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Model Reduction by Moment Matching under Explicit Filters: A Swapped Interconnection Perspective

Junyu Mao and Giordano Scarciotti

Abstract— **In this work, we propose a characterization of the moments of a linear system for generalized filters that do not have an implicit representation, *i.e.*, they do not satisfy a differential equation. We devote particular attention to the case in which the filter has a non-smooth convolution kernel. The notion of moment is extended using an integral matrix equation. We present a family of reduced-order models that achieves moment matching based on this generalized notion of moment. Finally, the developed results are demonstrated by means of a numerical example with smooth and non-smooth kernels.**

Index Terms— Model reduction, moment matching, time-domain, swapped interconnection, linear systems

I. Introduction

In a wide range of modern engineering problems, where large-scale systems are under consideration, high-order mathematical models are commonly constructed for model-based analysis, control and prediction. However, in practice, the high dimensionality of these models poses considerable computational challenges, despite continuous advancements in computational power. To ease this computational burden, the field of *model order reduction* aims at reducing the complexity (*e.g.* dimensionality) of dynamical models. Informally, model reduction can be described as the problem of approximating important properties of a system (*i.e.* input-output behaviour) by a simplified description, *e.g.*, a lower-order model. For linear systems, model reduction techniques have been extensively studied for decades, see, *e.g.*, [1], [2]. A popular family of methods is based on the interpolation framework and Krylov projection theory, see, *e.g.*, [3], [4]. This class is also commonly referred to as *moment matching* methods since the resulting reduced-order models match the so-called "moments" of the original system at specific frequencies. The reader is referred to [5] for more detail about linear model order reduction methods and to the monograph [6] for a treatise on the interpolation theory.

This paper focuses on the interconnection-based model order reduction method originally introduced in [7] and [8]. Therein the author has shown that moment matching can be recast as the problem of matching the steady-state responses of certain system interconnections. This point of view enabled several developments in the area of model reduction by moment matching, see, *e.g.*, [9]–[12], and [13] for a recent survey. Of particular relevance for the present paper is [10]. Therein the authors have extended the moment matching problem to the case in which the signal inputs are produced by generators in the so-called *explicit form*. The resulting reduced-order model preserves the steady-state response of the original system when this is excited by possibly non-continuous or non-differentiable input signals, such as triangular waves, pulse-width modulated signals, square waves and saw waves. This generalization has inspired research in many different directions, such as model order reduction for certain classes of hybrid systems [14], the definition of a discontinuous phasor transform [15], and an optimal control method [16].

In this paper, we provide a notion of "swapped" moment under generalized filters. We recall that the moments of a linear system at the eigenvalues of a matrix Q are in one-to-one relation with the steady-state response of the output of a so-called swapped interconnection in which the output of the system y is filtered by $\dot{\varpi} = Q\varpi + Ry$, which is a filter in implicit form (*i.e.* described by differential equations). In the present paper, we consider a filter in explicit form and provide a generalized notion of moment which characterizes the steady-state of this interconnection for non-smooth filters. The result of this development is a complete dual theory with respect to [10]. The significance of the results of this paper lies on the fact that it is well-known that two-sided moment matching can be achieved by combining the direct and swapped interconnection theory. A two-sided moment matching model is a model that interpolates double the number of signals while maintaining the same reduced order, thus producing a better reduced-order model for the same computational cost.

The remainder of this paper is structured as follows. In Section II we recall the theory of moment matching for linear systems and give a formal description of the problem addressed. Section III proposes a new definition of moment based on filters in explicit form. Section IV presents a family of reduced-order models that achieves moment matching. In Section V we illustrate the developed theory through a numerical example. The paper ends with some concluding remarks.

Notation: Throughout this paper we use standard notation. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real numbers and complex numbers, respectively. $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers. $\mathbb{C}_0$ ($\mathbb{C}_{<0}$) denotes the set of complex numbers with zero (negative) real part. The set of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$. The identity matrix is denoted by the symbol I, and $\sigma(A)$ denotes the spectrum of a square matrix A. $A^\top$ denotes the transpose of any matrix A. The symbol $\mathcal{L}(f(t))$ denotes the Laplace transform of the function $f(t)$ (provided that $f(t)$ is Laplace transformable) and, abusing the notation, $\sigma(\mathcal{L}(f(t)))$ denotes the set of poles

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of $\mathcal{L}(f(t))$. The symbol ι indicates the imaginary unit.

II. Preliminaries

In this section, we recall the "interconnection-based" model order reduction method and formulate the problem addressed in the paper.

A. On the Notion of Moment

Consider a linear, single-input, single-output, continuous-time system, described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. Let $W(s) = C(sI-A)^{-1}B$ be the associated transfer function and assume system (1) is minimal, *i.e.*, both reachable and observable. The moment of system (1) is defined as follows.

Definition 1. The 0-moment of system (1) at $s_i \in \mathbb{C} \setminus \sigma(A)$ is the complex number $\eta_0(s_i) = W(s_i)$. The k-moment of system (1) at s_i is the complex number $\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}W(s)\right]_{s=s_i}$, where $k \geq 1$ is an integer.

It has been shown in [7], [8], [17] that a given set of moments of system (1) can be characterized by means of Sylvester equations, as follows.

Theorem 1. Consider system (1) and $s_i \in \mathbb{C} \setminus \sigma(A)$, with $i = 1,\dots,\rho$. Let $S \in \mathbb{R}^{\nu\times\nu}$ and $Q \in \mathbb{R}^{\nu\times\nu}$ be any non-derogatory[1] matrices with characteristic polynomial

$$p(s) = \prod_{i=1}^{\rho}(s-s_i)^{k_i}, \tag{2}$$

with $\nu = \sum_{i=1}^{\rho} k_i$. Then, there exists a one-to-one relation between the matrix

$$\begin{bmatrix}\eta_0(s_1) & \dots & \eta_{k_1-1}(s_1) & \dots & \eta_0(s_\rho) & \dots & \eta_{k_\rho-1}(s_\rho)\end{bmatrix}$$

and

- the matrix $C\Pi$, in which $\Pi \in \mathbb{R}^{n\times\nu}$ is the unique solution of the Sylvester equation
$$\Pi S = A\Pi + BL, \tag{3}$$
with $L \in \mathbb{R}^{1\times\nu}$ any matrix such that the pair (L,S) is observable.
- the matrix ΥB, in which $\Upsilon \in \mathbb{R}^{\nu\times n}$ is the unique solution of the Sylvester equation
$$Q\Upsilon = \Upsilon A + RC, \tag{4}$$
with $R \in \mathbb{R}^{\nu\times 1}$ any matrix such that the pair (Q,R) is reachable.

Based on these one-to-one relations, the matrices $C\Pi$ and ΥB are called moments of system (1) at (S,L) and (Q,R), respectively. More importantly, this characterization of the moments enables, through the Sylvester equation (3) ((4), respectively), to establish a connection between 0-moments and the steady-state response of the output of certain system interconnections, shown in Fig. 1. This is summarized in the following result.

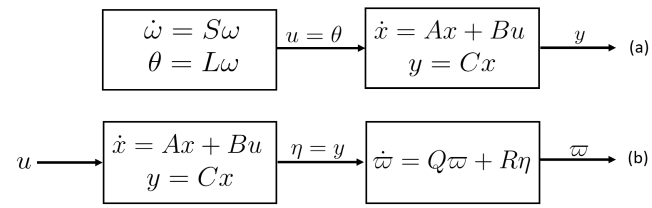

Fig. 1. Diagrammatic illustrations of the *direct* interconnection (a) and the *swapped* interconnection (b).

Theorem 2. Consider system (1) and $s_i \in \mathbb{C} \setminus \sigma(A)$, with $i = 1,\dots,\rho$ and suppose $\sigma(A) \subset \mathbb{C}_{<0}$. Let $S \in \mathbb{R}^{\nu\times\nu}$ and $Q \in \mathbb{R}^{\nu\times\nu}$ be any non-derogatory matrices with characteristic polynomial (2) with $k_i = 1$ for $i = 1,\dots,\rho$. Then, there exists a one-to-one relation between the moments $\eta_0(s_1), \dots, \eta_0(s_\rho)$ and

- the steady state of the output $y(t)$ of the *direct*[2] interconnection (Fig. 1(a)) of system (1) and the signal generator
$$\dot{\omega} = S\omega, \qquad \theta = L\omega, \tag{5}$$
via $u = \theta$, with L and $\omega(0)$ such that the triple $(L, S, \omega(0))$ is minimal [18].
- the steady state of the output $\varpi(t)$ of the *swapped*[3] interconnection (Fig. 1(b)) of system (1) and the filter
$$\dot{\varpi} = Q\varpi + R\eta, \tag{6}$$
via $\eta = y$, with R such that the pair (Q,R) is reachable, $x(0) = 0$, $\varpi(0) = 0$, and the input $u(t)$ of system (1) any non-zero signal[4] that decays to zero exponentially.

Theorem 2 provides two alternative "interconnection-based" characterizations of the moments of a linear system. This allowed extending the notion of moment to systems where a transfer function concept does not exist [13]. Model order reduction by moment matching changed from a problem of interpolation of points to a problem of interpolation of responses of system interconnections. In particular, the model

$$\dot{\xi} = (S-GL)\xi + Gu, \qquad \psi = C\Pi\xi, \tag{7}$$

matches the moments of (1) at (S,L) for any G such that $\sigma(S) \cap \sigma(S-GL) = \emptyset$. Likewise, the model

$$\dot{\xi} = (Q-RH)\xi + \Upsilon Bu, \qquad \psi = H\xi, \tag{8}$$

matches the moments of (1) at (Q,R) for any H such that $\sigma(Q) \cap \sigma(Q-RH) = \emptyset$. A model that matches the moments

[1] A matrix is non-derogatory if its characteristic and minimal polynomial coincide.

[2] This indicates that the output of the generator drives the system of interest.

[3] This indicates that the output of the system drives the filter.

[4] In [8], $u(t)$ was originally selected as the Dirac-delta function for simplicity. Note, however, that this selection is not necessary. Here we provide a relaxed choice of input that still preserves this one-to-one relation between the moments and the steady state of ϖ.

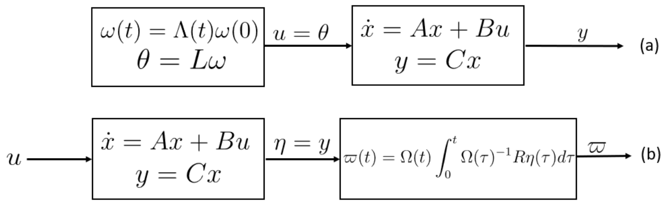

Fig. 2. Diagrammatic illustrations of the direct interconnection (a) with an explicit generator and the swapped interconnection (b) with an explicit filter.

of (1) at (S, L) and (Q, R) simultaneously (which, under certain conditions, is equivalent to matching the steady states for both interconnections) is called a two-sided moment matching model [19]. For instance, if $\sigma(S) \cap \sigma(Q) = \emptyset$, model (7) with

$$G = (\Upsilon\Pi)^{-1}\Upsilon B \tag{9}$$

matches the moments of (1) at (S, L) and (Q, R) simultaneously.

B. Problem Formulation

In the previous section we have recalled that moment matching preserves the long-time behaviour of the system to be reduced for certain input of interests. As highlighted in [10], the signal generator (5) (described by means of linear differential equations) cannot describe certain signals that may be of interest in various applications. For instance, non-smooth signals (*e.g.* the PWM waves) cannot be represented as the solution of smooth differential equations. Motivated by this observation, [10] has extended the direct interconnection theory to cases where the signal generator is generalized to the so-called explicit form[5] (see Fig. 2(a)). Note that the signal generator (5) can be written in explicit form as

$$\omega(t) = e^{S(t-t_0)}\omega_0,$$

where $e^{S(t-t_0)}$ is the transition matrix for linear time-invariant systems. Then [10] generalized this signal generator (5) to the explicit model

$$\omega(t) = \Lambda(t, t_0)\omega_0, \tag{10}$$

with $\Lambda(t, t_0) \in \mathbb{R}^{\nu\times\nu}$, invertible for all $t \in \mathbb{R}$, such that $\Lambda(t_0, t_0) = I$ and $\Lambda(t_2, t_1)\Lambda(t_1, t_0) = \Lambda(t_2, t_0)$ for any $t_0 \leq t_1 \leq t_2$. It has been noted in [10] that the explicit model (10) could generate a much more general class of signals, including those produced by the implicit model (5) or by time-varying models described by the equations

$$\dot{\omega} = S(t)\omega, \qquad u = L\omega, \tag{11}$$

with $S(t) \in \mathbb{R}^{\nu\times\nu}$. Notably, the explicit model (10) can also generate non-differentiable and discontinuous signals, such as those produced by hybrid systems or some special classes of nonlinear systems [22]. Under certain assumptions, the steady-state response of system (1) driven by the generator (10) is given by $x(t) = \Pi_\infty(t)\omega(t)$, with $\Pi_\infty(t)$ a matrix-valued function defined as

$$\Pi_\infty(t) = \left(\int_{-\infty}^{t} e^{A(t-\tau)}BL\Lambda(\tau)\,d\tau\right)\Lambda(t)^{-1}, \tag{12}$$

where $\Lambda(t)$ denotes $\Lambda(t, 0)$. Moreover, in any set $\mathcal{T} = (t_1, t_2)$ with $0 \leq t_1 < t_2$ where $\Lambda(t)$ is differentiable with respect to t, $\Pi_\infty(t)$ is the unique solution of the differential equation

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t)\Lambda(t)^{-1} \tag{13}$$

with initial condition $\Pi(t_1) = \Pi_\infty(t_1)$. The differential equation (13) can be seen as a generalization of the Sylvester equation (3). In particular, when $\Lambda(t) = e^{St}$, $\Pi_\infty(t)$ is constant and equal to the solution of the Sylvester equation (3).

In this work, our goal is to develop a swapped interconnection theory for filters in explicit form. The importance of this research direction is justified by the fact that a two-sided moment-matching model (*i.e.* a reduced-order model that matches the moments for both filter and generator) hinges upon the results developed from both the direct and the swapped interconnection perspectives. In fact, it is clear from (9) that a concept of "Υ_∞" is required to achieve two-sided matching. In other words, the results in this paper are preliminary to construct a reduced-order model that matches twice the number of explicit signals while maintaining the same (reduced) order.

III. Definition of Moments

We begin with generalizing the filter (6) by exploiting an idea similar to that introduced in Section II-B. First, note that this filter can be rewritten in explicit form as (recall that $\varpi(0) = 0$)

$$\varpi(t) = \int_0^t e^{Q(t-\tau)}R\eta(\tau)d\tau,$$

where e^{Qt} is the transition matrix for linear time-invariant systems (6). Then, we consider the model

$$\varpi(t) = \Omega(t)\int_0^t \Omega(\tau)^{-1}R\eta(\tau)d\tau, \tag{14}$$

with $\Omega(t) \in \mathbb{R}^{\nu\times\nu}$, invertible for all $t \in \mathbb{R}$, such that $\Omega(0) = I$ and $\Omega(-t) = \Omega(t)^{-1}$, as a natural generalization of the signal generator (6). Note that (14) can be interpreted as a model that filters the signal $\eta(t)$ via the convolution kernel $\Omega(t)$. To ensure the existence and the uniqueness of $\varpi(t)$, we introduce the following assumptions.

Assumption 1. $\Omega(t)$ and $\Omega(t)^{-1}$ are bounded for all $t \in \mathbb{R}_{\geq 0}$.

This technical assumption guarantees the boundedness of the signal ϖ as long as $\lim_{t\to+\infty} u(t) = 0$ quickly enough and system (1) satisfies the following assumption.

Assumption 2. System (1) is asymptotically stable, *i.e.*, $\sigma(A) \subset \mathbb{C}_{<0}$. The initial condition $x(0) = 0$.

Note that under the asymptotic stability of (1), the assumption that the initial condition of the system is zero is without loss of generality. We can now prove the main result of the paper.

[5]This terminology is adapted from [20], [21] and explained next.

Theorem 3. Consider the interconnection of system (1) and the explicit filter (14) via the equation $\eta = y$ (see Fig. 2(b)). Suppose Assumptions 1 and 2 hold. Assume that there exists $k > 0$ and $\alpha > 0$ such that $||u(t)|| \leq ke^{-\alpha t}$. $\lim_{t\to+\infty} u(t) = 0$ and $\Omega(t)$ is semi-differentiable. Then, the steady-state response of ϖ is given by

$$\varpi_\infty(t) = \Omega(t) \lim_{\hat{t}\to\infty} \int_0^{t+\hat{t}} \Omega(\tau)^{-1} \Upsilon_\infty(\tau) B u(\tau) d\tau, \quad (15)$$

with $\Upsilon_\infty(t) \in \mathbb{R}^{\nu\times n}$ the matrix-valued function defined as

$$\Upsilon_\infty(t) := \Omega(t) \int_t^{+\infty} \Omega(\tau)^{-1} RCe^{A(\tau-t)} d\tau. \quad (16)$$

Moreover, in a set $\mathcal{T}^* = (t_1, t_2)$ with $0 \leq t_1 < t_2$ where $\Upsilon_\infty(t)$ is differentiable, Υ_∞ is the solution of the differential equation

$$\dot{\Upsilon}(t) = \dot{\Omega}(t)\Omega(t)^{-1}\Upsilon(t) - RC - \Upsilon(t)A, \quad (17)$$

with initial condition $\Upsilon(t_1) = \Upsilon_\infty(t_1)$.

Theorem 3 indicates that for a given $\Omega(t)$, the function $\Upsilon_\infty(t)B$ is all that is needed to characterize the steady-state response of ϖ for any input signal $u(t)$ that decays fast enough. We are now in a position to give the definition of moment for the swapped interconnection under an explicit filter.

Definition 2. Consider the system (1) and the explicit filter (14). Suppose Assumptions 1 and 2 hold. We call the function $\Upsilon_\infty(t)B$, with $\Upsilon_\infty(t)$ defined as in (16), the *moment of system* (1) at (Ω, R) (or at (14)).

Definition 2 is justified by the observation that two models that have the same moment at (Ω, R) have also the same steady-state response of their filtered outputs through (14) for any input u that decays to zero fast enough.

We now restrict our focus to the case when the "kernel" Ω is periodic.

Corollary 1. Consider system (1) and the explicit filter (14). Suppose Assumptions 1 and 2 hold and Ω has the property

$$\Omega(t) = \Omega(t + T), \quad (18)$$

for all times, with T the period. Then,

$$\Upsilon_\infty(t) = \Omega(t) \int_t^{t+T} \Omega(\tau)^{-1} RCe^{A(\tau-t)} d\tau \, (I - e^{AT})^{-1}, \quad (19)$$

for any t, or equivalently,

$$\Upsilon_\infty(t) = \Omega(t) \int_{t-T}^{t} \Omega(\tau)^{-1} RCe^{A(\tau-t)} d\tau \, (e^{-AT} - I)^{-1}, \quad (20)$$

for any t.

Remark 1. A notable advantage of (19) and (20) with respect to (16) is that the integration is only computed over a finite time interval, which significantly reduces the required computational efforts.

IV. A Family of Reduced-Order Models

In this section, we present a family of models that match the moments of system (1) at (14). We begin with a definition.

Definition 3. Consider the system (1) and the explicit filter (14). Suppose Assumptions 1 and 2 hold and R is such that $\sigma(\mathcal{L}(\Omega(t)R)) = \sigma(\mathcal{L}(\Omega(t)))$. Then, the system described by the equations

$$\begin{aligned} \xi(t) =& \int_0^t K(t-\tau)G(\tau)u(\tau)d\tau, \\ \psi(t) =& H(t)\xi(t), \end{aligned} \quad (21)$$

with $\xi(t) \in \mathbb{R}^\nu$, $\psi(t) \in \mathbb{R}$, $K(t) \in \mathbb{R}^{\nu\times\nu}$, $G(t) \in \mathbb{R}^{\nu\times 1}$ and $H(t) \in \mathbb{R}^{1\times\nu}$, is a *model of system* (1) at (14), if the matrix-valued function

$$Y_\infty(t) = \Omega(t) \int_t^{+\infty} \Omega(\tau)^{-1} RH(\tau)K(\tau - t) d\tau \quad (22)$$

exists and is such that

$$\Upsilon_\infty(t)B = Y_\infty(t)G(t), \quad (23)$$

for all t, with $\Upsilon_\infty(t)$ defined in (16). If $\nu < n$, we call system (21) a *reduced-order model of system* (1) at (14).

Remark 2. The condition $\sigma(\mathcal{L}(\Omega(t)R)) = \sigma(\mathcal{L}(\Omega(t)))$ is a generalization of the condition that (Q, R) is reachable in Theorem 2 and it guarantees that all modes of ϖ are excited by the system output y. In fact, (Q, R) reachable is equivalent to the condition $\sigma(\mathcal{L}(e^{Qt}R)) = \sigma(Q)$.

Note that the existence of (22) ensures that the moments of (21) at (14) are well-defined and that (23) enforces the moment matching condition. As long as $K(t), G(t)$ and $H(t)$ are such that conditions (22) and (23) hold and that the origin of system (21) is asymptotically stable, (21) describes a family of parameterized models. Then $K(t), G(t)$ and $H(t)$ can be used to enforce additional properties or structure. For instance, consider the selection

$$K(t) = e^{\bar{F}t}, \qquad H(t) = \bar{H}, \quad (24)$$

with $\bar{F} \in \mathbb{R}^{\nu\times\nu}$ and $\bar{H} \in \mathbb{R}^{1\times\nu}$. With this selection, it can be shown that (21) has the implicit form

$$\dot{\xi}(t) = \bar{F}\xi(t) + G(t)u(t), \qquad \psi = \bar{H}\xi, \quad (25)$$

which is a linear system with a time-varying input gain $G(t)$. This allows one to easily enforce some important dynamical properties onto the reduced-order model. In the following we present a result that exploits the selection (24) to guarantee the stability of (21) for the case in which Ω is periodic.

Proposition 1. Consider the system (1) and the explicit filter (14) with the periodic property (18). Let $\bar{F} \in \mathbb{R}^{\nu\times\nu}$ be any Hurwitz matrix. Suppose Assumptions 1 and 2 hold and R is such that $\sigma(\mathcal{L}(\Omega(t)R)) = \sigma(\mathcal{L}(\Omega(t)))$. Then, the system described by the equations

$$\begin{aligned} \dot{\xi}(t) &= \bar{F}\xi(t) + Y_\infty(t)^{-1}\Upsilon_\infty(t)Bu(t), \\ \psi &= \bar{H}\xi, \end{aligned} \quad (26)$$

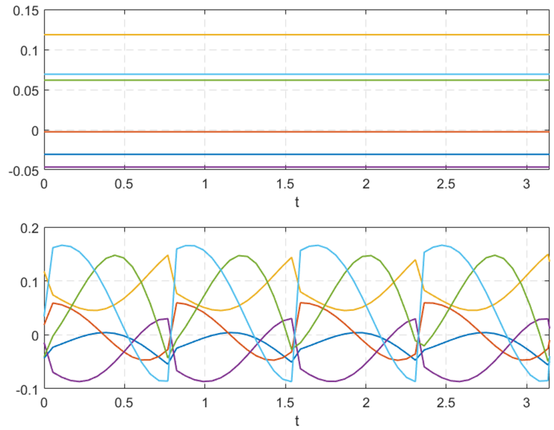

Fig. 3. Time history of the elements of $\Upsilon_{\sim}(t)$ (top) and $\Upsilon_{\sqcap}(t)$ (bottom).

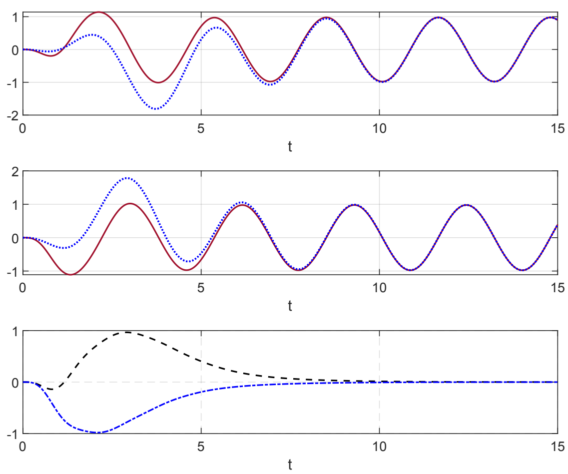

Fig. 4. Top: time history of the first component of the state of the filter (14) characterized by $\Omega_{\sim}$ driven by the original system (solid/red line) and by the reduced-order model $\Sigma_{\sim}^{\mathrm{ROM}}$ (dotted/blue line), respectively. Middle: same characteristics for the second component of the state of the filter. Bottom: time history of the error $\varpi^{\mathrm{FOM}} - \varpi^{\mathrm{ROM}}$.

with $\xi(t) \in \mathbb{R}^{\nu}$, $\bar{H} \in \mathbb{R}^{1\times\nu}$ and $\Upsilon_{\infty}(t)$ defined as in (19) (or equivalently (20)), is an asymptotically stable *model of system* (1) at (14), if

$$Y_{\infty}(t) = \Omega(t)\int_{t}^{t+T} \Omega(\tau)^{-1} R\bar{H} e^{\bar{F}(\tau-t)} d\tau \, (I - e^{\bar{F}T})^{-1} \tag{27}$$

is non-singular for all $t \in \mathbb{R}_{\geq 0}$.

V. Numerical Example

To demonstrate the developed results, we consider system (1) with matrices generated using the function `rss` of MATLAB R2021a as

$$\begin{aligned} A &= \begin{bmatrix} -2.439 & 2.337 & -1.776 \\ -2.933 & -1.096 & 4.221 \\ 0.09223 & -4.579 & -1.537 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & -0.7648 & -1.402 \end{bmatrix}^{T}, \\ C &= \begin{bmatrix} 0 & 0.4882 & 0 \end{bmatrix}. \end{aligned} \tag{28}$$

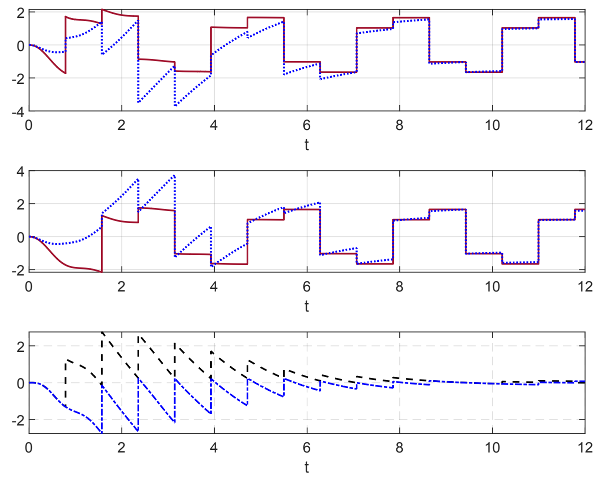

Fig. 5. Top: time history of the first component of the state of the filter (14) characterized by $\Omega_{\sqcap}$ driven by the original system (solid/red line) and by the reduced-order model $\Sigma_{\sqcap}^{\mathrm{ROM}}$ (dotted/blue line), respectively. Middle: same characteristics for the second component of the state of the filter. Bottom: time history of the error $\varpi^{\mathrm{FOM}} - \varpi^{\mathrm{ROM}}$.

In the rest of the paper, we select $R = [1\ 1]^{\top}$ and investigate two selections of $\Omega(t)$. The first is described as

$$\Omega_{\sim}(t) = \begin{bmatrix} \cos\left(\frac{2\pi}{T}t\right) & -\sin\left(\frac{2\pi}{T}t\right) \\ \sin\left(\frac{2\pi}{T}t\right) & \cos\left(\frac{2\pi}{T}t\right) \end{bmatrix}, \tag{29}$$

which is a smooth function with period T. The filter characterized by $\Omega_{\sim}(t,0)$ is a linear time-invariant system that has an implicit representation (6) with $\sigma(Q) = \left\{\pm\frac{2\pi}{T}\iota\right\}$. The second one is described as

$$\Omega_{\sqcap}(t) = \begin{bmatrix} \sqcap\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) & -\sqcap\left(\frac{2\pi}{T}t\right) \\ \sqcap\left(\frac{2\pi}{T}t\right) & \sqcap\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) \end{bmatrix}, \tag{30}$$

in which $\sqcap(t) = \operatorname{sign}(\sin(t))$ generates a square wave. Note that $\Omega_{\sqcap}(t,0)$ is a periodic, discontinuous function with jumps occurring at $t = \frac{T}{2} + kT$, $k \in \mathbb{Z}$.

Select $T = \pi$. Let $\Upsilon_{\sim}(t)$ and $\Upsilon_{\sqcap}(t)$ be the solutions of (16) for $\Omega_{\sim}$ and $\Omega_{\sqcap}$, respectively. Fig. 3 shows the evolution of the entries of $\Upsilon_{\sim}(t)$ (top) and $\Upsilon_{\sqcap}(t)$ (bottom). Observe that the components of $\Upsilon_{\sim}(t)$ are constant since, as mentioned, the filter reduces to a linear time-invariant system. In contrast, the components of $\Upsilon_{\sim}(t)$ are time-varying and periodic (as proven in Corollary 1).

To construct a reduced-order model, we generate the following data matrices randomly using the Matlab function `rss`

$$\bar{F} = \begin{bmatrix} -1.1008 & 0.3733 \\ 0.3733 & -0.9561 \end{bmatrix}, \ \bar{H} = \begin{bmatrix} -0.2256 & 1.117 \end{bmatrix}.$$

Note that $\sigma(\bar{F}) = [-1.4087, -0.6482]$. Following Proposition 1, two stable reduced-order models $\Sigma_{\sim}^{\mathrm{ROM}}$ and $\Sigma_{\sqcap}^{\mathrm{ROM}}$ of the form (26) are determined for $\Omega_{\sim}$ and $\Omega_{\sqcap}$, respectively. We select the input signal as $u(t) = 20e^{-t}$ and let ϖ^{FOM} and ϖ^{ROM} be the state of the filter (14) when connected with the full-order model and the reduced-order model, respectively. The top graph in Fig. 4 shows the time history of the first component of the state of the filter (14) characterized by $\Omega_\sim$ driven by the original system (solid/red line) and by the reduced-order model $\Sigma_\sim^{\text{ROM}}$ (dotted/blue line), respectively. The middle graph shows the same characteristics for the second component of the state of the filter. The error between ϖ^{FOM} and ϖ^{ROM} over time is shown in the bottom graph (blue/dash-dotted line for the first component and black/dashed line for the second). We can observe that as time t increases, the filtered responses of the full-order model and of the reduced-order model tend to coincide with each other and approach the same steady-state ϖ_∞, achieving moment matching at $(\Omega_\sim, R)$. Note that in this case, the reduced-order model obtained belongs to the family (8), *i.e.*, the standard theory is recovered. Finally, Fig. 5 shows analogous quantities for the non-smooth $\Omega_\sqcap$. Also in this case, as time t increases, the filtered responses of the full-order model and of the reduced-order model tend to coincide with each other, achieving moment matching at $(\Omega_\sqcap, R)$ and presenting new results with respect to the state-of-the-art.

VI. Conclusions

The notion of "swapped" moment for linear systems has been extended to the case in which the filter in the swapped interconnection is described in explicit form. Particular attention has been devoted to the case in which the filter has a non-smooth convolution kernel. By exploiting this definition, we have derived a family of reduced-order models that achieve moment matching in the sense of producing the same filtered output at steady state. The paper ends with a numerical example that illustrates the developed results. This work constitutes a stepping stone for developing a two-sided moment matching theory for explicit signal generators and filters, enabling the matching at double the number of discontinuous signals while maintaining the same reduced order.

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