal{X}_i, \tag{11}$$

with $l_1 = l_2 = l_a$ and $l_3 = l_4 = l_b$. The dual operator Ψ benefits from being bounded, in particular when $x \in \Sigma_1 \cup \Sigma_4$.

In order to express compactly the dynamics and the quadratic sector conditions related to Φ, we consider the augmented vectors $y, z \in \mathbb{R}^{n+3}$ given by $y = \begin{pmatrix} x^\top & \Phi & \dot{\Phi} & 1 \end{pmatrix}^\top$ and $z = \begin{pmatrix} x^\top & \Psi & \dot{\Psi} & 1 \end{pmatrix}^\top$, that are linked to one another by $y = N_i z$ over $x \in \mathcal{X}_i$, where

$$N_i = \begin{pmatrix} I_n & 0 & 0 & 0 \\ l_i K & 1 & 0 & 0 \\ l_i K(A + B l_i K) & l_i K B & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; i \in \mathcal{I}.$$

In addition, we have $\bar{x} = N_0 z$, with

$$N_0 = \begin{pmatrix} I_n & 0_{n\times 1} & 0_{n\times 1} & 0_{n\times 1} \\ 0_{1\times n} & 0 & 0 & 1 \end{pmatrix}.$$

It is thus possible to rewrite Dynamics (1) as $\dot{\bar{x}}(t) = \Gamma_i z(t)$, $x \in \mathcal{X}_i$, where, $i \in \mathcal{I}$,

$$\Gamma_i = \begin{pmatrix} A + B l_i K & B & 0_{n\times 1} & 0_{n\times 1} \\ 0_{1\times n} & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+3)}.$$

Let us now focus on the characteristics of the nonlinear operator. We can distinguish two kinds of conditions:

- Several conditions can be expressed as polyhedral constraints related to x, Ψ and also $\dot{\Psi}$. They correspond to a reformulation of the sets $\Sigma_i \in \mathbb{R}^{n+1}$, $i \in \mathcal{I}$, and furthermore to the fact that in Σ_2 and Σ_3, we have $\dot{\Phi} = 0$. It results to the polyhedral constraints $\{z \in \mathbb{R}^{n+3}, \; \bar{G}_i z \succeq 0\}$, with

$$\bar{G}_1 = \begin{bmatrix} l_1 K & 1 & 0 & 0 \\ 0_{1\times n} & -1 & 0 & -l_1 \rho_a \\ 0_{1\times n} & 1 & 0 & l_1(\rho_a + h) \end{bmatrix},$$

$$\bar{G}_4 = \begin{bmatrix} -l_4 K & -1 & 0 & 0 \\ 0_{1\times n} & -1 & 0 & l_4(\rho_b + h) \\ 0_{1\times n} & 1 & 0 & -l_4 \rho_b \end{bmatrix},$$

and for $i \in \mathcal{I}_0$,

$$\bar{G}_i = \begin{bmatrix} K & 0 & 0 & (i-2)\rho_b \\ -K & 0 & 0 & (3-i)\rho_a \\ l_i K & 1 & 0 & 0 \\ -l_i K & -1 & 0 & 0 \\ l_i K(A + B l_i K) & l_i K B & 1 & 0 \\ -l_i K(A + B l_i K) & -l_i K B & -1 & 0 \end{bmatrix}.$$

- Equation (2) allows sector conditions in Σ_1 and Σ_4. They are gathered in Lemma 5 which is proven in [7].

Lemma 5: In Σ_1, we have the following quadratic sector conditions, for any $\alpha_1 > 1$:

$$\dot{\Phi}\left(\Psi + l_1\left(\rho_a + h/2\right)\right) \leq 0, \quad \dot{\Phi}\left(\dot{\Phi} - \alpha_1 l_1 K \dot{x}\right) \leq 0, \tag{12}$$

which imply that the quadratic forms in z satisfy respectively $z^\top M_{1,1} z \leq 0$ and $z^\top M_{2,1} z \leq 0$, with

$$M_{1,1} = N_1^\top \begin{pmatrix} 0 & 0 & -K^\top l_1 & 0 \\ \star & 0 & 1 & 0 \\ \star & \star & 0 & l_1\left(\rho_a + \frac{h}{2}\right) \\ \star & \star & \star & 0 \end{pmatrix} N_1,$$

$$M_{2,1} = N_1^\top \begin{pmatrix} 0 & 0 & -\left(\alpha_1 l_1 K A\right)^\top & 0 \\ \star & 0 & -\left(\alpha_1 l_1 K B\right)^\top & 0 \\ \star & \star & 2 & 0 \\ \star & \star & \star & 0 \end{pmatrix} N_1.$$

In Σ_4, we have , for any $\alpha_4 > 1$:

$$\dot{\Phi}\left(\Psi - l_4\left(\rho_b + h/2\right)\right) \leq 0, \quad \dot{\Phi}\left(\dot{\Phi} - \alpha_4 l_4 K \dot{x}\right) \leq 0, \tag{13}$$

which imply that the quadratic forms in z satisfy respectively $z^\top M_{1,4} z \leq 0$ and $z^\top M_{2,4} z \leq 0$, with

$$M_{1,4} = N_4^\top \begin{pmatrix} 0 & 0 & -K^\top l_2 & 0 \\ \star & 0 & 1 & 0 \\ \star & \star & 0 & -l_2\left(\rho_b + \frac{h}{2}\right) \\ \star & \star & \star & 0 \end{pmatrix} N_4,$$

$$M_{2,4} = N_4^\top \begin{pmatrix} 0 & 0 & -\left(\alpha_4 l_4 K A\right)^\top & 0 \\ \star & 0 & -\left(\alpha_4 l_4 K B\right)^\top & 0 \\ \star & \star & 2 & 0 \\ \star & \star & \star & 0 \end{pmatrix} N_4. \qquad \square$$

IV. Main result

This section provides sufficient conditions, expressed in terms of parameterized LMIs, and gathered in Theorem 1. An optimization problem is presented to select a solution minimizing the size of the UUB set.

Theorem 1: Consider the system (1) that verifies Assumption 2. Assume that there exist a symmetric matrix $T \in \mathbb{R}^{(n+3)\times(n+3)}$, matrices U_i and W_i, $i \in \mathcal{I}$ with suitable dimensions and nonnegative components, positive scalars $\tau_{0,i}$, $\tau_{1,1}$, $\tau_{2,1}$, $\tau_{1,4}$, $\tau_{2,4}$, such that Inequalities (9), (10) and

$$\Pi_i + \bar{G}_i^\top W_i \bar{G}_i \prec 0, i \in \mathcal{I}_0, \quad (14)$$

$$\Pi_i - \tau_{1,i} M_{1,i} - \tau_{2,i} M_{2,i} + \bar{G}_i^\top W_i \bar{G}_i \prec 0, i \in \mathcal{I} \setminus \mathcal{I}_0, (15)$$

where, $i \in \mathcal{I}$, $\Pi_i = -\tau_{0,i}\left(\mathrm{diag}(0_n, 0, 0, 1) - N_0^\top \bar{P}_i N_0\right) + \mathrm{He}\left(\Gamma_i^\top \bar{P}_i N_0\right)$, hold. Then, for any choice of initial conditions $(Kx(0), \Phi[Kx](0))$ satisfying Assymption 1, the system (1) is UUB with the UUB-Lyapunov function V given by (5)–(6) and the UUB set $L_V = \{x \in \mathbb{R}^n, V(x) \leq 1\}$. □

Proof: Let us consider the function V having the structure detailed in (6). V is continuous over $\mathbb{R}^n$. The unit level set L_V is then closed. Thanks to (9) and (10), Lemmas 3 and 4 apply and thus V is a UUB-Lyapunov function candidate. Consider the structure of matrices Π_i, we have $\forall i \in \mathcal{I}$:

$$z^\top \Pi_i z = 2\dot{\bar{x}}^\top \bar{P}_i \bar{x} - \tau_{0,i}(1 - \bar{x}^\top \bar{P}_i \bar{x}) = \dot{V}(x) - \tau_{0,i}(1 - V(x)).$$

For $i \in \mathcal{I}_0$, Inequality (14) being strict, $\exists \delta_i > 0$ such that

$$\dot{V}(x) - \tau_{0,i}(1 - V(x)) + \delta_i \|x\|^2 \leq -\sum\nolimits_{j,k} W_{i,(j,k)}(\bar{G}_{i,(j)}z)(\bar{G}_{i,(k)}z),$$

leading to $\dot{V}(x) \leq -\frac{\delta_i}{\alpha} V(x)$ in $\mathcal{X}_i \setminus L_V$, $i \in \mathcal{I}_0$, due to the S-procedure. For $i \in \mathcal{I} \setminus \mathcal{I}_0$, the strict inequality (15) implies that there exists $\delta_i > 0$ such that

$$\dot{V}(x) - \tau_{0,i}(1 - V(x)) - \tau_{1,i} z^\top M_{1,i} z - \tau_{2,i} z^\top M_{2,i} z + \delta_i \|x\|^2 \leq -\sum\nolimits_{j,k} W_{i,(j,k)}(\bar{G}_{i,(j)}z)(\bar{G}_{i,(k)}z),$$

leading to $\dot{V}(x) \leq -\frac{\delta_i}{\alpha} V(x)$ in $\mathcal{X}_i \setminus L_V$, $i \in \mathcal{I} \setminus \mathcal{I}_0$. Finally with $\delta = \min_{i\in\mathcal{I}} \delta_i$, we have $\dot{V}(x) \leq -\frac{\delta}{\alpha} V(x)$, $\forall x \notin L_V$.

The latter inequality guarantees that, outside L_V, the function V decreases exponentially to reach (in finite time) the value 1 and that the set L_V is positively invariant by the dynamics (1) (see [14]). ■

Among all the solutions of Theorem 1, we would like to select the one minimizing the size of the level set L_V. The difficulty is to have a cost function of an optimization problem that is related to the size of a level set of a piecewise quadratic positive function. By following the discussion in [8, Section 7.1], the level set of a piecewise quadratic positive function consists in a collection of pieces of ellipsoids. We suggest to consider the average of the size of these ellipsoids as an approximation of the size of L_V. In other words, we would like to minimize Tr(Q), where $Q = Q^\top$ such that

$$\left[\begin{array}{c|cccc} Q & I_n & I_n & I_n & I_n \\ \hline \star & \multicolumn{4}{c}{\mathrm{diag}(P_1, P_2, P_3, P_4)} \end{array}\right] \succ 0, \quad (16)$$

which is equal, by a Schur complement, to $Q \succ \sum_{i\in\mathcal{I}} P_i^{-1}$. Let us provide the optimization problem that offers a solution to Problem 1.

$$\min_{T,Q,U_i,W_i,\tau_{0,i},\tau_{1,1},\tau_{2,1},\tau_{1,4},\tau_{2,4}} \mathrm{Tr}(Q), \quad \text{under (9), (10), (15), (14).}$$

Notice that the constraints in the optimization problem are LMIs if the parameters $\tau_{0,i}$ are given.

V. Numerical illustration

In order to illustrate our result and compare it with the available results in the literature, we consider the example coming from [7]. The linear system is a double integrator and the gain K satisfies Assumption 1.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ K = \begin{pmatrix} -2 & -3 \end{pmatrix}.$$

$$l_a = 1, \ l_b = 1.2, \ \rho_a = 0.1, \ \rho_b = 0.2, \ h = 0.2.$$

We impose $\tau_{0,i} = \tau_0$, $\forall i \in \mathcal{I}$, to have a single parameter in the line search. The optimal value of Tr(Q) depends on the parameter τ_0. This dependency is depicted in Figure 2. In order to numerically validate the fact that Tr(Q) approaches the area of L_V, we compute *a posteriori* the area of L_V after solving the optimization problem. We can see on Figure 2, that the two curves have the same shape. We can explain the curve as follows: for τ_0 tending to zero, our conditions are not feasible because A is not Hurwitz. In addition, τ_0 should be chosen such that $(A - Bl_k K + \frac{\tau_0}{2})$ is Hurwitz, $k \in \{a, b\}$, that is here $\tau_0 < 1.76$. We set below $\tau_0 = 0.6$, which is the argument of the minimum.

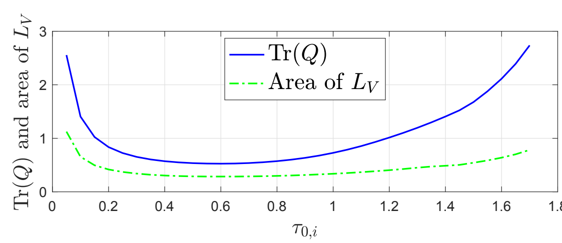

Fig. 2. Optimal value of Tr(Q) and area of L_V, depending on $\tau_{0,i}$.

We consider the initial condition $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$ and we choose $\Phi[Kx](0) = -2.5$, which verifies the constraint (3). We depict the time trajectory of the state $x(t)$ on Figure 3. It is shown that $V(t)$ decreases faster than linearly in the semilogy scape to reach 1 at $t = 2.7$s and after remains below 1. The value of the nonlinearity $\Phi[Kx](t)$ follows the charateristic of the nonlinear operator as depicted in Figure 4. Finally, we plot the state trajectory in the state space in Figure 5. It is clear that L_V (in red solid line) is a smaller set than the ellipsoid (in black solid line) generated by the conditions in [7] and consequently provides a less conservative solution to Problem 1. L_V is composed of pieces of ellipsoids in the sets $\mathcal{X}_i$, $i \in \mathcal{I}$ as expected.

For the initial condition $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$, the state-trajectory tends to the set L_V and converges to a limit cycle that lies into L_V. In Figure 5, two other trajectories are

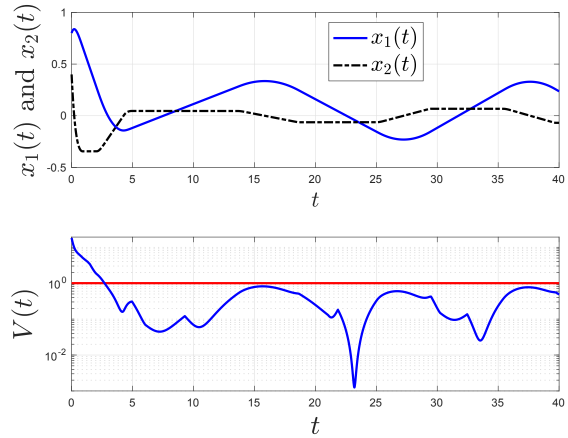

Fig. 3. Time-trajectory of the state $x(t)$ and the function $V(t)$.

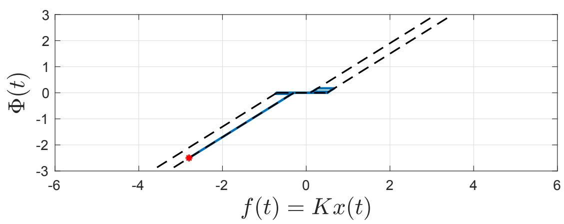

Fig. 4. Characteristic of the nonlinear operator. The starting point corresponds to $(Kx0, \Phi[Kx](0)) = (-2.8, -2.5)$, depicted with a red cross. The trajectory $(Kx(t), \Phi[Kx](t))$ is in blue. The characteristic of Figure 1 is recalled in back dashes.

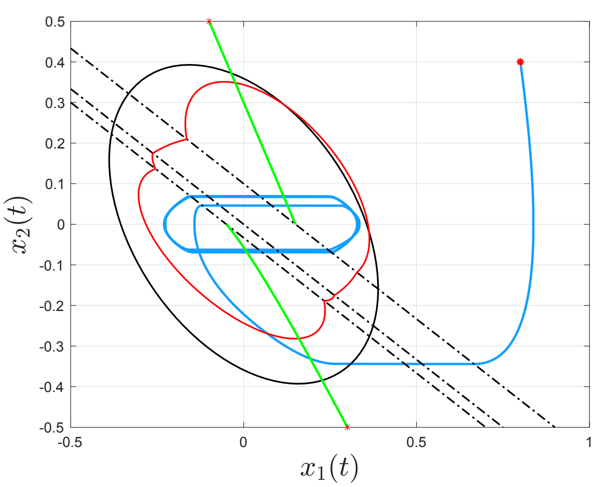

Fig. 5. Phase portrait of the system. The bounds of the polyhedral partition $\mathcal{X}_i$ are depicted with dashed-dot lines. The black ellipsoid is the UUB-set provided in [7] and the red curve depicted L_V from Theorem 1. Several trajectories are depicted starting from $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$ (in blue solid lines) and from $x_0 = \begin{pmatrix} 0.3 & -0.5 \end{pmatrix}^\top$ and $x_0 = \begin{pmatrix} -0.1 & 0.5 \end{pmatrix}^\top$ (in green solid lines).

depicted starting from $x_0 = \begin{pmatrix} 0.3 & -0.5 \end{pmatrix}^\top$ and $x_0 = \begin{pmatrix} -0.1 & 0.5 \end{pmatrix}^\top$. They respectively belong to $\mathcal{X}_4$ and $\mathcal{X}_1$ and reach (unstable) equilibrium points. Finally, we can emphasize that the trajectories may cross each other, due to the fact that Dynamics (1) does not depend only on the state, but also on the memory of the trajectory via Φ.

VI. Conclusion

A piecewise quadratic Lyapunov function approach has been used to cope with the uniform ultimately boundedness property of a system which is a linear plant that is feedback interconnected with a nonlinear operator exhibiting an asymmetric backlash and an asymmetric deadzone. The requirements of a Lyapunov function candidate are imposed thanks to linear matrix inequalities and the exponential uniform ultimately boundedness property is ensured thanks to parameterized linear matrix inequalities. The tractability of our sufficient conditions and the fact that our solution is less conservative than the results available in the literature are discussed on an academic illustration.

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