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Model Reference Adaptive Control with Proportional Integral Adaptation Law

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***Abstract*— In this contribution we extend the classical update law of the certainty equivalence based (indirect) model reference adaptive control (MRAC) with an additional feedthrough error term. We use a linear state-space representation of the parameter estimation error in order to construct a Lyapunov function. Without unstructured uncertainties, we can show asymptotic stability of the closed loop. Even for relatively high adaptation gains the algorithm can reduce the oscillations of the state variables. In case of unstructured input uncertainties, the additional term reduces the model following error drastically without increasing the control signal for low frequency unstructured uncertainties. For the simplified pure linear case we analyze the effect of the additional proportional error feedthrough in the adaptation law showing its similarity to a disturbance observer. As a simulation example the wing-rock dynamics are reconsidered.**

I. Introduction

The model reference adaptive control (MRAC) scheme is a well-established and widely used control algorithm to handle structured uncertainties with unknown parameters that linearly enter the system [1], [2]. The classical MRAC, however, shows certain disadvantages. First of all, high adaptation rates lead to undesired oscillations. Furthermore the robustness to unstructured uncertainties is minor since they lead to drifting of the estimated parameters. To overcome these drawbacks various extensions of the approach can be found in the literature, for example the σ-modification [3], the e-modification [4], the projection operator extension [5] and the $\mathcal{L}_1$-extension [6]. In the context of $\mathcal{L}_1$-adaptive control also proportional update laws can be found [7]. Other publications consider derivative-free MRAC approaches [8] or fractional-order update laws [9], [10], [11]. The results presented in [11] show that the robustness to unstructured uncertainties can be increased by using a low order of differentiation, hence, a proportional update law. However, with a proportional update law the tracking error does not converge anymore. For these reasons the aim of this work is to combine the classical MRAC with a proportional part in the update law, introducing a transfer zero, so to speak.

For incorporating this feedthrough term in the adaptation law we introduce a linear higher-order state-space representation of the assumed (piecewise) constant parameters. Compared to [12] the order of this representation can be chosen by the practitioner. This is important since it allows us to reduce the bounds on the model following error because additional negative terms in the derivative of the Lyapunov function can be generated. The main effect of the proposed algorithm stems from the feedthrough term contained in the higher order state-space representation. It allows for increased adaptation rates and improves the transient performance significantly without additional implementation requirements.

The paper is structured as follows. After formulating the problem in Section II we recall the derivation of the classical indirect MRAC algorithm, as we will follow similar steps to derive a higher-order adaptive-law subsequently. Applying the extended representation allows to derive a higher-order adaptation law including a proportional term in Section III. These extensions increase the robustness of the control algorithm with respect to unstructured input uncertainties. This is made evident by inspecting the derivative of the Lyapunov function and shall also be explained in the frequency domain for a simplified linear example in Section III-B. The results are illustrated in Section IV by controlling the wing-rock dynamics, presented in [13]. Finally, we draw our conclusions in Section V.

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II. Problem Formulation

Let us consider the system given by

$$\Sigma\colon \begin{cases} \dot{x}(t) = Ax(t) + B\left(u(t) + \Theta^\top \beta(x) + \eta(t)\right) & \text{(1a)} \\ y(t) = Cx(t) & \text{(1b)} \end{cases}$$

with state $x(t) \in \mathbb{R}^n$, scalar input and output $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, respectively, and corresponding known matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. We suppose the pair (A, B) is controllable. The matched disturbance is separated into a structured part with a known regressor $\beta(x) \in \mathbb{R}^p$ and unknown, linearly entering constant parameters $\Theta \in \mathbb{R}^p$, and the unstructured but bounded uncertainty $\eta(t) \in \mathbb{R}$ with $|\eta(t)| \leq \bar{\eta} < \infty$ for all $t \geq 0$. The regressor is bounded if the state is bounded, i.e.

$$x(\cdot) \in \mathcal{L}_\infty \quad \Longrightarrow \quad \beta(x(\cdot)) \in \mathcal{L}_\infty. \tag{2}$$

Remark 1. *The assumption that the matrix A is known can be relaxed if the uncertainty of the matrix ΔA can be parameterized in terms of the input matrix, i.e.*

$$A_{\text{real}} = A + \Delta A \qquad \text{with} \qquad \Delta A = \alpha B, \quad \alpha \in \mathbb{R}.$$

By this parameterization the additional uncertainty can be included in the regressor $\bar{\beta}(x) = \begin{pmatrix}\beta^\top(x) & x^\top\end{pmatrix}^\top$. With a similar extension as shown in [14] the assumptions on the input matrix can be relaxed as well, such that only the sign of the input matrix is required. In addition to that an extension towards multiple inputs is straightforward.

The control law contains a nominal and adaptive part, i.e.

$$
\begin{aligned}
u(t) &= u_{\mathrm{nom}}(t) + u_{\mathrm{ad}}(t) \\
&= Kx(t) + Fr(t) - \hat{\Theta}^\top(t)\beta(x(t)).
\end{aligned} \tag{3}
$$

The desired closed-loop dynamics are defined by the reference model

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{4a}$$

$$y_m(t) = C x_m(t). \tag{4b}$$

The dynamics of the error $e(t) = x(t) - x_m(t)$ result in

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{x}_m(t) \\
&= Ax(t) + B(Kx(t) + Fr(t) + \eta(t) + \Theta^\top\beta(x)) \\
&\quad - B\hat{\Theta}^\top(t)\beta(x) - A_m x_m(t) - B_m r(t)
\end{aligned} \tag{5}
$$

with $A_m = A + BK$, $B_m = BF$, $\tilde{\Theta}(t) = \hat{\Theta}(t) - \Theta$ s.t.

$$\dot{e}(t) = A_m e(t) - B\tilde{\Theta}^\top(t)\beta(x) + B\eta(t). \tag{6}$$

A. Classical Indirect MRAC

Up to this point we only show the standard procedure to derive a certainty equivalence based adaptive controller [13], [1]. To show the stability of the classical indirect MRAC a Lyapunov function is applied which depends on the model following error $e(t)$ and the estimation error $\tilde{\Theta}(t)$

$$V_{\mathrm{MRAC}}(e,\tilde{\Theta}) = e^\top(t)Pe(t) + \tilde{\Theta}^\top(t)\,\Gamma^{-1}\tilde{\Theta}(t) \tag{7}$$

with matrix $P = P^\top > 0$ and learning rate $\Gamma = \Gamma^\top > 0$. Its time derivative reads

$$
\begin{aligned}
\dot{V}_{\mathrm{MRAC}}(e,\tilde{\Theta}) &= e^\top\big(A_m^\top P + PA_m\big)e \\
&\quad + 2e^\top PB\big(-\tilde{\Theta}^\top\beta(x) + \eta\big) + 2\tilde{\Theta}^\top\Gamma^{-1}\dot{\tilde{\Theta}}.
\end{aligned} \tag{8}
$$

As the unknown parameters Θ are constant, the derivative of the parameter estimation error is given by the estimation dynamics, i.e. $\dot{\tilde{\Theta}}(t) = \dot{\hat{\Theta}}(t)$. The nominal part is covered by the linear Lyapunov equation $A_m^\top P + PA_m = -Q$ and as A_m is Hurwitz $Q = Q^\top > 0$ is positive definite. Inserting this in (8) leads to

$$
\begin{aligned}
\dot{V}_{\mathrm{MRAC}}(e,\tilde{\Theta}) &= -e^\top Qe + 2e^\top PB\eta(t) \\
&\quad + 2\tilde{\Theta}^\top \underbrace{\left(-\beta(x)e^\top PB + \Gamma^{-1}\dot{\hat{\Theta}}\right)}_{=0}.
\end{aligned} \tag{9}
$$

This results in the parameter update law

$$\dot{\tilde{\Theta}}(t) = \dot{\hat{\Theta}}(t) = \Gamma\beta(x)e^\top(t)PB \tag{10}$$

which renders the derivative of the Lyapunov function negative semidefinite in the undisturbed case, $\eta \equiv 0$, i.e.

$$\dot{V}_{\mathrm{MRAC}}(e,\tilde{\Theta}) = -e^\top Qe \le 0.$$

As the derivative of this Lyapunov function is only negative semidefinite, we shall apply Barbălat's lemma [15] to show that the error e will converge asymptotically to zero.

In the case of bounded unstructured uncertainties the adaptation law has to be extended in order to avoid the drifting of the estimation parameters and guarantee the uniform ultimate boundedness of the error. Different approaches are given in the literature. The σ-modification [3] introduces a damping to the adaptation law

$$\dot{\hat{\Theta}}(t) = \Gamma\left(\beta(x)e^\top(t)PB - \sigma\hat{\Theta}(t)\right) \tag{11}$$

with $\sigma > 0$. This decreases the adaptation rates, however it stabilizes the parameter estimates. Further approaches extend this σ-modification with a nonlinear term [4] or rely on the projection operator to keep the parameter estimates within a predefined range [5].

III. Higher-Order Indirect MRAC

In comparison to the standard MRAC we now consider an extended state-space to describe the evolution of the parameter estimation error $\tilde{\Theta}(t)$. Yet we still require (piecewise) constant parameters $\Theta(t)$.

The constant parameters are understood as the output of an artificial linear system with a separate state $z_i \in \mathbb{R}^q$ for each unknown parameter Θ_i $(i = 1,\dots,p)$ such that we have

$$\dot{z}_i(t) = A_z z_i(t) + B_z f_i(e,x) \tag{12a}$$

$$\Theta_i(t) = C_z z_i(t) + D_z f_i(e,x), \tag{12b}$$

with

$$
\begin{aligned}
&A_z = \mathrm{diag}\left(\alpha_1,\dots,\alpha_q\right), \quad \alpha_i < 0, i = 2,\dots,q \\
&B_z = \left(1,\dots,1\right)^\top \\
&C_z = \left(1, c_2 \dots, c_q\right), \quad c_i > 0, \\
&D_z \ge 0.
\end{aligned}
$$

To generate a constant output we set $\alpha_1 = 0$, $f_i(e,x) = 0$ and the initial state to $z_i(0) = (\Theta_i, 0,\dots,0)^\top$. With different initial conditions also time varying parameters can be represented. The dynamics of the estimated parameters are given by a similar system, that is

$$\dot{\hat{z}}_i(t) = A_z\hat{z}_i(t) + B_z\hat{f}_i(e,x) \tag{13a}$$

$$\hat{\Theta}_i(t) = C_z\hat{z}_i(t) + D_z\hat{f}_i(e,x). \tag{13b}$$

Hence the estimation error is given by

$$
\begin{aligned}
\tilde{\Theta}_i(t) &= \hat{\Theta}_i(t) - \Theta_i(t) \\
&= C_z(\hat{z}_i(t) - z_i(t)) + D_z\Big(\hat{f}_i(e,x) - \cancelto{0}{f_i(e,x)}\Big) \\
&= C_z\tilde{z}_i(t) + D_z\hat{f}_i(e,x)
\end{aligned} \tag{14}
$$

with the extended dynamics

$$\dot{\tilde{z}}_i(t) = A_z\tilde{z}_i(t) + B_z\hat{f}_i(e,x). \tag{15}$$

Note that higher order adaptation laws can also be applied in the case of time-varying parameters [12].

In order to derive a stabilizing adaptation law $\hat{f}(e,x)$, consider the Lyapunov function

$$V_{\mathrm{HO}}(e,\tilde{z}) = e^\top Pe + \sum_{i=1}^{p} C_z\gamma_i^{-1}\tilde{z}_i \odot \tilde{z}_i \;>\; 0, \tag{16}$$

where $\odot$ signifies the element-wise Hadamard product. The first derivative of this Lyapunov function then reads

$$
\begin{aligned}
\dot{V}_{\mathrm{HO}}(e,z) &= e^\top\left(A_m^\top P + PA_m\right)e + 2e^\top PB\left(\eta - \tilde{\Theta}^\top\beta(x)\right) \\
&\quad + 2\sum_{i=1}^{p} C_z\gamma_i^{-1}\tilde{z}_i \odot \left(A_z\tilde{z}_i + B_z\hat{f}_i(e,x)\right).
\end{aligned}
$$

By design of the nominal control, the matrix P satisfies the Lyapunov equation $A_m^\top P + PA_m = -Q$ with $Q = Q^\top > 0$. With the structure of A_z (diagonal) and B_z (only ones) the Hadamard product may be simplified, yielding

$$\begin{aligned}\dot V_{\mathrm{HO}}(e,\tilde z) = &- e^\top Qe + 2e^\top PB\eta \\ &- 2e^\top PB\tilde\Theta^\top\beta(x) + 2\sum_{i=1}^p C_z\gamma_i^{-1}A_z\tilde z_i \odot \tilde z_i \\ &+ 2\sum_{i=1}^p \gamma_i^{-1}C_z\tilde z_i\hat f_i(e,x).\end{aligned}$$

So as to render the derivative independent of the unknown estimation error $\tilde\Theta$ we insert (14) and obtain

$$\begin{aligned}\dot V_{\mathrm{HO}}(e,\tilde z) = &- e^\top Qe + 2e^\top PB\eta \\ &- 2e^\top PB\tilde\Theta^\top\beta(x) + 2\sum_{i=1}^p C_z\gamma_i^{-1}A_z\tilde z_i \odot \tilde z_i \\ &+ 2\sum_{i=1}^p \gamma_i^{-1}\left(\tilde\Theta_i - D_z\hat f_i(e,x)\right)\hat f_i(e,x).\end{aligned}$$

In contrast to the classical approach, here we use an integrated version of the estimation error. Therefore it is essential that the real parameters can be described by the underlying artificial dynamics leading to an integrator in A_z. Using vector notation $\hat f(e,x) = (\hat f_1(e,x) \ \ldots \ \hat f_p(e,x))^\top$ and $\Gamma = \mathrm{diag}(\gamma_1,\ldots,\gamma_p)$ gives

$$\begin{aligned}\dot V_{\mathrm{HO}}(e,z) = &- e^\top Qe + 2e^\top PB\eta \\ &- 2\tilde\Theta^\top\beta(x)e^\top PB + 2\sum_{i=1}^p C_z\gamma_i^{-1}A_z\tilde z_i \odot \tilde z_i \\ &+ 2\tilde\Theta^\top\Gamma^{-1}\hat f(e,x) - 2D_z\hat f^\top(e,x)\Gamma^{-1}\hat f(e,x).\end{aligned}$$

Hence the adaptation function $\hat f$ is similar to the classical MRAC and renders the derivative independent of the unknown estimation error $\tilde\Theta$, i.e.

$$\hat f(e,x) = \Gamma\beta(x)e^\top(t)PB \tag{17}$$

such that the derivative of the Lyapunov function for $\eta = 0$ is negative semidefinite, see

$$\begin{aligned}\dot V_{\mathrm{HO}}(e,z) = &- e^\top Qe + \underbrace{2\sum_{i=1}^p C_z\gamma_i^{-1}A_z\tilde z_i \odot \tilde z_i}_{\le 0} \\ &+ 2\tilde\Theta^\top\underbrace{\left(-\beta(x)e^\top PB + \Gamma^{-1}\Gamma\beta(x)e^\top PB\right)}_{=0} \\ &\underbrace{-2D_z\left(\Gamma\beta(x)e^\top(t)PB\right)^\top\left(\beta(x)e^\top(t)PB\right)}_{<0}.\end{aligned}$$

Note that the semidefiniteness only occurs due to the integrator mode $\alpha_1 = 0$ in matrix A_z and the derivative does not depend on the first states $\tilde z_i$. Therefore the convergence of the model following error is again a result of the application of Barbălat's lemma. Compared to the classical MRAC this equation features additional negative terms which can be used to dominate the unstructured uncertainty and reduce the ultimate bounds of the model following error.

Remark 2. *For $A_z = 0$, $C_z = 1$ and $D_z = 0$ the classical MRAC is recovered. For $A_z = -\sigma < 0$, $C_z = 1$ and $D_z = 0$ the σ-modification [3] is recovered. As the integration mode ($\alpha_1 = 0$) is missing, the σ-modified MRAC will only lead to uniform ultimate boundedness of the model following error. Even in the case without input disturbances, $\eta(t) = 0$, the σ-modification will not lead to asymptotic stability.*

Remark 3. *Without the feedthrough term in* (13) *($D_z = 0$) this approach can be interpreted as an approximation of the fractional-order adaptation law presented in [11]*

$$\mathcal{D}^\alpha\hat\Theta(t) = \Gamma\beta(x)e^\top(t)PB.$$

The states z_i represent isolated modes of the distributed state ζ_i mimicking the infinite memory of the nonlocal fractional-order operator

$$\dot\zeta_i(\omega,t) = -\omega\zeta_i(\omega,t) + \Gamma\beta_i(x)e^\top(t)PB, \omega\in[0,\infty).$$

The fractional-order integral is then given by an integral with respect to ω

$$\hat\Theta_i(t) = \mathcal{I}^\alpha\left(\Gamma\beta_i(x)e^\top(t)PB\right) = \int_0^\infty \underbrace{\frac{\sin(\alpha\pi)}{\pi\omega^\alpha}}_{\mu_\alpha(\omega)}\zeta(\omega,t)\mathrm{d}\omega.$$

Again, the multiplication with the output matrix C_z can be interpreted as discretized version of this integral with the kernel $\mu_\alpha(\omega)$.

Remark 4 (Proportional-Integral MRAC)**.** *In order to keep the computational costs low, we will focus on the simple choice of parameters $A_z = 0$ and $D_z = K_p \ge 0$, combining an integral and the proportional adaptation law*

$$\dot{\hat\Theta}(t) = \Gamma\beta(x(t))e^\top(t)PB + \Gamma K_p\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta(x(t))e^\top(t)PB\right)$$

with only two remaining tuning parameters $\Gamma > 0$ and K_p.

A. Boundedness

As discussed in [16] additional unstructured uncertainties $\eta \not\equiv 0$ in classical indirect MRAC lead to drifting of the estimation parameters $\hat\Theta$, rendering the model following error unstable. Regarding the proposed higher order adaptation law (13) with (17) this effect is only reduced because a major part of the adaptation is covered by the proportional part. However, the pure integrator remains unstable. For this reason we have to change the integrator in the matrix A_z to achieve (uniform) ultimate boundedness (as defined in [15]) of the model following error e and the estimation error $\hat z_i$.

In order to stabilize the estimator states $\hat z$ the first entry of the diagonal matrix A_z in (13a) has to be modified. This leads to

$$\dot{\hat z}_i(t) = (A_z + A_\sigma)\hat z_i + B_z\hat f_i(e,x) \tag{18}$$

with $\sigma > 0$ and $A_\sigma = \mathrm{diag}\left(-\sigma, 0, \ldots, 0\right) \in \mathbb{R}^{q\times q}$. This changes the dynamics of $\tilde z_i$ to

$$\begin{aligned}\dot{\tilde z}(t) &= \dot{\hat z}(t) - \dot z(t) \\ &= (A_z + A_\sigma)\hat z_i(t) + B_z\hat f_i(e,x) - A_z z_i(t) \\ &= A_z\tilde z_i(t) + B_z\hat f_i(e,x) + A_\sigma z_i(t).\end{aligned} \tag{19}$$

Note that the real parameters are constant leading to $\Theta_i = z_{i,1} =$ const. With this modification the derivative of the Lyapunov function (16) becomes

$$\dot{V}_{\mathrm{HO}}(e,z) = e^\top \left(A_m^\top P + PA_m\right) e + 2e^\top PB \left(\eta - \tilde{\Theta}^\top \beta(x)\right) + 2\sum_{i=1}^{p} C_z \gamma_i^{-1} \tilde{z}_i \odot \left(\tilde{A}_z \tilde{z}_i + B_z \hat{f}_i(e,x) + A_\sigma z_i\right).$$

With $A_\sigma z_i = \begin{pmatrix} -\sigma z_{i,1} & 0 & \dots & 0 \end{pmatrix}^\top$ it simplifies to

$$\begin{aligned}\dot{V}_{\mathrm{HO}}(e,\tilde{z}) =& -e^\top Q e + 2e^\top PB\eta \\ & - 2e^\top PB\tilde{\Theta}^\top \beta(x) + 2\sum_{i=1}^{p} C_z \gamma_i^{-1} A_z \tilde{z}_i \odot \tilde{z}_i \\ & + 2\sum_{i=1}^{p} \gamma_i^{-1} C_z \tilde{z}_i \hat{f}_i(e,x) - 2\sum_{i=1}^{p} \gamma_i^{-1} \tilde{z}_{i,1} \sigma z_{i,1}.\end{aligned}$$

Let the first element of C_z be normalized. Then we have $z_{i,1} = \Theta_i$. As the output equation (13b) remains unchanged, we use the same steps as previously which leads to

$$\begin{aligned}\dot{V}_{\mathrm{HO}}(e,\tilde{z}) =& -e^\top Q e + 2e^\top PB\eta - 2\sum_{i=1}^{p} \gamma_i^{-1} \tilde{z}_{i,1} \sigma \Theta_i \\ & - 2e^\top PB\tilde{\Theta}^\top \beta(x) + 2\sum_{i=1}^{p} C_z \gamma_i^{-1} A_z \tilde{z}_i \odot \tilde{z}_i \\ & + 2\sum_{i=1}^{p} \gamma_i^{-1} \left(\tilde{\Theta}_i - D_z \hat{f}_i(e,x)\right) \hat{f}_i(e,x).\end{aligned}$$

We change to vector notation by introducing a diagonal matrix $\mathrm{diag}(C_z) \in \mathbb{R}^{q \times q}$ the elements of which are made up by the output row vector C_z. Inserting the adaptation law (17) gives

$$\begin{aligned}\dot{V}_{\mathrm{HO}}(e,\tilde{z}) =& -e^\top Q e + 2e^\top PB\eta - 2\sigma \sum_{i=1}^{p} \gamma_i^{-1} \tilde{z}_{i,1} \Theta_i \\ & - 2\sum_{i=1}^{p} \gamma_i^{-1} \tilde{z}_i^\top \,\mathrm{diag}(C_z) |A_z| \tilde{z}_i \\ & - 2D_z \left(\Gamma \beta(x) e^\top PB\right)^\top \left(\beta(x) e^\top PB\right).\end{aligned}$$

The terms of unknown sign are bounded form above and the lower bounds are introduced for the negative terms to derive an upper bound of the derivative

$$\begin{aligned}\dot{V}_{\mathrm{HO}}(e,\tilde{z}) \leq& -\lambda_{\min}(Q)\|e\|_2^2 + 2\bar{\eta}\|PB\|_2\|e\|_2 \\ & + 2\sigma \sum_{i=1}^{p} \gamma_i^{-1} |\Theta_i| \|\tilde{z}_i\|_2 \\ & - 2\sum_{i=1}^{p} \gamma_i^{-1} \underbrace{\lambda_{\min}\left(\mathrm{diag}(C_z)|A_z|\right)}_{a_1 > 0} \|\tilde{z}_i\|_2^2 \\ & - 2D_z \underbrace{\lambda_{\min}\left(\beta^\top(x) \Gamma \beta(x) PBB^\top P\right)}_{=0} \|e\|_2^2.\end{aligned}$$

Note that in general ($n > 1$) for the single input case the last term has no influence, as the minimal eigenvalue of $\beta^\top(x)\Gamma\beta(x)PBB^\top P$ is zero. Hence the proportional term D_z has no influence on these conservative bounds. Finally we have

$$\begin{aligned}\dot{V}_{\mathrm{HO}}(e,\tilde{z}) \leq& -\left(\sqrt{\lambda_{\min}(Q)}\|e\| - \frac{\bar{\eta}\|PB\|_2}{\sqrt{\lambda_{\min}(Q)}}\right)^2 + \\ & - 2\sum_{i=1}^{p} \gamma_i^{-1} \left(\sqrt{a_1}\,\|\tilde{z}_i\|_2 - \frac{|\sigma|\,|\Theta|_i}{2\sqrt{a_1}}\right)^2 + \\ & + 2\sum_{i=1}^{p} \gamma_i^{-1} \frac{\sigma^2\,|\Theta|_i^2}{4a_1} + \frac{\bar{\eta}^2\|PB\|_2^2}{\lambda_{\min}(Q)}.\end{aligned}$$

Hence the derivative of the Lyapunov function is negative if

$$\|e\|_2 > r_e + \varepsilon_e \quad \text{or} \quad \|\tilde{z}_i\|_2 > r_{\tilde{z},i} + \varepsilon_{\tilde{z},i}, \quad i = 1,\dots,p,$$

where ε_e ans $\varepsilon_{\tilde{z},i}$ are arbitrary small positive constants and respective bounds are given by

$$r_e = \sqrt{2\sum_{i=1}^{p} \gamma_i^{-1} \frac{\sigma^2\,|\Theta|_i^2}{4a_1\lambda_{\min}(Q)} + \frac{\bar{\eta}^2\|PB\|_2^2}{\lambda_{\min}^2(Q)}} + \frac{\bar{\eta}\|PB\|_2}{\lambda_{\min}(Q)},$$

$$r_{\tilde{z},i} = \sqrt{2\sum_{i=1}^{p} \gamma_i^{-1} \frac{\sigma^2\,|\Theta|_i^2}{4a_1^2(Q)} + \frac{\bar{\eta}^2\|PB\|_2^2}{a_1\lambda_{\min}(Q)}} + \frac{\sigma\,|\Theta|_i}{2a_1}.$$

The ultimate bound on the error is given by

$$\|e\|_2^2 \leq \frac{1}{\lambda_{\min}(P)} \left(\frac{\lambda_{\max}(P)}{r_e^{-2}} + \sum_{i=1}^{p} \gamma_i^{-1} \max(c_i) r_{\tilde{z},i}^2\right). \quad (20)$$

Besides the nominal control defining P, the main influence can be attributed to the learning rates γ_i. With larger values of γ_i the second term in (20) is reduced. In addition, for large γ_i the bound r_e is mostly dominated by the upper bound of the unstructured uncertainty $\bar{\eta}$.

B. Linear Case — Frequency Domain Analysis

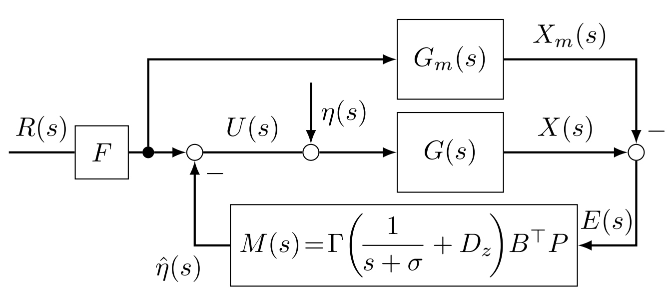

Fig. 1. Block diagram for the linear MRAC with proportional error feedback ($D_z \neq 0$) and σ-modification ($\sigma \neq 0$).

Following the ideas presented in [17] we analyze the effect of the additional proportional term of the adaptation law and the closed-loop dynamics in the frequency domain. This is only possible when the model parameters A and B are known and no further nonlinearities are present, such that the remaining regressor is one, i.e. $\beta(x) = 1$. This amounts to the control loop structure presented in Figure 1 with $G_m(s) = G(s) = (sI - A_m)^{-1}B$. Although the approach requires constant parameters, we may interpret this single unknown parameter as an input disturbance $\Theta + \eta(t) = \bar{\eta}(t)$.

Hence we can compute the transfer function relating the input disturbance $\bar{\eta}(t)$ with its estimate $\hat{\eta}(t)$, i.e.

$$G_{\hat{\eta}\bar{\eta}}(s) = \frac{M(s)G(s)}{1 + M(s)G(s)}, \quad M(s) = \Gamma\left(\frac{1}{s+\sigma} + D_z\right)B^\top P. \tag{21}$$

Here we have three tuning parameters: σ determines the pole location, D_z influences the transfer zero and the learning rate Γ scales the stationary gain of $M(s)$. Further simplifying the analysis we consider the first-order process [17]

$$G(s) = G_m(s) = \frac{b}{s+a} = \left[\begin{array}{c|c} -a & b \\ \hline 1 & 0 \end{array}\right] \tag{22}$$

with $P = 1$ leading to

$$G_{\hat{\eta}\bar{\eta}}(s) = \frac{\Gamma(D_z s + D_z\sigma + 1)b^2}{s^2 + (a + \sigma + \Gamma D_z b^2)s + (\sigma a + \Gamma(D_z\sigma + 1)b^2)}.$$

Note that only without the σ-modification the stationary gain is one, i.e.

$$G_{\hat{\eta}\bar{\eta}}(0) = \frac{\Gamma b^2(D_z\sigma + 1)}{\sigma a + \Gamma b^2(D_z\sigma + 1)} \overset{\sigma=0}{=} 1.$$

The stationary gain gives some guidance towards the tuning of the parameters. Only with large $\Gamma \gg 0$ or $D_z \gg 0$ and small $\sigma \rightarrow 0^+$ low frequency disturbances can be estimated with sufficient accuracy, see Figure 2. For $\sigma > 0$ the deviation for low frequencies is clearly visible if not counteracted by a higher learning rate Γ or feed-through D_z.

For the classical MRAC ($D_z = 0$ and $\sigma = 0$), increasing Γ leads to two under-damped poles as shown in Figure 2 (upper Bode diagram). The proportional term D_z as well as the damping term σ are counteracting this effect as both terms add damping to the poles

$$s_{p,1/2} = -\frac{a + \sigma + b^2\Gamma D_z}{2} \pm \sqrt{\frac{(a + \sigma + b^2\Gamma D_z)^2}{4} - (\sigma a + \Gamma(D_z\sigma + 1)b^2)}.$$

This added damping renders better tracking performance for high adaptation gains, i.e. reducing oscillations and overshooting. Furthermore as D_z increases the first pole tends to zero. The transfer zero $s_0 = -(D_z\sigma + 1)/D_z$ tends towards $s_0|_{D_z\rightarrow\infty} = -\sigma$. As σ is small, this leads to a pseudo compensation of the low frequency pole. Hence the cutoff frequency of this filter quasi shifted to the fast pole and the new approach is able to compensate disturbances of higher frequencies. For $a = 2$ and $b = 1$ the amplitude response is shown in Figure 2 for different combinations of the parameters Γ, σ and D_z.

The pure σ-modification reduces the resonance peak slightly without increasing the bandwidth. For a large damping, higher values of σ are required, which decreases the stationary gain significantly (see Figure 2).

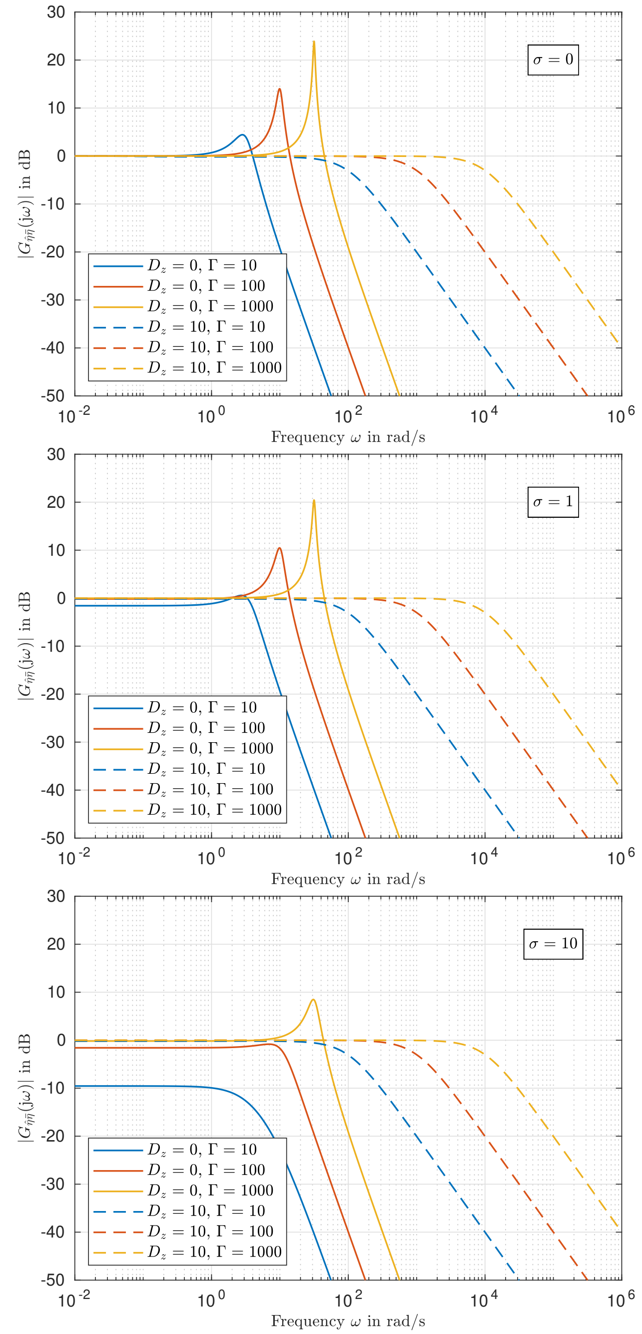

Fig. 2. Amplitude response of the disturbance observer interpretation of the extended MRAC algorithm for different values of D_z, Γ and σ.

In addition to that the proportional term D_z increases the bandwidth of the disturbance observer $\bar{G}_{\hat{\eta}\bar{\eta}}(s)$. As the estimated disturbance $\hat{\eta}(s)$ is directly fed back as an input, the proportional extension leads to an increased control effort for high frequency disturbances.

IV. Example

For illustration we consider the wing rock dynamics, as investigated in [13], given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(u(t) + \Theta^\top \beta(x)\right)$$
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

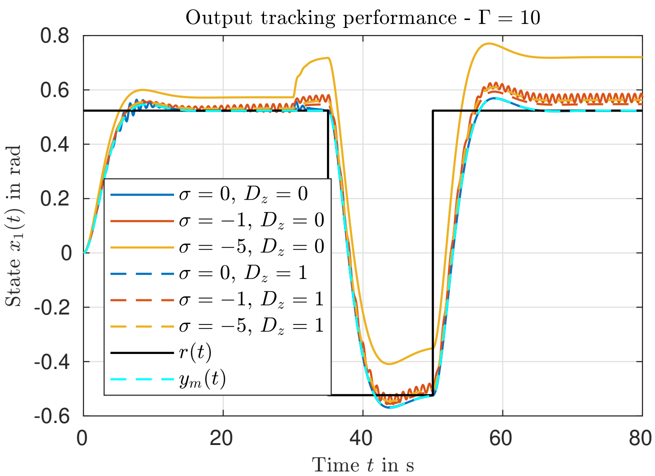

Fig. 3. Transients of the first state $x_1(t)$ for $\Gamma = 10$ and different combinations of D_z and σ without unstructured uncertainties ($\eta(t) = 0$).

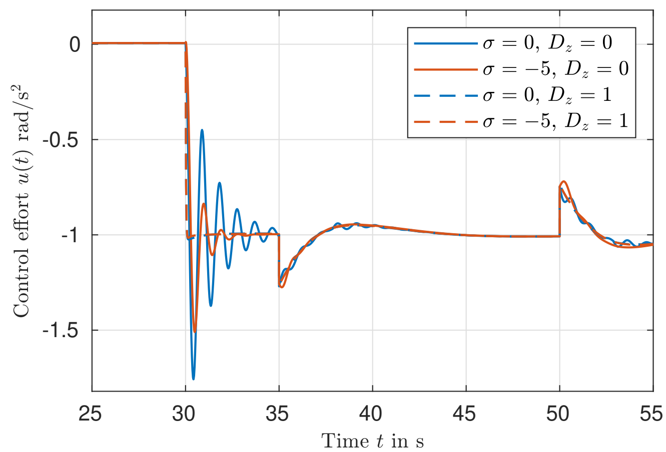

Fig. 4. Control effort for $\Gamma = 10$ and different combinations of D_z and σ without unstructured uncertainties ($\eta(t) = 0$).

with regressor $\beta^\top(x) = \begin{pmatrix}1 & x_1 & x_2 & |x_1|x_2 & |x_2|x_2 & x_1^3\end{pmatrix}$. In our simulation we set the real parameter vector Θ to be piecewise constant

$$\Theta(t) = \begin{cases}\Theta_1, & t < 30\\ \Theta_2, & t \geq 30.\end{cases}$$

with $\Theta_1^\top = \begin{pmatrix}0 & -0.018 & 0.015 & -0.062 & 0.0095 & 0.021\end{pmatrix}$ and $\Theta_2^\top = \begin{pmatrix}1 & -0.018 & 0.015 & -0.062 & 0.0095 & 0.021\end{pmatrix}$. The desired model dynamics are given by

$$\dot{x}_m(t) = \begin{pmatrix}0 & 1\\ -\omega_0^2 & -2\zeta\omega_0\end{pmatrix} x_m(t) + \begin{pmatrix}0\\ \omega_0^2\end{pmatrix} r(t) \tag{23}$$

with $\omega_0 = 0.5\,\mathrm{rad\,s^{-1}}$ and $\zeta = \frac{1}{\sqrt{2}}$. The adaptation rate is set to $\Gamma = 10$ and the reference model and the process start with zero initial conditions. We first consider the nominal case without unstructured uncertainties, that is $\eta(t) = 0$. The results are shown in Figure 3 where the reference $r(t)$ and the response of the reference model $y_m(t)$ are included. Despite the relatively low adaptation rate of $\Gamma = 10$ the classical MRAC ($\sigma = 0$ and $D_z = 0$, solid blue line) shows significant oscillations on all signals, including the control effort depicted in Figure 4 (for better visibility only the cases $\sigma = 0$ and $\sigma = -5$ are included). The pure σ-modification (solid lines) here requires relatively large damping to reduce these oscillations leading to an undesired stationary offset. The additional feed-through term $D_z \neq 0$ (dashed lines) lowers the oscillation and counteracts the negative effect of the added damping. Also it reduces the stationary error significantly.

Even small values of D_z already decrease the peaks and oscillations in the control effort significantly. The model following error is reduced for large D_z without increasing the control effort drastically, as low frequencies dominate the applied reference signal. Note that the σ-modification leads to a stationary deviation, which can be reduced with the feedthrough term D_z as well.

Next we apply a multi-sine signal as an unstructured input disturbance $\eta(t) = \sin(t) + \frac{4}{5}\sin(2t) + \frac{3}{5}\sin(6t) + \frac{4}{5}\sin(10t)$. The frequencies are chosen in the range of 1 rad/s to 10 rad/s. For the investigated parameters (listed in Table I) the bandwidth of the corresponding transfer function $G_{\hat{\eta}\bar{\eta}}$ (compare with Equation (21)) lies within this range. As performance indices we investigate the maximum error, the 2-norm of the output model following error $e_y = Ce$ (representing the root mean square error - RMSE) and the peak of the control effort. The results are given in Table I. The lowest performance indices are highlighted in green for each adaptation rate. The numbers show that the peak of the control effort is mainly influenced by the adaptation gain Γ, whereas the effect of σ and D_z is minor. The error on the other hand is significantly reduced with higher values of Γ and D_z, whereas σ has only a minor influence and a clear tendency is not visible.

Figure 5 shows the first component of the model following error for selected sets of parameters with a nonzero input disturbance. With an increased D_z the error can be reduced significantly even for small values of σ. The corresponding control effort is shown in Figure 6. Overall the feed-through term does not increase the amplitude or oscillations on the control signal for the shown disturbance. The feedhrough term only accelerates the adaptation as the phase lag introduced by the σ-modification (red line) is reduced in such a way that the adaptation is also faster compared to the classical MRAC (blue line). Independent of σ the control signal is faster for larger values of D_z.

V. Conclusions

Within this work, we have extended the well-established certainty equivalence based model reference adaptive control towards higher order adaptation laws including a proportional term. We describe the estimation error using a linear state-space representation and use a quadratic Lyapunov function similar to the classical approach. The derivative shows two additional negative parts which reduce the ultimate bounds of the tracking error. For a simple linear example we can analyze the effects of the proportional term in the frequency domain. The proportional integral MRAC not only reduces the oscillations of the control signal as a effect of high

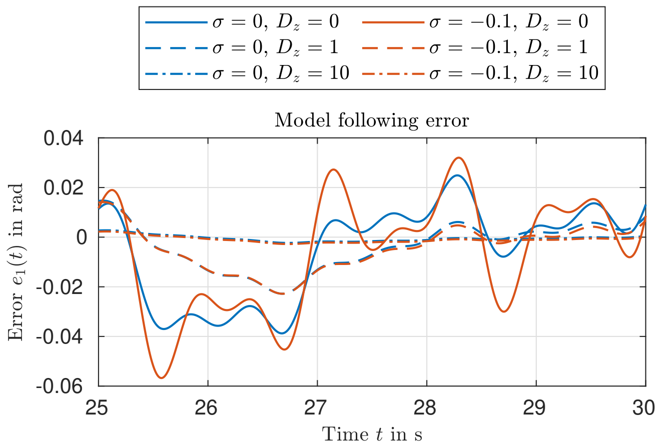

Fig. 5. Model following error for $\Gamma = 10$ different values of σ and D_z in presence of unstructured uncertainties.

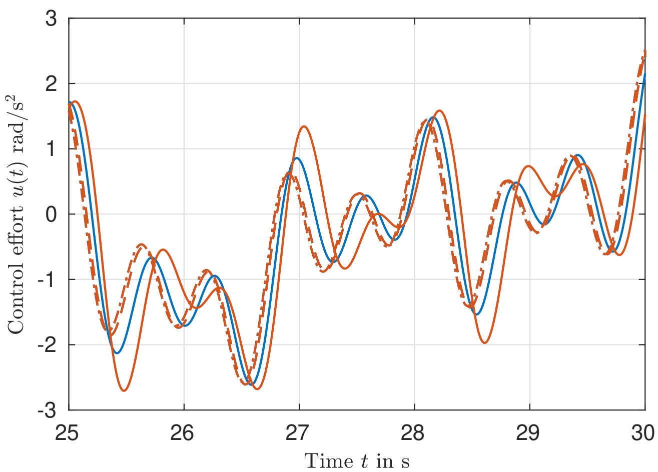

Fig. 6. Control effort for $\Gamma = 10$ and different values of σ and D_z in presence of unstructured uncertainties.

TABLE I

PERFORMANCE INDICES FOR DIFFERENT PARAMETERS.

Parameter			$e_{\max}$	RMSE^2	$u_{\max}$
Γ	$-\sigma$	D_z	$= \|e\|_\infty$	$= \|e_y\|_2^2$	$= \|u\|_\infty$
1	0	0	0.9502	51.7261	3.5201
		1	0.4640	31.7583	3.5485
		10	0.0635	5.6554	3.6077
	0.01	0	0.9482	49.7527	3.5080
		1	0.4639	31.9856	3.5443
		10	0.0640	5.8388	3.6071
	0.1	0	0.9299	62.6542	3.5494
		1	0.4706	37.4902	3.5196
		10	0.0651	7.9108	3.6079
10	0	0	0.5413	8.5287	3.8737
		1	0.0692	3.6495	3.6281
		10	0.0064	0.7978	3.5975
	0.01	0	0.5424	7.7889	3.8728
		1	0.0694	5.1693	3.6278
		10	0.0064	0.8224	3.5975
	0.1	0	0.5505	8.4294	3.8621
		1	0.0699	3.7572	3.6271
		10	0.0063	1.1214	3.5975

adaptation rates, but also improves the disturbance rejection as the proportional term increases the cutoff frequency with the consequence that also high frequency disturbances can be compensated.

The theoretical results are illustrated controlling the wing-rock phenomenon. The simulation results show significant improvements: The oscillations and peaks of the control signals are reduced, the tracking error is reduced even with unstructured disturbances, without increasing the control effort significantly.

A drawback of the presented framework is the required full state information. Therefore future work will consider the output feedback case as well as non constant parameters. In addition, the adaptive control signal might exhibit high frequencies to compensate the unstructured uncertainties which may impact the actuators.

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