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On the distance to infeasibility in DC power grids with constant-power loads

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Abstract— **This paper is concerned with the feasibility of the power flow in DC power grids with constant power loads. We introduce the notion of distance to infeasibility as a voltage stability index and robustness measure for power flow feasibility. In particular, we study the p-norm distance to infeasibility in the domain of the constant power loads, and show how this distance may be expressed as a mathematical program. Necessary and sufficient matrix inequalities are presented that guarantee a minimal p-norm distance between a given vector of power demands and the boundary of infeasibility. For the cases 1-norm and ∞-norm distance we show that the condition can be formulated as (multiple) linear matrix inequalities, whereas in all other cases the matrix inequalities are strictly concave and thus non-convex. For the 2-norm distance we show that the distance to infeasibility may be computed via bilinear matrix inequalities. A numerical example for the ∞-norm distance to infeasibility for the 39 bus New England power grid is provided.**

I. Introduction

Over the last decade, DC power grids have found an increasing interest among applications such as smart grids and high-voltage DC (HVDC) transmission. A major concern in DC power grids is the presence of constant-power loads, which demand a constant amount of power from the grid. Such loads are known to destabilize the grid by selfishly extracting more power from the grid, which can lead to a rapid decrease of nodal voltages, which is known as voltage collapse. In particular, voltage collapse occurs if constant power demands cannot be met at steady state [1].

The DC power flow feasibility problem studies under which conditions the constant-power demands in a DC power grid can be satisfied at steady state. Sufficient conditions for this problem have been presented in [2], [3], [4], [5]. The problem has been fully characterized in [6], [7], resulting in necessary and sufficient conditions for power flow feasibility. Although these results are able to determine the power flow feasibility when all system parameters are known, one shortcoming is that the robustness to parametric uncertainties of power flow feasibility cannot be guaranteed. Such a guarantee is often required since power grids should not be operated close to the feasibility boundary, especially since system parameters are often uncertain.

The term *voltage stability index* [8] commonly refers to a measure for how close a power grid is to voltage collapse. Such indices provide a measure for the robustness of the power flow. More practically, they function as a warning signal for when the power flow in the system is close to infeasibility and an intervention is needed. Classically, such indices are phrased for AC power grids, and rely on either the power flow Jacobian, or on local branch or nodal variables. We refer to [8] for an overview of such indices.

The focus of this work is to formalize a voltage stability index that measures how close the power flow in a DC power grid is to being infeasible. For this purpose, we propose to study the p-norm distance in the power domain of the vector of power demands to boundary of power flow infeasibility. We show that this distance can be formulated a mathematical program. A converse to this problem is to demonstrate the robustness of power flow feasibility. To this end we want to find conditions such that all power demands within a p-norm ball of a given radius, centered around a given vector of power demands, give rise to a feasible power flow. We present necessary and sufficient matrix inequalities that guarantee this robust property.

The presented distance-to-infeasibility metric of constant power loads is convenient when only bounds on the constant power loads are available. A similar problem that considers bounded uncertainty in line conductances of DC power grids was studied in [9], where it was shown that power flow feasibility is subject to the so-called Braess paradox, meaning that adding lines or increasing their conductances may cease the power flow feasibility in a DC power grid. There, a sufficient condition for the feasibility of all systems within the uncertainty bounds was proposed. We note that the results presented in this paper can be used in tandem with the sufficient condition in [9] to deal with bounded uncertainties in both line conductances and constant power loads.

We also remark the analogy between the reactive power flow in lossless decoupled AC grids and the power flow in DC power grids, *c.f.* [2]. In this analogy, the proposed distance-to-infeasibility measure corresponds to the distance to infeasibility of the reactive power flow of the lossless AC power grid. It therefore a measure of the reactive power reserve [10, Ch. 4]. The study and comparison of these systems and the reactive power reserve lies outside the scope of this paper.

The remainder of this paper is structured as follows. In Section II we introduce the power grid model and discuss power flow feasibility. Section III introduces the distance-to-feasibility measure, presents the problem of computing this measure, and presents the problem of guaranteeing robustness of the power flow. In Section IV we present

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matrix inequalities that guarantee power flow robustness, and present a mathematical program to compute the distance-to-infeasibility measure. In Section V we address the computational aspects of the presented methods for $\inf$-norm, 1-norm and 2-norm distance to infeasibility. Section VI presents a numerical simulation for the computation of the ∞-norm distance to infeasibility in a benchmark power grid. The paper is concluded in Section VII.

Notation and matrix definitions

For a vector $x = \begin{pmatrix} x_1 & \cdots & x_k \end{pmatrix}^\top$ we denote

$$[x] := \operatorname{diag}(x_1, \ldots, x_k).$$

We let $\mathbb{1}$ and $\mathbb{0}$ denote the all-ones and all-zeros vector, respectively. We let their dimensions follow from their context. All vector and matrix inequalities are taken to be element-wise. We let e_i denote the i-th column of the identity matrix. We write $A \succeq B$ when $A - B$ is a symmetric positive semi-definite matrix, and $A \succneqq B$ when in addition $A \neq B$. We let $\|x\|_p$ denote the p-norm of $x \in \mathbb{R}^k$.

II. THE DC POWER GRID MODEL

This paper considers DC power grids with constant-power loads at steady state, which are modeled as a resistive circuit. Nodes (buses) in the grid are either sources (S) or loads (L). A source is a node at which the nodal voltage potentials of the network are fixed, such as a slack bus. A load is a node that demands a given quantity of power from the grid, often known as a P-load. The power flow feasibility problem asks if the nodal voltage potentials at the loads can be chosen such that all the power demands are satisfied.

To give a mathematical formulation of this problem we define the following quantities. We let $V = \begin{pmatrix} V_L^\top & V_S^\top \end{pmatrix}^\top \in \mathbb{R}^{n+m}$ be the voltage potentials at the nodes, which we assume to be positive. The vectors V_L and V_S correspond to the voltage potentials of the loads and source, respectively. We let $Y \in \mathbb{R}^{(n+m)\times(n+m)}$ denote the Kirchhoff matrix of the power grid (*e.g.*, see [6]), which relates the voltage potentials in the grid to the nodal current $I \in \mathbb{R}^{n+m}$ injected into the network by $I = YV$. The power that is injected into the network at the loads is therefore given by

$$P_L = [V_L](Y_{LL}V_L + Y_{LS}V_S). \tag{1}$$

Let $P_c \in \mathbb{R}^n$ denote the power demands of the load nodes. The power flow feasibility problem asks if the demands P_c can be satisfied for some $V_L > \mathbb{0}$, in which case $P_L + P_c = \mathbb{0}$. We formalize this as follows.

Definition 2.1: Given Y and V_S, we say that P_c is feasible if there exists $V_L > \mathbb{0}$ such that

$$[V_L](Y_{LL}V_L + Y_{LS}V_S) + P_c = \mathbb{0}. \tag{2}$$

The set of all feasible P_c is denoted by $\mathcal{F}$. In addition, we say that P_c is feasible under small perturbation if P_c does not lie on $\partial\mathcal{F}$, the boundary of $\mathcal{F}$.

Note that lines between sources do not contribute to the power flow problem, as the submatrix Y_{SS} does not appear in the equation (2).

A. Power flow feasibility as an LMI feasibility problem

It has been shown in [7, Thm. 5.1] that the set $\mathcal{F}$ is closed and convex, and that feasibility of P_c is equivalent to the infeasibility of a linear matrix inequality (LMI) concerning the matrix

$$Q_{P_c}(\lambda) := \begin{pmatrix} \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}[\lambda]) & \frac{1}{2}[\lambda]Y_{LS}V_S \\ \frac{1}{2}([\lambda]Y_{LS}V_S)^\top & \lambda^\top P_c \end{pmatrix}.$$

We repeat the result for the sake of completeness.

Theorem 2.2 (LMI for power flow infeasibility): Given Y and V_S, the vector P_c is not feasible (under small perturbation) if and only if there exists a nonzero vector λ such that $Q_{P_c}(\lambda)$ is positive definite (positive semi-definite). If λ exists, it satisfies $\lambda > 0$.

Theorem 2.2 tells us that power flow infeasibility is equivalent to the feasibility of an LMI. It was also shown in [7, Thm. 5.2] that this equivalence may be rephrased in terms of the alternative of the LMI, *e.g.*, see [11]. It is worth noting that [7, Thm. 5.2] can be slightly relaxed by taking into account that the vector λ in Theorem 2.2 is positive.

Theorem 2.3 (Alternative LMI for power flow feasibility): Given Y and V_S, the vector P_c is feasible if and only if there exists a nonzero positive semi-definite matrix $Z = Z^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ such that for all $i = 1, \ldots, n$,

$$\operatorname{trace}(ZQ_{P_c}(e_i)) \leq 0.$$

Proof: Define the matrices $A_0 = 0$ and $A_i = Q_{P_c}(e_i)$ for $i = 1, \ldots, n$ and apply Lemma A.1 to [7, Thm. 5.2]. ■

III. PROBLEM FORMULATION

In this paper we are interested in obtaining a metric for how close the power flow in a DC power grid is to being infeasible. Since voltage collapse occurs when the power flow is close to being infeasible, this metric functions as a voltage stability index. For our metric we focus on the constant-power demands, and ask what is the distance of a feasible vector of power demands to the boundary of feasibility in the power domain. In this paper we will restrict ourselves to the p-norm distances given by

$$d_p^S(x, y) := \|S^\top(x - y)\|_p.$$

where $p \in [1, \infty]$ and S is a square nonsingular shearing matrix. We assume that inverse of S is a nonnegative matrix, as is the case for nonsingular M-matrices [12].

Assumption 3.1: The inverse of S satisfies $S^{-1} \geq 0$.

The associated distance of a vector of power demands P_c to the infeasibility set $\mathcal{F}^c := \mathbb{R}^n \setminus \mathcal{F}$ is given by

$$d_p^S(P_c, \mathcal{F}^c) := \inf_{y \in \mathcal{F}^c} d_p^S(P_c, y).$$

It should be noted that, if P_c lies on the boundary of feasibility, or is infeasible, then we naturally have $d_p^S(P_c, \mathcal{F}^c) = 0$. This paper is concerned with the computational feasibility of the distances $d_p^S(P_c, \mathcal{F}^c)$.

Problem 1 (Distance problem): Find a computationally feasible method to compute $d_p^S(P_c, \mathcal{F}^c)$.

We are also interested in formulating a condition that guarantees a degree of robustness of power flow, and if it is computational feasible. By this notion of robustness we mean that all power demands in a sufficiently large neighbourhood of P_c lead to a feasible power flow, and small fluctuation and inaccuracies in P_c do not affect the power flow feasibility. More specifically, we ask under which conditions a ball around a given P_c is fully contained in the feasibility set $\mathcal{F}$. To this end, we let the p-norm ball with radius γ, with shearing matrix S and centered at y be defined by

$$\mathcal{B}^S_{p,\gamma}(y) := \left\{\, z \,\middle|\, \|S^\top(y-z)\|_p \le \gamma \,\right\}.$$

Problem 2 (Robustness problem): Find a computationally feasible method to verify that $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c}) \subseteq \mathcal{F}$.

Naturally, these two problems are related in the sense that the distance $d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c})$ is the largest radius γ such that the ball $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c})$ is contained in $\mathcal{F}$, provided that P_c is feasible.

To the best of the authors knowledge, Problem 1 and Problem 2 have not been considered before in the literature, and are a novel perspective to measure the degree of feasibility of the power flow equations.

In the remainder of the paper we show that Problem 1 and Problem 2 can be formulated as a mathematical program in terms of matrix inequalities. The presented mathematical programs may be considered in control applications to guarantee the robust feasibility of the power flow, or to decide to intervene if the power flow is to close to infeasibility, similar to any other voltage stability index.

IV. Reformulation of the distance and robustness problem

A. Robust power flow feasibility guarantee as a matrix inequality

In this section we address Problem 2. Theorem 2.2 states how power flow feasibility is equivalent to the infeasibility of an LMI. Similarly, we show that the robust feasibility is equivalent to a matrix inequality involving the matrix map

$$Q^S_{P_\mathrm{c},q,\gamma}(\lambda) := \begin{pmatrix} \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}[\lambda]) & \frac{1}{2}[\lambda]Y_{LS}V_S \\ \frac{1}{2}([\lambda]Y_{LS}V_S)^\top & \lambda^\top P_\mathrm{c} + \gamma\|S^{-1}\lambda\|_q \end{pmatrix}$$

which not linear in λ due to the presence of $\|S^{-1}\lambda\|_q$.

Theorem 4.1: Let $p, q \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. All power demands in the ball $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c})$ are feasible if and only if there does not exist λ such that $Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succ 0$. Similarly, all power demands in the ball $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c})$ are feasible under small perturbation if and only if there does not exist a nonzero λ such that $Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succeq 0$.

Proof: We will prove the latter equivalence. The proof of former equivalence is analogous.

We note from Theorem 2.2 that all power demands in $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c})$ are feasible under small perturbation if and only if for each δ such that $\|S^\top\delta\|_p \le \gamma$ there does not exist a nonzero λ such that $Q_{P_\mathrm{c}+\delta}(\lambda) \succeq 0$.

Let λ be any nonzero vector. Suppose that $\|S^\top\delta\|_p \le \gamma$. By Hölder's inequality [13] we have that

$$\lambda^\top\delta = (S^{-1}\lambda)^\top(S^\top\delta) \le \|S^\top\delta\|_p\|S^{-1}\lambda\|_q \le \gamma\|S^{-1}\lambda\|_q,$$

which implies that $Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succeq Q_{P_\mathrm{c}+\delta}(\lambda)$. Suppose there does not exist a nonzero $\widehat{\lambda}$ such that $Q^S_{P_\mathrm{c},q,\gamma}(\widehat{\lambda})$ is positive semi-definite. Then $Q_{P_\mathrm{c}+\delta}(\lambda)$ is not positive semi-definite.

Conversely, let λ be any nonzero vector. By the tightness of Hölder's inequality [13] there exists a nonzero vector δ such that

$$\lambda^\top\delta = (S^{-1}\lambda)^\top(S^\top\delta) = \|S^\top\delta\|_p\|S^{-1}\lambda\|_q.$$

We scale δ such that $\|S^\top\delta\|_p = \gamma$, implying that

$$\lambda^\top\delta = \gamma\|S^{-1}\lambda\|_q. \tag{3}$$

Suppose that for each $\widehat{\delta}$ such that $\|S^\top\widehat{\delta}\|_p \le \gamma$ there does not exist a nonzero $\widehat{\lambda}$ such that $Q_{P_\mathrm{c}+\widehat{\delta}}(\widehat{\lambda}) \succeq 0$. Then $Q_{P_\mathrm{c}+\delta}(\lambda) = Q^S_{P_\mathrm{c},q,\gamma}(\lambda)$ is not positive semi-definite, where we substituted (3). ■

The computational tractability of the matrix inequalities $Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succ 0$ and $Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succeq 0$ in Theorem 4.1 for different values of q is treated in Sections V-A, V-B and V-C.

B. The distance to infeasibility as a mathematical program

In this section we address Problem 1. Based on Theorem 4.1 we show that the distance to infeasibility may be expressed as mathematical program, and, more specifically, as a semi-definite program with a (generally) non-convex cost function. To this end, we define

$$\mathcal{A}_{P_\mathrm{c}}(\lambda) := \begin{pmatrix} \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}[\lambda]) & \frac{1}{2}[\lambda]Y_{LS}V_S \\ \frac{1}{2}([\lambda]Y_{LS}V_S)^\top & \lambda^\top P_\mathrm{c} + 1 \end{pmatrix}.$$

Note that $\mathcal{A}_{P_\mathrm{c}}(\lambda) \gneqq Q_{P_\mathrm{c}}(\lambda)$.

Theorem 4.2: Let $p, q \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $r \in (0,\infty]$ be the optimal value of the maximization problem

$$\begin{aligned} &\max_{\lambda\in\mathbb{R}^n}\ \|S^{-1}\lambda\|_q \\ &\text{subject to } \mathcal{A}_{P_\mathrm{c}}(\lambda) \succeq 0 \end{aligned} \tag{4}$$

then $d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c}) = \frac{1}{r}$.

Proof: The distance $d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c})$ equals the supremum of all γ such that $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c}) \subseteq \mathrm{int}(\mathcal{F})$. Conversely, $d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c})$ equals the smallest γ such that $\mathcal{B}^S_{p,\gamma}(P_\mathrm{c}) \nsubseteq \mathrm{int}(\mathcal{F})$. By Theorem 4.1 this means that

$$\begin{aligned} d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c}) = \ &\min_{\lambda\neq\mathbb{0}}\ \gamma \\ &\text{subject to } Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succeq 0 \end{aligned}$$

Since $Q^S_{P_\mathrm{c},q,\gamma}(\alpha\lambda) = \alpha Q^S_{P_\mathrm{c},q,\gamma}(\lambda)$, we are free to scale λ in the minimization problem. We scale λ such that $\gamma\|S^{-1}\lambda\|_q = 1$, leading to the minimization problem

$$\begin{aligned} d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c}) = \ &\min_{\lambda\neq\mathbb{0}}\ \gamma \\ &\text{subject to } Q^S_{P_\mathrm{c},q,\gamma}(\lambda) \succeq 0,\ \gamma\|S^{-1}\lambda\|_q = 1. \end{aligned}$$

By virtue of $\gamma\|S^{-1}\lambda\|_q = 1$, we have $Q^S_{P_\mathrm{c},q,\gamma}(\lambda) = \mathcal{A}_{P_\mathrm{c}}(\lambda)$ and $\gamma = \|S^{-1}\lambda\|_q^{-1}$, which therefore gives us

$$\begin{aligned} d^S_p(P_\mathrm{c}, \mathcal{F}^\mathrm{c}) = \ &\min_{\lambda\neq\mathbb{0}}\ \|S^{-1}\lambda\|_q^{-1} \\ &\text{subject to } \mathcal{A}_{P_\mathrm{c}}(\lambda) \succeq 0. \end{aligned} \tag{5}$$

By inverting the optimization function in (5) we obtain the maximization problem (4), but where $\lambda \neq \mathbb{0}$. Lemma B.1 in the Appendix implies that there always exists a nonzero $\widehat{\lambda}$ such that $\mathcal{A}_{P_{\mathrm{c}}}(\widehat{\lambda}) \succeq 0$. Hence, maximizing $\|S^{-1}\lambda\|_q$ over $\mathcal{A}_{P_{\mathrm{c}}}(\lambda) \succeq 0$ always gives a nonzero value, meaning that $\lambda = \mathbb{0}$ is never a maximizer of the problem. We therefore may include $\lambda = \mathbb{0}$ in the maximization problem without altering its optimal value. Thus, the optimal value of (4), which we called r in the statement of the Theorem, coincides with the inverse of the optimal value of the minimization problem in (5), implying that $d_p^S(P_{\mathrm{c}}, \mathcal{F}^{\mathrm{c}}) = \frac{1}{r}$. ∎

The computational tractability of the mathematical program in Theorem 4.2 for different choices of p is treated in Sections V-A, V-B and V-C.

V. Specific p-norm distances to infeasibility

A. The ∞-norm distance to infeasibility

In this section we show that the ∞-norm distance to infeasibility leads to a convex semi-definite optimization problem. The ∞-norm distance is useful when there is uncertainty in the power demands and only bounds for each individual power demand is available. Let $\underline{P_{\mathrm{c}}}, \overline{P_{\mathrm{c}}}$ be vectors that represent a bounded uncertainty in the power demands P_{c}, described by

$$\underline{P_{\mathrm{c}}}_{,i} \leq P_{\mathrm{c},i} \leq \overline{P_{\mathrm{c}}}_{,i} \quad \text{for all } i.$$

Assuming that $\underline{P_{\mathrm{c}}} < \overline{P_{\mathrm{c}}}$, we then have that such P_{c} are described by the ∞-norm ball

$$\mathcal{B}_{\infty,1}^{\mathrm{diag}(\frac{1}{2}(\overline{P_{\mathrm{c}}}-\underline{P_{\mathrm{c}}}))^{-1}}(\tfrac{1}{2}(\underline{P_{\mathrm{c}}}+\overline{P_{\mathrm{c}}})).$$

In general, the feasibility of all power demands in the ∞-norm ball $\mathcal{B}_{\infty,\gamma}^S(P_{\mathrm{c}})$ can be evaluated through an LMI.

Theorem 5.1: All power demands in the ∞-norm ball $\mathcal{B}_{\infty,\gamma}^S(P_{\mathrm{c}})$ are feasible (under small perturbation) if and only if there does not exist a nonzero λ such that $Q_{P_{\mathrm{c}}+\gamma S^{-\top}\mathbb{1}}(\lambda) \succ 0$ ($\succeq 0$).

Proof: We consider Theorem 4.1 with $p = \infty$, $q = 1$. It suffices to show that

$$\lambda^\top P_{\mathrm{c}} + \gamma\|S^{-1}\lambda\|_1 = \lambda^\top(P_{\mathrm{c}} + \gamma S^{-\top}\mathbb{1}). \tag{6}$$

If $Q_{P_{\mathrm{c}},1,\gamma}^S(\lambda) \succeq 0$ then also $[\lambda]Y_{LL} + Y_{LL}[\lambda] \succeq 0$. By [6, Lemma B.9] this means that $\lambda \geq \mathbb{0}$. Since $S^{-1} \geq 0$ by Assumption 3.1, we have that $S^{-1}\lambda \geq \mathbb{0}$ and $\|S^{-1}\lambda\|_1 = \mathbb{1}^\top S^{-1}\lambda$ which proves that (6) holds. ∎

Comparing the statement of Theorem 5.1 with Theorem 2.2, it is clear that the feasibility of the ball $\mathcal{B}_{\infty,\gamma}^S(P_{\mathrm{c}})$ is equivalent to the feasibility of the vector $P_{\mathrm{c}} + \gamma S^{-\top}\mathbb{1}$. This is no surprise, as the vector $P_{\mathrm{c}} + \gamma S^{-\top}\mathbb{1}$ is a tight elementwise upper bound for the ball $\mathcal{B}_{\infty,\gamma}^S(P_{\mathrm{c}})$, and [7, Lemma 7.1] states that any vector P_{c} that is bounded by a feasible vector $\widehat{P}_{\mathrm{c}}$ is also feasible.

The alternative LMI to Theorem 5.1 is stated as follows.

Corollary 5.2: All power demands in the ∞-norm ball $\mathcal{B}_{\infty,\gamma}^S(P_{\mathrm{c}})$ are feasible if and only if there exists a nonzero positive semi-definite matrix Z such that

$$\mathrm{trace}(ZQ_{P_{\mathrm{c}}+\gamma S^{-\top}\mathbb{1}}(e_i)) \leq 0 \text{ for } i = 1, \ldots, n.$$

Proof: The inclusion $\mathcal{B}_{\infty,\gamma}^S(P_{\mathrm{c}}) \subseteq \mathcal{F}$ is equivalent to the feasibility of $P_{\mathrm{c}} + \gamma S^{-\top}\mathbb{1}$. We apply Theorem 2.3. ∎

In the case of the ∞-norm distance to infeasibility, we may compute the distance by a semi-definite program with a linear cost function. Hence, in this case we obtain a computationally tractable method to compute this distance.

Theorem 5.3: The ∞-norm distance to infeasbility $d_\infty^S(P_{\mathrm{c}}, \mathcal{F}^{\mathrm{c}})$ may be computed via the optimization problem

$$\begin{aligned} d_\infty^S(P_{\mathrm{c}}, \mathcal{F}^{\mathrm{c}})^{-1} = \quad & \max_{\lambda\in\mathbb{R}^n} \ \mathbb{1}^\top S^{-1}\lambda \\ & \text{subject to } \mathcal{A}_{P_{\mathrm{c}}}(\lambda) \succeq 0 \end{aligned}$$

which is a convex problem, since its cost function is linear and the constraints are a semi-definite program.

Proof: From the proof of Theorem 4.2 we know that the optimal λ is nonzero. Any $\lambda \neq \mathbb{0}$ which satisfies $\mathcal{A}_{P_{\mathrm{c}}}(\lambda) \succeq 0$ also satisfies $[\lambda]Y_{LL} + Y_{LL}[\lambda] \succeq 0$. By [6, Lemma B.9] this means that $\lambda > \mathbb{0}$. Since also $S^{-1} \geq 0$ by Assumption 3.1, we have that $\|S^{-1}\lambda\|_1 = \mathbb{1}^\top S^{-1}\lambda$. ∎

Since the optimization problem Theorem 5.3 is a convex semi-definite problem, we may also consider its dual to express the ∞-norm distance to infeasibility.

Corollary 5.4: The ∞-norm distance to infeasbility $d_\infty^S(P_{\mathrm{c}}, \mathcal{F}^{\mathrm{c}})$ may be computed via a convex semi-definite program by

$$\begin{aligned} d_\infty^S(P_{\mathrm{c}}, \mathcal{F}^{\mathrm{c}})^{-1} = \quad & \min_{Z\in\mathbb{S}_{\succeq 0}^{n+1}} \ Z_{n+1,n+1} \\ & \text{subject to } \mathrm{trace}(ZQ_{P_{\mathrm{c}}}(e_i)) = -(S^{-\top}\mathbb{1})_i \\ & \qquad \text{for } i = 1, \ldots, n \end{aligned}$$

Proof: Following Section 2.2 of [11], the problems in Theorem 5.3 and Corollary 5.4 are each other's convex dual. By Lemma B.1 we have that the (primal) problem in Theorem 5.3 is strictly feasible. Hence, by [11, Thm. 8], the optimal values of both problems coincide. ∎

B. The 1-norm distance to infeasibility

We continue by showing that the 1-norm distance to infeasibility may be computed by evaluating n semi-definite programs. In particular, the robust feasibility of a 1-norm ball corresponds to the infeasibility of n LMIs.

Theorem 5.5: All power demands in the 1-norm ball $\mathcal{B}_{1,\gamma}^S(P_{\mathrm{c}})$ are feasible (under small perturbation) if and only if for all $i = 1, \ldots, n$ there does not exist a nonzero λ such that $Q_{P_{\mathrm{c}}+\gamma S^{-\top}e_i}(\lambda) \succ 0$ ($\succeq 0$).

Proof: If $Q_{P_{\mathrm{c}},1,\gamma}^S(\lambda) \succeq 0$ then also $[\lambda]Y_{LL}+Y_{LL}[\lambda] \succeq 0$. By [6, Lemma B.9] this implies that $\lambda \geq \mathbb{0}$. We consider Theorem 4.1 with $p = \infty$, $q = 1$. Since $S^{-1} \geq 0$ by Assumption 3.1, we have $S^{-1}\lambda \geq \mathbb{0}$ and therefore $\|S^{-1}\lambda\|_\infty = (S^{-1}\lambda)_i e_i$ for some i. Hence

$$\lambda^\top P_{\mathrm{c}} + \gamma\|S^{-1}\lambda\|_1 = \lambda^\top(P_{\mathrm{c}} + \gamma S^{-\top}e_i). \tag{7}$$

Therefore, $Q_{P_{\mathrm{c}},\infty,\gamma}^S(\lambda) \succ 0$ ($\succeq 0$) if and only if there exists an i so that $Q_{P_{\mathrm{c}}+\gamma S^{-\top}e_i}(\lambda) \succ 0$ ($\succeq 0$). ∎

Comparing the statement of Theorem 5.5 with Theorem 2.2, it is clear that the feasibility of the ball $\mathcal{B}_{1,\gamma}^S(P_{\mathrm{c}})$ coincides with the feasibility of n vectors $P_{\mathrm{c}} + \gamma S^{-\top}e_i$. This can be explained by the observation that the vectors

$P_\mathrm{c} + \gamma S^{-\top} e_i$ form the corners of the ball $\mathcal{B}^S_{1,\gamma}(P_\mathrm{c})$, and by convexity of $\mathcal{F}$ and the domination property of [6, Lemma 7.1], it is necessary and sufficient that these corner points are feasible.

Similar to the ∞-norm case, we may rephrase the n LMI conditions in Theorem 5.5 in terms of their alternatives.

Corollary 5.6: All power demands in the 1-norm ball $\mathcal{B}^S_{1,\gamma}(P_\mathrm{c})$ are feasible if and only if for all $i = 1, \ldots, n$ there exist a nonzero positive semi-definite matrix Z such that

$$\mathrm{trace}(ZQ_{P_\mathrm{c}+\gamma S^{-\top} e_i}(e_j)) \leq 0 \text{ for } j = 1, \ldots, n.$$

Proof: The inclusion $\mathcal{B}^S_{1,\gamma}(P_\mathrm{c}) \subseteq \mathcal{F}$ is equivalent to the feasibility of $P_\mathrm{c} + \gamma S^{-\top} e_i$ for $i = 1, \ldots, n$. Apply Theorem 2.3. ∎

In the case of the 1-norm distance to infeasibility, we may compute the distance by computing n semi-definite programs with a linear cost function. Hence, in this case we obtain a computationally tractable method to compute this distance. However, computing the 1-norm distance to infeasibility for large n is significantly more costly compared to the ∞-norm.

Theorem 5.7: The 1-norm distance to infeasbility $d^S_1(P_\mathrm{c}, \mathcal{F}^\mathrm{c})$ may be computed via the mathematical program

$$\begin{aligned} d^S_1(P_\mathrm{c}, \mathcal{F}^\mathrm{c})^{-1} = \max_i \max_{\lambda \in \mathbb{R}^n} \ & (S^{-1}\lambda)_i \\ \text{subject to } & \mathcal{A}_{P_\mathrm{c}}(\lambda) \succeq 0 \end{aligned}$$

which is comprised of n convex semi-definite programs.

Proof: From the proof of Theorem 4.2 we know that the optimal λ is nonzero. Any $\lambda \neq \mathbb{0}$ which satisfies $\mathcal{A}_{P_\mathrm{c}}(\lambda) \succeq 0$ also satisfies $[\lambda] Y_{LL} + Y_{LL}[\lambda] \succeq 0$. By [6, Lemma B.9] this means that $\lambda > \mathbb{0}$. Since also $S^{-1} \geq 0$ by Assumption 3.1, we have that $S^{-1}\lambda \geq \mathbb{0}$ and $\|S^{-1}\lambda\|_\infty = \max_i (S^{-1}\lambda)_i$. Interchanging the maximization over i with the maximization over λ leads to the presented semi-definite program. ∎

The 1-norm distance may also be computed by taking the convex dual of each semi-definite problem in Theorem 5.7.

Corollary 5.8: The 1-norm distance to infeasbility $d^S_1(P_\mathrm{c}, \mathcal{F}^\mathrm{c})$ may be computed through the convex semi-definite program

$$\begin{aligned} d^S_1(P_\mathrm{c}, \mathcal{F}^\mathrm{c})^{-1} = \max_i \min_{Z \in \mathbb{S}^{n+1}_{\succeq 0}} \ & Z_{n+1,n+1} \\ \text{subject to } & \mathrm{trace}(ZQ_{P_\mathrm{c}}(e_j)) = -(S^{-1})_{ij} \\ & \text{for } j = 1, \ldots, n \end{aligned}$$

Proof: Following Section 2.2 of [11], the problems in Theorem 5.7 and Corollary 5.8 are each other's convex dual. By Lemma B.1 we have that the (primal) problem in Theorem 5.3 is strictly feasible. Hence, by [11, Thm. 8], the optimal values of both problems coincide. ∎

C. The p-norm distance to infeasibility for $p \neq 1, \infty$

We will now show that the p-norm distance to infeasibility with $p \neq 1, \infty$ gives rise to a non-convex optimization problem. For vectors $x, y \in \mathbb{R}^n$ which are not each other's scalar multiple, we have for $p \in (1, \infty)$ and $\alpha \in (0, 1)$ that [14, Ch. 11]

$$\|\alpha x + (1-\alpha) y\|_p < \alpha \|x\|_p + (1-\alpha)\|y\|_p.$$

Hence, a minimization over a p-norm with $p \neq 1, \infty$ leads to a strictly convex problem. However, for a maximization problem, as in the case of Theorem 4.2, the same inequality means that the problem is strictly concave, and therefore non-convex. This means that the maximization problem in Theorem 4.2 with $p \neq 1, \infty$ can have multiple optimizers, and that a naive solver for such a problem can converge to a local optimum that is not globally optimal.

The above means that for $p \in (1, \infty)$, computing the p-norm distance to infeasibility is less computationally tractable compared to the ∞-norm and 1-norm case. However, in mathematical programming for control, many problems can be phrased using bilinear matrix inequalities (BMIs), and computational packages for solving these mathematical programs are available [15]. We show that the 2-norm distance to infeasibility can be phrased using BMIs.

Theorem 5.9: The 2-norm distance to infeasbility $d^S_2(P_\mathrm{c}, \mathcal{F}^\mathrm{c})$ may be computed via the semi-definite program

$$\begin{aligned} d^S_2(P_\mathrm{c}, \mathcal{F}^\mathrm{c})^{-2} = \max_{\lambda, \omega, \mu \in \mathbb{R}^n} \ & \sum_i \mu_i \\ \text{subject to } & \mathcal{A}_{P_\mathrm{c}}(\lambda) \succeq 0 \\ & \lambda_i - \sum_j (SS^\top)_{ij} \omega_j \geq 0, \\ & \sum_j (SS^\top)_{ij} \omega_j - \lambda_i \geq 0 \\ & \mu_i - \lambda_i \omega_i \geq 0, \quad \lambda_i \omega_i - \mu_i \geq 0. \end{aligned}$$

Proof: The final line of the semi-definite program are bilinear (matrix) inequalities. The inequalities imply that $\mu_i = (S^{-1}\lambda)_i^2$, and hence the cost function equals $\|S^{-1}\lambda\|_2^2$. The claim follows from Theorem 4.2. ∎

VI. Numerical example

We consider the 39 bus New England system distributed with MATPOWER 7.0 [16] and consider the decoupled reactive power flow in that system [2]. That is, we assume that this AC power grid is lossless by disregarding the line resistances, and we assume that the phase angles between neighbouring buses is zero. We only consider the reactive power flow, which is independent from the active power demands in the system. The reactive power flow in the network corresponds to the power flow in a DC network, where line impedances, reactive power demands and source voltage magnitudes of the 39 bus system correspond to the line conductances in the Kirchhoff matrix Y, the power demands P_c and source voltages V_S of the DC power grid, respectively. We use MOSEK and the Julia programming language to implement the convex semi-definite program of Theorem 5.3 and compute the ∞-norm distance to infeasibility, taking $S = I_{n\times n}$. After 1.5 seconds, the program returns that the ∞-norm distance to infeasibility of P_c is $d^{I_{n\times n}}_\infty(P_\mathrm{c}, \mathcal{F}^\mathrm{c}) = 2.1838930$ p.u.. This implies that we may increase the power demand at each bus by 2.0 p.u. and still have a feasible power flow. By doing so we compute $d^{I_{n\times n}}_\infty(P_\mathrm{c} + 2.0\mathbb{1}, \mathcal{F}^\mathrm{c}) = 0.1838941$ p.u., which is 2.0 p.u. less than the original distance to infeasibility, as we anticipated. When we increase all power demands by 2.3

p.u., the solver tells us that the problem is not feasible, meaning that the power flow is not feasible.

VII. Conclusion

This paper introduced a distance-to-infeasibility metric that functions as a voltage stability index for how close a system is to voltage collapse. We showed that the p-norm distance to infeasibility can be formulated as a mathematical program, which for $p = \infty$ and $p = 1$ is equivalent to (a/multiple) optimization problem(s) with linear cost and LMI constraints, and therefore computationally tractable. For $p \in (1, \infty)$ the mathematical program is non-convex, but for $p = 2$ is equivalent to an optimization problem(s) with linear cost and BMI constraints. This paper also introduced a guarantee for the robustness of power flow feasibility, which is shown to be equivalent to the infeasibility of a matrix inequality. For $p = \infty$ and $p = 1$ we showed that this guarantee can be phrased as (multiple) linear matrix inequality(s). Alternatives and dual problems for the aforementioned convex semi-definite problems were also given. Finally we gave a numerical example of the computation of the ∞-norm distance to infeasibility for a benchmark power grid.

Appendix I

Throughout this section we let $A_0, \ldots, A_k \in \mathbb{R}^{N \times N}$ be symmetric matrices. For $x \in \mathbb{R}^K$ we define

$$\mathcal{A}(x) := A_0 + \textstyle\sum_{i=1}^K A_i x_i$$

Lemma A.1: One and only one of the following holds:

i) There exists a vector $x > \mathbb{0}$ such that $\mathcal{A}(x) \succ 0$;

ii) There exists a positive semi-definite symmetric matrix Z satisfying $\operatorname{trace}(ZA_i) \le 0$ for $i = 0, \ldots, k$.

Proof: Let $\widehat{A}_0, \ldots, \widehat{A}_k \in \mathbb{R}^{(N+K)\times(N+K)}$ and $\widehat{\mathcal{A}}(x)$ by

$$\widehat{A}_0 := \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}; \ \widehat{A}_i := \begin{pmatrix} A_i & 0 \\ 0 & [e_i] \end{pmatrix}; \ \widehat{\mathcal{A}}(x) := \widehat{A}_0 + \sum_{i=1}^K \widehat{A}_i x_i.$$

By [11, Thm. 1], one and only one of the following holds:

iii) There exists a vector $x \in \mathbb{R}^k$ such that $\widehat{\mathcal{A}}(x) \succ 0$;

iv) There exists a positive semi-definite symmetric matrix $\widehat{Z}$ satisfying $\operatorname{trace}(\widehat{Z}\widehat{A}_0) \le 0$ and $\operatorname{trace}(\widehat{Z}\widehat{A}_i) = 0$ for $i = 1, \ldots, k$.

Clearly i) and iii) are equivalent. Consider the partition

$$\widehat{Z} = \begin{pmatrix} \widehat{Z}_{11} & \widehat{Z}_{12} \\ \widehat{Z}_{21} & \widehat{Z}_{22} \end{pmatrix}$$

with $\widehat{Z}_{11} \in \mathbb{R}^{N \times N}$, and note that

$$\operatorname{trace}(\widehat{Z}\widehat{A}_0) = \operatorname{trace}(\widehat{Z}_{11} A_0);$$
$$\operatorname{trace}(\widehat{Z}\widehat{A}_i) = \operatorname{trace}(\widehat{Z}_{11} A_i) + Z_{22,ii}.$$

If iv) holds then $Z_{22,ii} \ge 0$ and it follows that iv) implies ii) by taking $Z = Z_{11}$. Conversely, if ii) holds then by taking $\widehat{Z}$ such that $\widehat{Z}_{11} = Z$, $\widehat{Z}_{12} = \widehat{Z}_{21}^\top = 0$ and

$$\widehat{Z}_{22} = -\operatorname{diag}(\operatorname{trace}(ZA_1), \ldots, \operatorname{trace}(ZA_k)),$$

we have that Z is positive semi-definite and that iv) holds. Hence ii) and iv) are equivalent, which proves the claim. ■

Appendix II

Lemma B.1: For all $P_c \in \mathbb{R}^n$ there exists a λ such that

$$\mathcal{A}_{P_c}(\lambda) \succ 0.$$

Proof: If P_c is not feasible, then by Theorem 2.2 there exists a λ such that $Q_{P_c}(\lambda) \succ 0$. Since $\mathcal{A}_{P_c}(\lambda) \succeq Q_{P_c}(\lambda)$, we have that also $\mathcal{A}_{P_c}(\lambda) \succ 0$.

Suppose P_c is feasible. We define

$$P_{c,\max} := \tfrac{1}{4}[Y_{LS}V_S] Y_{LL}^{-1} Y_{LS} V_S. \tag{8}$$

By [6, Lemma 2.18] we have $\mathbb{1}^\top(P_{c,\max} - P_c) \ge 0$. Let $\alpha := \mathbb{1}^\top(P_{c,\max} - P_c) + \varepsilon$ for some $\varepsilon > 0$ and $\lambda = \alpha^{-1}\mathbb{1}$, then

$$\alpha \mathcal{A}_{P_c}(\lambda) = \begin{pmatrix} Y_{LL} & \frac{1}{2} Y_{LS} V_S \\ \frac{1}{2}(Y_{LS}V_S)^\top & \mathbb{1}^\top P_{c,\max} + \varepsilon \end{pmatrix}.$$

Note that $\alpha > 0$. The Schur complement of $\alpha\mathcal{A}_{P_c}(\lambda)$ with respect to the positive definite submatrix Y_{LL} is positive definite since

$$\mathbb{1}^\top P_{c,\max} - \tfrac{1}{4}(Y_{LS}V_S)^\top Y_{LL}^{-1} Y_{LS} V_S + \varepsilon = \varepsilon > 0,$$

where we substituted (8). Thus $\mathcal{A}_{P_c}(\lambda) \succ 0$ holds. ■

References

[1] D. K. Molzahn, I. A. Hiskens *et al.*, "A survey of relaxations and approximations of the power flow equations," *Found. trends electr. energy syst.*, vol. 4, no. 1-2, pp. 1–221, 2019.

[2] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Voltage collapse in complex power grids," *Nat. Commun.*, vol. 7, no. 10790, 2016.

[3] S. Bolognani and S. Zampieri, "On the existence and linear approximation of the power flow solution in power distribution networks," *IEEE Trans. Power Syst.*, vol. 31, no. 1, pp. 163–172, 2015.

[4] Z. Liu, M. Su, Y. Sun, W. Yuan, H. Han, and J. Feng, "Existence and stability of equilibrium of dc microgrid with constant power loads," *IEEE Trans. Power Syst.*, vol. 33, no. 6, pp. 6999–7010, 2018.

[5] Z. Xia, M. Su, Z. Liu, R. Liu, and Y. Liu, "Existence conditions and stability for the power-flow of dc micro-grids with cpls," *IEEE Transactions on Smart Grid*, vol. 13, no. 6, pp. 4284–4299, 2022.

[6] M. Jeeninga, C. De Persis, and A. Van der Schaft, "DC Power Grids with constant-power loads—Part I: A full characterization of power flow feasibility, long-term voltage stability, and their correspondence," *IEEE Trans. Automat. Contr.*, vol. 68, no. 1, pp. 2–17, 2022.

[7] ——, "DC power grids with constant-power loads—Part II: Nonnegative power demands, conditions for feasibility, and high-voltage solutions," *IEEE Trans. Automat. Contr.*, vol. 68, no. 1, pp. 18–30, 2022.

[8] J. Modarresi, E. Gholipour, and A. Khodabakhshian, "A comprehensive review of the voltage stability indices," *Renewable and Sustainable Energy Reviews*, vol. 63, pp. 1–12, 2016.

[9] M. Jeeninga, "Braess' paradox for power flow feasibility and parametric uncertainties in dc power grids with constant-power loads," *Systems & Control Letters*, vol. 161, p. 105146, 2022.

[10] V. Ajjarapu, *Computational techniques for voltage stability assessment and control*. Springer, 2007.

[11] V. Balakrishnan and L. Vandenberghe, "Semidefinite programming duality and linear time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 1, pp. 30–41, 2003.

[12] M. Fiedler, *Special matrices and their applications in numerical mathematics*. Kluwer Academic Publishers, 1986.

[13] S. Roman, *Advanced linear algebra*. Springer, 2008, vol. 3.

[14] N. Carothers, *A short course on Banach space theory*. Cambridge University Press, 2004, vol. 64.

[15] J. G. VanAntwerp and R. D. Braatz, "A tutorial on linear and bilinear matrix inequalities," *Journal of process control*, vol. 10, no. 4, pp. 363–385, 2000.

[16] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "Matpower: Steady-state operations, planning, and analysis tools for power systems research and education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, 2011.