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Stability of consensus quantum networks in general dimension

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***Abstract*— In this study, we deal with quantum networked systems consisting of N quantum subsystems having D states. In previous studies, it was proved that a symmetric-state consensus (SSC) can be attained by an algorithm composed of local measurements and a local feedback control in cases of $D = 2$ and 3. However, a proof for general D remained an open problem. In this study, we rigorously prove that an SSC can be attained by the same algorithm in the case of general D.**

I. INTRODUCTION

In recent years, quantum information systems such as quantum computers, quantum cryptography, and quantum communications have attracted much attention from researchers and engineers. While quantum information theory has been developing, the realization of such systems has also proceeded, and quantum computers and other devices are now used in specific tasks.

In order to utilize the properties of such quantum information systems efficiently, it is necessary to generate and operate entangled quantum states composed of large numbers of qubits. However, the generation and operation of such quantum states through the use of conventional methods require large and complex equipment, so realization is difficult.

As an idea to get past this issue, [7] introduced a quantum network system composed of multiple quantum systems corresponding to qubits, which are connected via a network, and proposed an algorithm that locally exchanges states between neighboring quantum systems. This algorithm can construct long-bit consensus states, each called a symmetric-state consensus (SSC), by a simple device. However, [4] pointed out that in this algorithm, the purity of the quantum system decreases and the uncertainty increases during operations. In order to solve this problem, [4] introduced a modified algorithm that repeats local observation and feedback control, and showed via numerical simulation that SSC can be constructed without decreasing the purity.

These studies [7] and [4] are theoretically interesting in comparison with the consensus problem for classical systems [8] because in quantum systems, local operations and controls may change the quantum state of the whole system and it is difficult to show that the proposed algorithm can generate the consensus states.

Furthermore, [9], [10], [6] provided theoretical proof of convergence of the algorithm [4] to the target state SSC, which was not given in [4]. However, the proof is restricted to the case that the number of quantum states D is $D = 2$ or 3, and the proof of convergence in the case of general D was left as an open problem. From both practical and theoretical viewpoints, a proof for the case of general D has been urgently needed.

With this background, this paper addresses the problem of realizing SSC by a consensus algorithm [4], [9], [10], [6] for the case of general D and gives a rigorous proof of convergence to an SSC. In the proof, we utilize the PBH rank test to determine the observability of a hypothetical control system.

This paper consists of five sections. Section II describes the fundamentals of stochastic systems and quantum theory. Section III introduces quantum network systems and the consensus algorithm. Section IV gives a main result in which a rigorous proof of the convergence to an SSC for a general D with the consensus algorithm is provided. Section V is the conclusion of this paper and describes future work.

Notation

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$: sets of real numbers, complex numbers, integers, and natural numbers, respectively; $\mathbb{P}[\bullet]$: probability of $\bullet$; $\mathbb{E}[\bullet]$: expectation of $\bullet$; $\otimes$: Kronecker product; $\bullet^\top$, $\bullet^\dagger$: transpose and complex conjugate of $\bullet$, respectively; $\mathrm{tr}(\bullet)$: trace of $\bullet$; diag$(\bullet)$, blkdiag$(\bullet)$: diagonal matrix of $\bullet$ and block diagonal matrix of $\bullet$, respectively; $\bullet \succeq 0$: $\bullet$ is positive semi-definite.

II. PRELIMINARIES

In this section, we introduce the fundamentals on the stability of stochastic systems and quantum theory that will be used in the subsequent sections. The contents of this section are based on [5], [1], [2], [9], [6].

A. Stability of stochastic system

Let $\{X_t\}_{t\in\mathbb{Z}}$ and $X_t \in \mathbb{C}^n$ be a Markov process and the state of a stochastic system, respectively. Define a stability of a stochastic system as follows.

Definition 2.1: For an arbitrary positive number ϵ, if

$$\lim_{t\to\infty} \mathbb{P}[\min_{x\in\mathcal{X}_\mathrm{I}} \|X_t - x\| \geq \epsilon] = 0 \tag{1}$$

is satisfied, then X_t is said to converge to the set $\mathcal{X}_\mathrm{I} \subset \mathbb{C}^n$ in probability.

Definition 2.2: A set $\mathcal{C}$ is called an invariant set of a system if the following condition is satisfied:

$$\{x_t, t = 1, 2, \ldots | \forall x_0 \in \mathcal{C}\} \subseteq \mathcal{C}. \tag{2}$$

The following proposition is convenient for proving convergence in probability.

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Proposition 2.1: [5] Let $\{X_t\}_{t\in\mathbb{Z}}$, $X_t \in \mathbb{C}^n$ be a Markov process and assume that there exist bounded non-negative functions $V(x)$ and $k(x)$ satisfying

$$\mathbb{E}\{V(X_t)|X_{t-1}\} - V(X_{t-1}) = -k(X_{t-1}). \tag{3}$$

Then, $\lim_{t\to\infty} k(X_t) = 0$ for almost all paths. Moreover, let $\mathcal{M} = \{x \in \mathcal{C}^n | k(x) = 0\}$ and $\tilde{\mathcal{M}}$ represent the maximum invariant set in $\mathcal{M}$. Then, X_t converges to $\tilde{\mathcal{M}}$ in probability.

B. Quantum theory

Let $\mathcal{D}_o(n)$ be the set of quantum states ψ in $\mathbb{C}^n$:

$$\mathcal{D}_o(n) = \{\psi \in \mathbb{C}^n | \psi^\dagger \psi = 1\}. \tag{4}$$

Similarly, the density operator ρ is represented as an element of the following set:

$$\mathcal{D}(n) = \{\rho \in \mathbb{C}^{n\times n} | \rho = \rho^\dagger \succeq 0,\ \mathrm{tr}(\rho) = 1\}. \tag{5}$$

The density operator corresponds to the probability distribution of a quantum system and is also called the state of the system. Suppose that the state of a quantum system is represented by a density operator on the Hilbert space $\mathbb{C}^D$, $D \in \mathbb{N}$, $D \geq 2$. Denoting the D orthonormal basis of the Hilbert space as $|1\rangle, |2\rangle, \ldots, |D\rangle$, the state of the total system composed of N quantum systems is represented by the following density operator on the Hilbert space:

$$\mathbb{C}^{D^N} = \underbrace{\mathbb{C}^D \otimes \mathbb{C}^D \otimes \cdots \otimes \mathbb{C}^D}_{N\text{ set}}. \tag{6}$$

This is called an N-dimensional quantum system. Its basis is given by

$$\{|d_1\rangle \otimes |d_2\rangle \otimes \cdots \otimes |d_N\rangle\},\ |d_i\rangle \in \{|1\rangle, |2\rangle, \ldots, |D\rangle\} \tag{7}$$

and the elements can be also simply represented as

$$|d_1\rangle \otimes |d_2\rangle \otimes \cdots \otimes |d_N\rangle =: |d_1 d_2 \cdots d_N\rangle. \tag{8}$$

Physical quantities are represented by Hermite operators σ called observables. Let s_n and Π_n be the eigenvalue of σ and its projection operator, respectively. Then, σ is uniquely decomposed as

$$\sigma = \sum_n s_n \Pi_n, \tag{9}$$

where

$$\sum_n \Pi_n = I. \tag{10}$$

The above is called spectral decomposition. In general, for an observable σ, if the state before the measurement is ρ, the probability p_n of obtaining the measurement value s_n is given by

$$p_n = \mathrm{tr}(\rho \Pi_n), \tag{11}$$

and the state after the measurement changes to

$$\rho \mapsto \frac{\Pi_n \rho \Pi_n}{\mathrm{tr}(\rho \Pi_n)}. \tag{12}$$

The state operation of an N-dimensional quantum system is performed by unitary operator $U \in \mathcal{U}(D^N)$, where $\mathcal{U}(D^N)$ represents the set of all unitary operators on D^N-dimensional quantum states, and a quantum state is transformed as

$$\rho \mapsto U \rho U^\dagger. \tag{13}$$

C. Non-negative matrix, probability matrix, and graph

A matrix whose elements are non-negative is called a non-negative matrix. Then, a nonnegative matrix whose each row sums to 1 is called a stochastic matrix, and a stochastic matrix whose each column sums to 1 is called a doubly stochastic matrix. We define $\mathcal{B}(n)$ as the set of all n-dimensional square matrices and $\mathcal{S}(n)$ as the set of all n-dimensional doubly stochastic matrices.

We next define the irreducibility and the period of a matrix as follows.

Definition 2.3: For $A \in \mathcal{B}(n)$, A is said to be reducible if $H^\top A H$ can be decomposed into two or more block triangular matrices using permutation matrix H. A square matrix that is not reducible is said to be irreducible. Note that when $A \in \mathcal{B}(1)$, A is reducible if $A = 0$ and irreducible if $A \neq 0$.

Definition 2.4: For $A \in \mathcal{B}(n)$ an irreducible matrix, let $\mathcal{G} = (\mathcal{V}_A, \mathcal{E}_A)$ be a directed graph associated with A, where $\mathcal{V}_A$ represents the set of vertices and $\mathcal{E}_A$ the set of edges. Then, the greatest common divisor of the number of edges of all directed closed paths in the graph $\mathcal{G}$ is called the cycle of the matrix A. A matrix with cycle 1 is called aperiodic.

It is known that application of the Perron-Frobenius theorem leads to the following proposition.

Proposition 2.2: If $A \in \mathcal{S}(n)$ is irreducible and aperiodic, the following holds:

$$\lim_{m\to\infty} A^m = \frac{1}{n}\mathrm{ones}(n), \tag{14}$$

$$\mathrm{ones}(n) := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{B}(n). \tag{15}$$

III. Quantum Network Systems and Consensus Algorithm

The contents of this section are from [9], [10], [6]. In this study, we regard a symmetric quantum state as a consensus state. This notion comes from the property of the consensus state in a classical multi-agent system that the state of a whole system remains unchanged even if the agents exchange their state variables with each other.

Definition 3.1 ([7]): Let $\mathcal{P}(n)$ be the set of all permutation matrices in $\mathcal{B}(n)$, π be a permutation of N integers, and $U_\pi \in \mathcal{P}(n)$ be a permutation matrix such that

$$\begin{aligned} &U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_N) \\ &= x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(N)},\ x_k \in \mathbb{C}^D. \end{aligned} \tag{16}$$

Then, if $\rho \in \mathcal{D}(D^N)$ satisfies $U_\pi \rho U_\pi^\dagger = \rho$ for an arbitrary π, ρ is called in a *symmetric state consensus* (SSC).

In the following, "SSC" is also used to represent the set of states in anSSC, such as writing $\rho \in$ SSC to mean ρ is in an SSC.

In the algorithm given in [4], it is assumed that local measurement/feedback control is applied in order on paired quantum subsystems among an N-dimensional quantum system. Let $\mathcal{V}_N$ be the set of vertices and $\mathcal{E}_N$ be the set of edges, where each vertex represents a quantum subsystem in an N-dimensional system and each edge between two vertices represents a paired relation of two quantum subsystems. Then, a graph $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$ represents the relations of paired quantum subsystems for the N-dimensional quantum system. We assume the following in this paper.

Assumption 3.1: Graph $\mathcal{G}_N$ is connected.

Next, permutation matrix $S_{ij} \in \mathcal{P}(D^N)$ that permutes the states of the i-th quantum subsystem and the j-th quantum subsystem is defined as follows:

$$\begin{aligned} &S_{ij}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_N) \\ &= x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_N,\ x_k \in \mathbb{C}^D. \quad (17) \end{aligned}$$

Note that S_{ij} satisfies the following:

$$S_{ij}^{\dagger} = S_{ij},\ S_{ij}^2 = I_{D^N}. \quad (18)$$

Using S_{ij}, we define observable σ_{ij} as a local measurement for the edge (i, j).

Definition 3.2: The observable σ_{ij} for the pair comprising the i-th and j-th subsystems is given as follows:

$$\sigma_{ij} = pP_{ij} + qQ_{ij},\ p \neq q \in \mathbb{R} \quad (19)$$

$$P_{ij} = \frac{1}{2}(I + S_{ij}),\ Q_{ij} = \frac{1}{2}(I - S_{ij}) \quad (20)$$

When value p is obtained by the measurement, the quantum state is projected onto the eigenspace with eigenvalue 1 of P_{ij}, and the quantum system is invariant to the permutation between the i-th subsystem and the j-th subsystem. Therefore, if we obtain the measurement value p for every pair, the whole system is symmetric and an SSC is achieved. On the other hand, when value q is obtained by a measurement, the quantum state is projected onto the eigenspace with eigenvalue 1 of Q_{ij}, and the quantum system is anti-symmetric with respect to the permutation between the i-th and j-th subsystems. In this case, we consider applying a local control action by the following unitary operator.

Definition 3.3: Define a unitary operator $U_{ij} \in \mathcal{U}(D^N)$ on N-dimensional quantum network systems such that it applies the operator $u_{ij} \in \mathcal{B}(D^2)$ defined below to the pair comprising the i-th and j-th subsystems and the identity to the rest of the subsystems.

$$u_{ij} = \mathrm{blkdiag}_{k=1}^{D}(u'_{ij}(k)) \quad (21)$$

$$u'_{ij}(k) = \mathrm{diag}(\underbrace{-1, -1, \ldots, -1}_{k-1}, \underbrace{1, 1, \ldots, 1}_{D-k+1}) \quad (22)$$

Note that this U_{ij} satisfies

$$Q_{ij} U_{ij} Q_{ij} = O. \quad (23)$$

After this control operation U_{ij}, the probability to obtain the measurement value p with σ_{ij} becomes 1. This implies that the pair comprising the i-th and j-th subsystems is symmetrized.

With the above measurement and control operation, in the previous study [4], the consensus algorithm defined below was proposed and a numerical simulation demonstrated the convergence of the quantum state to an SSC for arbitrary initial states. Note that the time evolution of ρ or ψ is indexed as ρ_t or ψ_t, $t = 0, 1, \ldots,$ in the following algorithm.

Algorithm 3.1 (Consensus algorithm [4]):

1. For graph $\mathcal{G}_N$, select an edge (i, j) from $\mathcal{E}_N$ in order or at random such that all the edges are selected in a cycle period.
2. Measure the quantum system in the quantum state ρ_t or ψ_t with the observable σ_{ij}.
3. When a measurement value p is obtained, do nothing and return to 1. Otherwise, when a measurement value q is observed, apply control operation U_{ij} to the i-th subsystem and the j-th subsystem and return to 1. In both cases, time is incremented as $t = t + 1$.

Remark 3.1: The algorithm is composed of local measurements and local control operations, so it is regarded as a decentralized feedback control.

The proof of the convergence of Algorithm 3.1 to an SSC was not given in [4]. This is because the repeated measurement is a complex stochastic process and it is difficult to follow the state transitions of the quantum system. A strict proof of convergence when $D = 2$ or 3 was given in [9], [10], [6] by application of the Lyapunov stability theorem of Proposition 2.1.

IV. Main results

In this section, we extend the results in [9], [10], [6] on the convergence to an SSC of Algorithm 3.1 to the case of general D, which is the main result of the present paper.

A. Main theorem and lemmas for the proof

Theorem 4.1: For Algorithm 3.1, the states ρ_t, $t = 0, 1, \ldots,$ of the N-dimensional quantum system converge to an SSC with probability 1 for any initial states.

In the remainder of this section, we show several lemmas, and give the proof of the above theorem in Section IV-C.

First, we rearrange the basis of the quantum state for the later proofs. Let $\{|d_1 d_2 \ldots d_N\rangle\}$ be the D^N basis and assign its elements to equivalence classes such that each permutation of states between subsystems is a binary relation. That is, each element in an equivalence class can be transformed to the other elements in the same equivalence class by permutations. Let k_i be the numbers of values i, $i = 1, 2, \ldots, D$, in an equivalence class; then, each equivalence class can be identified by the set of k_i: $(k_1, k_2, \ldots, k_D) =: \mathbf{k}$. Give an index m to identify each equivalence class $\mathbf{k}$ as $\mathbf{k}_m$ and let $\mathcal{F}_{\mathbf{k}_m}$ be the set of the elements in equivalence class $\mathbf{k}_m$. If we let $\mathbf{K}$ be the set of all the equivalence classes $\{\mathbf{k}_m\}$, then it is known that the number of elements of $\mathbf{K}$ is $\frac{(D+N-1)!}{(D-1)!N!} =: M$.

Next, define $l_{\mathbf{k}}$ by

$$l_{\mathbf{k}}(i) := \left|\left\{|d_1 d_2 \cdots d_{N-1}\, i\rangle \,\middle|\, |d_1 d_2 \cdots d_{N-1}\, i\rangle \in \mathcal{F}_{\mathbf{k}}\right\}\right|. \tag{24}$$

Similarly, define $l_{\mathbf{k}}(i,j)$ by

$$l_{\mathbf{k}}(i,j) := \left|\left\{|d_1 d_2 \cdots j\, i\rangle \,\middle|\, |d_1 d_2 \cdots j\, i\rangle \in \mathcal{F}_{\mathbf{k}}\right\}\right|. \tag{25}$$

These satisfy the following:

$$l_{\mathbf{k}} = \frac{N!}{\prod_{i=1}^{N}(k_i!)}, \tag{26}$$

$$l_{\mathbf{k}}(i) = \frac{(N-1)!}{(k_i-1)!\prod_{i'=1,i'\neq i}^{N}(k_{i'}!)} = \frac{k_i}{N} l_{\mathbf{k}}, \tag{27}$$

$$\begin{aligned} l_{\mathbf{k}}(i,j) &= \frac{(N-2)!}{(k_i-1)!(k_j-1)!\prod_{i'=1,i'\neq i,j}^{N}(k_{i'}!)} \\ &= \frac{k_i k_j}{N(N-1)} l_{\mathbf{k}} = \frac{k_j}{N-1} l_{\mathbf{k}}(i),\ i \neq j, \end{aligned} \tag{28}$$

$$\begin{aligned} l_{\mathbf{k}}(i,i) &= \frac{(N-2)!}{(k_i-2)!\prod_{i'=1,i'\neq i}^{N}(k_{i'}!)} \\ &= \frac{k_i(k_i-1)}{N(N-1)} l_{\mathbf{k}} = \frac{k_i-1}{N-1} l_{\mathbf{k}}(i). \end{aligned} \tag{29}$$

Next consider arranging the basis elements of $\mathcal{F}_{\mathbf{k}}$ in ascending order with respect to d_N, similarly rearrange the basis elements having the same d_N in ascending order of d_{N-1}, and continue this rearrangement up to d_1,

Example 4.1: Take the case of $D = 3$ and $N = 3$. Then, the $D^N = 3^3 = 27$ basis elements can be divided into the following 10 equivalence classes:

$$\begin{aligned} &\{|111\rangle\}, \{|222\rangle\}, \{|333\rangle\}, \\ &\{|112\rangle, |121\rangle, |211\rangle\}, \{|122\rangle, |212\rangle, |221\rangle\}, \\ &\{|113\rangle, |131\rangle, |311\rangle\}, \{|133\rangle, |313\rangle, |331\rangle\}, \\ &\{|223\rangle, |232\rangle, |322\rangle\}, \{|233\rangle, |323\rangle, |332\rangle\}, \\ &\{|123\rangle, |132\rangle, |213\rangle, |231\rangle, |312\rangle, |321\rangle\}. \end{aligned} \tag{30}$$

For example, the rearrangement of the basis elements of the equivalence class $\{|123\rangle, |132\rangle, \ldots\}$ is given by

$$|321\rangle, |231\rangle, |312\rangle, |132\rangle, |213\rangle, |123\rangle. \tag{31}$$

Next, constitute permutation matrix $T \in \mathcal{P}(D^N)$ with the bases of $\mathcal{F}_{\mathbf{k}_1}, \mathcal{F}_{\mathbf{k}_2}, \ldots, \mathcal{F}_{\mathbf{k}_M}$ such that

$$T = \begin{bmatrix} T_1 & \cdots & T_{l_{\mathbf{k}_1}} & T_{l_{\mathbf{k}_1}+1} & \cdots & T_{l_{\mathbf{k}_1}+l_{\mathbf{k}_2}} & \cdots \end{bmatrix} \tag{32}$$

where $\{T_1, \ldots, T_{l_{\mathbf{k}_1}}\}$, $\{T_{l_{\mathbf{k}_1+1}}, \ldots, T_{l_{\mathbf{k}_1}+l_{\mathbf{k}_2}}\}$, $\ldots$, are the bases of $\mathcal{F}_{\mathbf{k}_1}$, $\mathcal{F}_{\mathbf{k}_2}$, $\ldots$, respectively. By using T, the permutation matrix S_{ij} can be decomposed as follows:

$$T^\top S_{ij} T = \mathrm{blkdiag}_{m=1}^{M}(S_{ij}(m)),\ S_{ij}(m) \in \mathcal{P}(l_{\mathbf{k}_m}). \tag{33}$$

Similarly, matrices P_{ij}, Q_{ij}, and unitary operator U_{ij} can be decomposed as follows:

$$\begin{aligned} T^\top P_{ij} T &= \mathrm{blkdiag}_{m=1}^{M}(P_{ij}(m)),\ P_{ij}(m) \in \mathcal{B}(l_{\mathbf{k}_m}) \\ T^\top Q_{ij} T &= \mathrm{blkdiag}_{m=1}^{M}(Q_{ij}(m)),\ Q_{ij}(m) \in \mathcal{B}(l_{\mathbf{k}_m}) \\ T^\top U_{ij} T &= \mathrm{blkdiag}_{m=1}^{M}(U_{ij}(m)),\ U_{ij}(m) \in \mathcal{B}(l_{\mathbf{k}_m}) \end{aligned} \tag{34}$$

Corresponding to this decomposition, a quantum state ψ can be also represented as

$$\begin{aligned} T^\top \psi &= \begin{bmatrix} \psi^\top(1) & \cdots & \psi^\top(m) & \ldots & \psi^\top(M) \end{bmatrix}^\top \\ &\quad \psi(m) \in \mathbb{C}^{l_{\mathbf{k}_m}}. \end{aligned} \tag{35}$$

Then, the observable σ_{ij} and the unitary operator U_{ij} on ψ are considered to be the composition of the independent observable $\sigma_{ij}(m) := pP_{ij}(m) + qQ_{ij}(m)$ and the unitary operator $U_{ij}(m)$ on $\psi(m)$, $m = 1, 2, \cdots, M$, respectively. We call $\psi(m)$ the m-th mode in the dynamics.

In [9], [10], [6], it was shown that the following Lemmas 4.1–4.7 hold for general D and Lemmas 4.8 and 4.9 hold for $D = 2$ and 3.

Lemma 4.1: The equation

$$\begin{aligned} &\lim_{n\to\infty} \underbrace{\left(\prod_{i\neq j\in\mathcal{V}_N} P_{ij}\right)\left(\prod_{i\neq j\in\mathcal{V}_N} P_{ij}\right)\cdots\left(\prod_{i\neq j\in\mathcal{V}_N} P_{ij}\right)}_{n} \\ &= \tilde{P}_N \end{aligned} \tag{36}$$

holds for any order of multiplication in each $\prod_{i\neq j\in\mathcal{V}_N} P_{ij}$ in (36), where

$$\tilde{P}_N := T\left(\mathrm{blkdiag}_{m=1}^{M}\tilde{P}_N(m)\right)T^\top, \tag{37}$$

$$\tilde{P}_N(m) := \frac{1}{l_{\mathbf{k}_m}}\mathrm{ones}(l_{\mathbf{k}_m}). \tag{38}$$

Note that in the proof of Lemma 4.1, it is shown that matrix $\prod_{i\neq j\in\mathcal{V}_N} P_{ij}$ satisfies the conditions in Proposition 2.2 and (14) is employed.

The matrix $\tilde{P}_N$ satisfies the following two lemmas.

Lemma 4.2: For arbitrary $i \neq j \in \mathcal{V}_N$, the following hold:

$$\begin{aligned} &\tilde{P}_N P_{ij} = P_{ij}\tilde{P}_N = \tilde{P}_N,\ \tilde{P}_N Q_{ij} = Q_{ij}\tilde{P}_N = O, \\ &\tilde{P}_N^2 = \tilde{P}_N,\ \mathrm{tr}(\rho P_{ij}) = 0 \Rightarrow \mathrm{tr}\left(\rho\tilde{P}_N\right) = 0. \end{aligned}$$

Lemma 4.3: For $\rho \in \mathcal{D}(D^N)$, if $\mathrm{tr}\left(\rho\tilde{P}_N\right) = 1$, then $\rho \in$ SSC.

Related to $\tilde{P}_N$, we also define a graph $\mathcal{G}_{N-1} = (\mathcal{V}_{N-1}, \mathcal{E}_{N-1})$ such that $\mathcal{G}_{N-1}$ is a subgraph of $\mathcal{G}_N$ obtained by deleting a vertex v_n and the edges which are connected to v_n from $\mathcal{G}_N$, where v_n is selected such that $\mathcal{G}_{N-1}$ is connected. Without loss of generality, assume that v_n is the N-th component of the N-dimensional quantum system, $v_n = v_N$, and also that one of the vertices connected to v_N is the $(N-1)$-th component, v_{N-1}. With this graph $\mathcal{G}_{N-1}$, matrices $\tilde{P}_{N-1}$ and $\tilde{P}_{N-1}(m)$ are defined similarly and (36) also holds if N is replaced by $N-1$. The above implies the following:

$$\begin{aligned} \tilde{P}_{N-1} &= T\left(\mathrm{blkdiag}_{m=1}^{M}(\tilde{P}_{N-1}(m))\right)T^\top, \\ &\quad \tilde{P}_{N-1}(m) \in \mathcal{B}(l_{\mathbf{k}_m}). \end{aligned} \tag{39}$$

Hereafter, we use the following Lyapunov function:

$$V(\rho) = 1 - \mathrm{tr}\left(\rho\tilde{P}_N\right). \tag{40}$$

Let ρ and ρ' represent the quantum states before and after the measurement of σ_{ij}, respectively, and

$$\Delta V := \mathbb{E}\{V(\rho')|\rho\} - V(\rho) \tag{41}$$

represent the conditional expectation of the increment of $V(\rho)$. Then, we get the following lemma.

Lemma 4.4:

$$\Delta V \leq 0 \tag{42}$$

$$\Delta V = 0. \Leftrightarrow \operatorname{tr}\left(U_{ij}Q_{ij}\rho Q_{ij}U_{ij}^{\dagger}\tilde{P}_N\right) = 0 \tag{43}$$

This lemma suggests finding the largest invariant set $\tilde{\mathcal{M}}$ satisfying $\Delta V = 0$ for the proof of Theorem 4.1. For this purpose, we define the following sets:

$$\mathcal{M}_1 := \left\{\rho \in \mathcal{D}(D^N) | \operatorname{tr}\left(\rho\tilde{P}_N\right) = 1\right\}, \tag{44}$$

$$\mathcal{M}_2 := \left\{\rho \in \mathcal{D}(D^N) | 0 < \operatorname{tr}\left(\rho\tilde{P}_N\right) < 1\right\}, \tag{45}$$

$$\mathcal{M}_3 := \left\{\rho \in \mathcal{D}(D^N) | \operatorname{tr}\left(\rho\tilde{P}_N\right) = 0\right\}. \tag{46}$$

These sets are mutually exclusive and such that $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 = \mathcal{D}(D^N)$. From Lemma 4.3, it is known that $\mathcal{M}_1 \subseteq \tilde{\mathcal{M}}$. Then, from Proposition 2.1, it is enough to prove $\tilde{\mathcal{M}} \cap (\mathcal{M}_2 \cup \mathcal{M}_3) = \emptyset$ in order to show $\mathcal{M}_1 = \tilde{\mathcal{M}}$ for the proof of Theorem 4.1, and the remainder of this section will show this fact.

The following lemma also holds.

Lemma 4.5: For the sequence of quantum states $\{\rho_t\}_{t\in\mathbb{N}\cup\{0\}} \subset \mathcal{D}(D^N)$, $\rho_0 \in \mathcal{D}(D^N)$ generated by Algorithm 3.1, the following hold:

$$\rho_0 \in \mathcal{M}_1 \cap \tilde{\mathcal{M}} \Rightarrow \rho_t \in \mathcal{M}_1 \cap \tilde{\mathcal{M}},\ \forall t \in \mathbb{N} \cup \{0\}, \tag{47}$$

$$\rho_0 \in \mathcal{M}_3 \cap \tilde{\mathcal{M}} \Rightarrow \rho_t \in \mathcal{M}_3 \cap \tilde{\mathcal{M}},\ \forall t \in \mathbb{N} \cup \{0\}. \tag{48}$$

This lemma implies the following lemma.

Lemma 4.6:

$$\mathcal{M}_2 \cap \tilde{\mathcal{M}} = \emptyset. \tag{49}$$

Thus, we next consider how to show $\mathcal{M}_3 \cap \tilde{\mathcal{M}} = \emptyset$. The following lemma also holds.

Lemma 4.7: Let $\tilde{\mathcal{M}}^{\mathrm{v}}$ be the set of state vectors ψ such that $\psi\psi^{\dagger} \in \tilde{\mathcal{M}}$. Similarly, define

$$\mathcal{M}_3^{\mathrm{v}} = \left\{\psi \in \mathcal{D}_o(D^N) | \left\|\tilde{P}_N\psi\right\| = 0\right\}. \tag{50}$$

Then the following holds:

$$\mathcal{M}_3 \cap \tilde{\mathcal{M}} \neq \emptyset \Leftrightarrow \mathcal{M}_3^{\mathrm{v}} \cap \tilde{\mathcal{M}}^{\mathrm{v}} \neq \emptyset. \tag{51}$$

From the previous lemma, it is sufficient to show $\mathcal{M}_3^{\mathrm{v}} \cap \tilde{\mathcal{M}}^{\mathrm{v}} = \emptyset$. For this purpose, we will use the following lemma.

Lemma 4.8: Let $\psi_0 \in \mathcal{D}_o(D^N)$ be an initial state. Then, if

$$\tilde{P}_{N-1}\psi_0 \neq 0, \tag{52}$$

$$\tilde{P}_N\tilde{P}_{N-1}\psi_0 = 0, \tag{53}$$

there exists $0 \leq d < D$ such that

$$\tilde{P}_N(U_{N-1,N}Q_{N-1,N}\tilde{P}_{N-1})^d\psi_0 \neq 0. \tag{54}$$

The previous lemma is the key to proving the final lemma and the main theorem. Lemma 4.8 was proved for cases $D =$ 2 and 3 in [9], [10], [6]; however, the proof for the case of general D was left as an open problem. We give the proof for the general case in Section IV-B.

Using Lemma 4.8, we get the following last lemma.

Lemma 4.9: Let $\psi_0 \in \mathcal{D}_o(D^N)$ be an initial state. Then, if

$$\tilde{P}_{N-1}\psi_0 \neq 0,\ \tilde{P}_N\psi_0 = 0, \tag{55}$$

there exists $0 < \tau$ such that

$$\tilde{P}_N\psi_\tau \neq 0. \tag{56}$$

The proof of this lemma for the case of the general D can be easily obtained using the same derivation as in [6] for the cases $D = 2$ and 3, so we omit the proof here.

B. Proof of Lemma 4.8

Without loss of generality, we assume that the m-th mode $\psi_0(m)$ satisfying $\tilde{P}_{N-1}(m)\psi_0(m) \neq 0$ includes D different values, similar to as in [6]. Then, the poof of this lemma is completed by showing that there does not exist $\psi_0(m)$, $\tilde{P}_{N-1}(m)\psi_0(m) \neq 0$ satisfying the following equations simultaneously:

$$\begin{aligned}
\tilde{P}_N(m)\tilde{P}_{N-1}(m)\psi_0(m) &= 0,\\
\tilde{P}_N(m)U_{N-1,N}(m)Q_{N-1,N}(m)\tilde{P}_{N-1}(m)\psi_0(m) &= 0,\\
\tilde{P}_N(m)(U_{N-1,N}(m)Q_{N-1,N}(m)\tilde{P}_{N-1}(m))^2\psi_0(m) &= 0,\\
&\vdots\\
\tilde{P}_N(m)(U_{N-1,N}(m)Q_{N-1,N}(m)\tilde{P}_{N-1}(m))^{D-1}\psi_0(m) &= 0.
\end{aligned} \tag{57}$$

From $(\tilde{P}_{N-1})^2\psi_0 = \tilde{P}_{N-1}\psi_0$ and (53), $\tilde{P}_{N-1}(k)\psi_0(k)$ can be represented as

$$\begin{aligned}
&\tilde{P}_{N-1}(m)\psi_0(m)\\
&= [\underbrace{a_1\, a_1\, \cdots\, a_1}_{l_{\mathbf{k}}(1)}\ \underbrace{a_2\, a_2\, \cdots\, a_2}_{l_{\mathbf{k}}(2)}\ \cdots\ \underbrace{a_D\, a_D\, \cdots\, a_D}_{l_{\mathbf{k}}(D)}]^{\top}.
\end{aligned} \tag{58}$$

Note that the basis corresponding to the element a_i in (58) is $|* * \cdots * i\rangle$. Hereafter, we represent the value a_i in $\psi_t(m)$ as $a_i^{(t)}$. Then, the process that Algorithm 3.1 generates $\psi_t(m)$, $t = 0, 1, \ldots, D-2$, is as follows.

1) Apply $Q_{N-1,N}(m)$ and $U_{N-1,N}(m)$ to

$$\begin{aligned}
&\tilde{P}_{N-1}(m)\psi_t(m)\\
&= [\underbrace{a_1^{(t)}\, \ldots\, a_1^{(t)}}_{l_{\mathbf{k}}(1)}\ \underbrace{a_2^{(t)}\, \ldots\, a_2^{(t)}}_{l_{\mathbf{k}}(2)}\ \cdots\ \underbrace{a_D^{(t)}\, \ldots\, a_D^{(t)}}_{l_{\mathbf{k}}(D)}]^{\top}
\end{aligned} \tag{59}$$

to get

$$\begin{aligned}
&\psi_{t+1}(m)\\
&= U_{N-1,N}(m)Q_{N-1,N}(m)\tilde{P}_{N-1}(m)\psi_t(m)\\
&= [\underbrace{b_{11}^{(t)} \cdots b_{11}^{(t)}}_{l_{\mathbf{k}}(1,1)}\underbrace{b_{12}^{(t)} \cdots b_{12}^{(t)}}_{l_{\mathbf{k}}(1,2)}\cdots\underbrace{b_{1D}^{(t)} \cdots b_{1D}^{(t)}}_{l_{\mathbf{k}}(1,D)}\\
&\quad \cdots\underbrace{b_{21}^{(t)} \cdots b_{21}^{(t)}}_{l_{\mathbf{k}}(2,1)}\underbrace{b_{22}^{(t)} \cdots b_{22}^{(t)}}_{l_{\mathbf{k}}(2,2)}\cdots\underbrace{b_{2D}^{(t)} \cdots b_{2D}^{(t)}}_{l_{\mathbf{k}}(2,D)}\\
&\quad \vdots\\
&\quad \cdots\underbrace{b_{D1}^{(t)} \cdots b_{D1}^{(t)}}_{l_{\mathbf{k}}(D,1)}\underbrace{b_{D2}^{(t)} \cdots b_{D2}^{(t)}}_{l_{\mathbf{k}}(D,2)}\cdots\underbrace{b_{DD}^{(t)} \cdots b_{DD}^{(t)}}_{l_{\mathbf{k}}(D,D)}]^\top,
\end{aligned} \tag{60}$$

where

$$b_{ij}^{(t)} = \begin{cases} \frac{a_j^{(t)} - a_i^{(t)}}{2}, & i < j,\\ 0, & i = j,\\ \frac{a_i^{(t)} - a_j^{(t)}}{2}, & i > j. \end{cases} \tag{61}$$

Recall that $l_{\mathbf{k}}(i,j)$ defined by (28) or (29) represents the number of basis elements $|* * \cdots * \, i \, j\rangle$.

2) Apply $\tilde{P}_N(m)$ to $\psi_{t+1}(m)$ to get

$$\tilde{P}_N(m)\psi_{t+1}(m) = \left[\underbrace{\frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)} \quad \cdots \quad \frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)}}_{l_{\mathbf{k}}}\right]^\top. \tag{62}$$

3) From (62), $\tilde{P}_N(m)\psi_{t+1}(m) = 0$ implies

$$\frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)} = 0. \tag{63}$$

On the other hand,

$$\begin{aligned}
&\tilde{P}_{N-1}(m)\psi_{t+1}(m)\\
&= [\underbrace{\tilde{a}_1^{(t)} \ldots \tilde{a}_1^{(t)}}_{l_{\mathbf{k}}(1)}\,\underbrace{\tilde{a}_2^{(t)} \ldots \tilde{a}_2^{(t)}}_{l_{\mathbf{k}}(2)} \cdots \underbrace{\tilde{a}_D^{(t)} \ldots \tilde{a}_D^{(t)}}_{l_{\mathbf{k}}(D)}]^\top,
\end{aligned} \tag{64}$$

$$\tilde{a}_i^{(t)} := \frac{1}{l_{\mathbf{k}}}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)}, \tag{65}$$

and so

$$\begin{aligned}
&\tilde{P}_N(m)\tilde{P}_{N-1}(m)\psi_{t+1}(m)\\
&= \left[\frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D} l_{\mathbf{k}}(i)\tilde{a}_i^{(t)} \cdots \frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D} l_{\mathbf{k}}(i)\tilde{a}_i^{(t)}\right]^\top\\
&= \left[\frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)} \cdots \frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)}\right]^\top.
\end{aligned} \tag{66}$$

Therefore, we get $\tilde{P}_N(m)\psi_{t+1}(m) = 0 \Leftrightarrow \tilde{P}_N(m)\tilde{P}_{N-1}(m)\psi_{t+1}(m) = 0$. From the definition of $a_i^{(t+1)}$, $a_i^{(t+1)} = \tilde{a}_i^{(t)}$ and we have the following recursive formula on $a_i^{(t)}$, $t = 1, 2, \ldots$:

$$\begin{aligned}
a_i^{(t+1)} &= \frac{1}{l_{\mathbf{k}}(i)}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)}\\
&= \frac{1}{N-1}\left[-\left(\sum_{j>i} k_j - \sum_{j<i} k_j\right) a_i^{(t)}\right.\\
&\quad \left. + \left(\sum_{j>i} k_j a_j^{(t)} - \sum_{j<i} k_j a_j^{(t)}\right)\right].
\end{aligned} \tag{67}$$

Then, the equations in (57) are equivalent to the following:

$$\begin{aligned}
&\begin{cases} \frac{1}{N}\sum_{i=1}^{D} k_i a_i^{(0)} = 0\\ \frac{1}{l_{\mathbf{k}}}\sum_{i=1}^{D}\sum_{j=1}^{D} l_{\mathbf{k}}(i,j)b_{ij}^{(t)} = 0, t = 0, 1, \ldots, D-2 \end{cases}\\
\Leftrightarrow &\begin{cases} \sum_{i=1}^{D} k_i a_i^{(0)} = 0\\ \sum_{i=1}^{D}\left(\sum_{j>i} k_j - \sum_{j<i} k_j\right) k_i a_i^{(t)} = 0,\\ \qquad\qquad t = 0, 1, \ldots, D-2 \end{cases}\\
\Leftrightarrow &A\begin{bmatrix} k_1 a_1^{(0)}\\ k_2 a_2^{(0)}\\ \vdots\\ k_D a_D^{(0)} \end{bmatrix} = A\begin{bmatrix} k_1 a_1\\ k_2 a_2\\ \vdots\\ k_D a_D \end{bmatrix} = O,
\end{aligned} \tag{68}$$

where A is a coefficient matrix.

Next consider deriving matrix A in (68) and showing $\det A \neq 0$ in order to prove Lemma 4.8. First, from (67), represent $a_i^{(t)}$ as a linear function of a_h, $h = 1, 2, \ldots, D$, as

$$a_i^{(t)} = \sum_{h=1}^{D} a_{ih}^{(t)} a_h, \tag{69}$$

and then,

$$\begin{aligned}
&\sum_{i=1}^{D}\left(\sum_{j>i} k_j - \sum_{j<i} k_j\right) k_i a_i^{(t)}\\
&= \sum_{h=1}^{D} \frac{\sum_{i=1}^{D}\left(\sum_{j>i} k_j - \sum_{j<i} k_j\right) k_i a_{ih}^{(t)}}{k_h} k_h a_h.
\end{aligned} \tag{70}$$

Therefore, matrix A can be represented as

$$A = \begin{bmatrix} 1 & 1 & \ldots & 1\\ A_{01} & A_{02} & \ldots & A_{0D}\\ A_{11} & A_{12} & \ldots & A_{1D}\\ \vdots & \vdots & \ddots & \vdots\\ A_{D-2,1} & A_{D-2,2} & \ldots & A_{D-2,D} \end{bmatrix}, \tag{71}$$

where

$$\begin{aligned}
A_{th} &= \frac{\sum_{i=1}^{D}\left(\sum_{j>i} k_j - \sum_{j<i} k_j\right) k_i a_{ih}^{(t)}}{k_h},\\
&\quad 0 \leq t \leq D-2,\ 1 \leq h \leq D.
\end{aligned} \tag{72}$$

By using (69), (67) can be represented as

$$
\begin{aligned}
&(N-1)a_i^{(t+1)}\\
&=\sum_{h=1}^{D}\left[-\left(\sum_{j>i}k_j-\sum_{j<i}k_j\right)a_{ih}^{(t)}\right.\\
&\qquad\left.+\sum_{j>i}k_j a_{jh}^{(t)}-\sum_{j<i}k_j a_{jh}^{(t)}\right]a_h\\
&=\sum_{h=1}^{D}\left[\sum_{j>i}k_j\left(a_{jh}^{(t)}-a_{ih}^{(t)}\right)-\sum_{j<i}k_j\left(a_{jh}^{(t)}-a_{ih}^{(t)}\right)\right]a_h.
\end{aligned}
\tag{73}
$$

Then, we get the following:

$$
\begin{aligned}
&a_{ih}^{(0)}=\delta_{ih}\\
&a_{ih}^{(t+1)}\\
&=\frac{1}{N-1}\left(\sum_{j>i}k_j\left(a_{jh}^{(t)}-a_{ih}^{(t)}\right)-\sum_{j<i}k_j\left(a_{jh}^{(t)}-a_{ih}^{(t)}\right)\right).
\end{aligned}
\tag{74}
$$

Equation (74) can be represented by the following recurrence relation by using matrix $\mathcal{A}^{(t)}$, $\left(\mathcal{A}^{(t)}\right)_{ih}:=a_{ih}^{(t)}$:

$$
\begin{aligned}
&\mathcal{A}^{(t+1)}=\frac{1}{N-1}K\mathcal{A}^{(t)},\ \mathcal{A}^{(0)}=I_D,\\
&K:=\begin{bmatrix}
-\sum_{j>1}k_j & k_2 & \cdots & k_D\\
-k_1 & \sum_{j<2}k_j-\sum_{j>2}k_j & \cdots & k_D\\
-k_1 & -k_2 & \cdots & k_D\\
\vdots & \vdots & \ddots & \vdots\\
-k_1 & -k_2 & \cdots & \sum_{j<D}k_j
\end{bmatrix},
\end{aligned}
\tag{75}
$$

and we get

$$\mathcal{A}^{(t)}=(N-1)^{-t}K^t. \tag{76}$$

On the other hand, nonsingularity of A is identical to nonsingularity of the following A', which is obtained by multiplying the i-th column of A by k_i, first row by $\frac{1}{2}$, and j-th row by $(N-1)^{-(j-2)}$:

$$
\begin{aligned}
&A'=\begin{bmatrix}
\frac{k_1}{2} & \frac{k_2}{2} & \cdots & \frac{k_D}{2}\\
A'_{01} & A'_{02} & \cdots & A'_{0D}\\
A'_{11} & A'_{12} & \cdots & A'_{1D}\\
\vdots & \vdots & \ddots & \vdots\\
A'_{D-2,1} & A'_{D-2,2} & \cdots & A'_{D-2,D}
\end{bmatrix},\\
&A'_{th}=(N-1)^{-t}\sum_{i=1}^{D}\left(\sum_{j>i}k_j-\sum_{j<i}k_j\right)k_i a_{ih}^{(t)},\\
&\qquad 0\le t\le D-2,\ 1\le l\le D.
\end{aligned}
\tag{77}
$$

From (76), A' can be represented as

$$
A'=\begin{bmatrix} g\\ -gK\\ -gK^2\\ \vdots\\ -gK^{D-1}\end{bmatrix},\ g:=\begin{bmatrix}\frac{k_1}{2} & \frac{k_2}{2} & \cdots & \frac{k_D}{2}\end{bmatrix}. \tag{78}
$$

Note that A' can be regarded as the observability matrix of the pair K and g. Then, by applying the PBH rank test, we get the following equivalent conditions:

$$A' \text{ is nonsingular} \Leftrightarrow \operatorname{rank}\begin{bmatrix}2g\\ \lambda I_D-K\end{bmatrix}=D,\ \forall\lambda\in\mathbb{C} \tag{79}$$

We can show that the PBH rank test in (79) is true from the further calculation, however, we omit it from the page limitation. From the above, Lemma 4.8 holds. ■

C. Proof of Theorem 4.1

We can see that Lemma 4.9 holds by replacing N by $N-1$, $N-2$, …, and the condition $\tilde{P}_{N-1}\psi_0\neq 0$ is true at $N-1=2$, that is, $\tilde{P}_2\psi_0\neq 0$, by the feedback law. Therefore, even if the initial ψ_0 is in $\mathcal{M}_3^{\mathrm{v}}$, ψ_t leaves $\mathcal{M}_3^{\mathrm{v}}$ at some finite time, which implies $\tilde{\mathcal{M}}^{\mathrm{v}}\cap\mathcal{M}_3^{\mathrm{v}}=\emptyset$, that is, $\tilde{\mathcal{M}}\cap\mathcal{M}_3=\emptyset$ from Lemma 4.7. From the above and Lemma 4.6, $\tilde{\mathcal{M}}\subseteq\mathcal{M}_1$, but on the other hand, $\tilde{\mathcal{M}}\supseteq\mathcal{M}_1$ from Lemma 4.3, so we conclude $\tilde{\mathcal{M}}=\mathcal{M}_1$. Finally, from Lemma 4.3 and Proposition 2.1, the system converges to an SSC in probability for arbitrary initial states. Also, from [9], [10], if the quantum state converges to an SSC in probability, it is an almost convergence, in other words, the quantum state converges to an SSC with probability 1. This completes the proof of Theorem 4.1. ■

V. Conclusion

In this study, we showed that for general D-state N-dimensional quantum network systems, consensus states can be generated by local measurements and a local feedback control. It was shown in [9], [10] that the consensus algorithm analyzed in the present study can generate the W-state, which is one of the usable entangled states. Therefore, one of our future tasks is the physical implementation of a W-state generator.

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