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Control Barrier Functions for Stochastic Systems under Signal Temporal Logic Tasks

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***Abstract*— Signal Temporal Logic (STL) offers an expressive formalism for describing complex high-level tasks in dynamical systems. This paper introduces a time-varying Control Barrier Function (CBF) for control-affine nonlinear stochastic systems to fulfill STL specifications. The CBFs are used in a robust optimization problem to provide a lower bound on the satisfaction probability of a given STL specification with a predetermined robustness level. We present an online control synthesis approach to minimize a performance function while having the required satisfaction guarantee. We finally provide a tractable solution to the robust optimization for STL with linear and quadratic predicate functions. To illustrate the effectiveness of the method, we apply it to a simple linear case study and to the path-planning problem for a nonlinear wheeled mobile robot.**

I. Introduction

Control synthesis for dynamical systems against complex tasks has recently garnered significant attention [1], [2], [3]. Temporal Logic has been considered a popular method for encoding complex specifications involving spatial and/or time requirements for dynamical systems [4]. Signal Temporal Logic (STL) is a formal language that allows us to encode time-constrained tasks by offering a closer connection to the physical system in the sense of having robust semantics as a function of system trajectories [5], [6]. However, control synthesis involving temporal logic typically leads to mixed-integer programming problems, which, regrettably, can entail computationally expensive numerical solutions [7].

Barrier functions offer Lyapunov-like guarantees on system behavior for safety verification [8]. In particular, Control Barrier Functions (CBFs) have proven valuable in synthesizing control strategies for safety-critical systems [9], [10]. In stochastic systems, CBFs are designed using the concept of supermartingales [11]. This concept provides a probabilistic guarantee of maintaining the barrier function within a desired sub-level for a finite-time horizon, given that the CBF has a gradual increment or decrement in expectation in each time step [12]. In the context of control under STL specifications, CBFs have been widely considered, as they offer a computationally efficient framework for encoding STL tasks while avoiding the complexity associated with mixed-integer optimization [13].

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Contributions. This paper considers discrete-time control-affine nonlinear stochastic systems and aims to synthesize a controller for the system that satisfies a given STL specification with the desired robustness and a desired satisfaction probability. Motivated by [14], a time-varying CBF is designed that fulfills a temporal behavior resulting from the STL specification. We enforce the CBF to be supermartingales for the dynamical system in order to establish the probabilistic guarantee of STL satisfaction using a robust optimization problem. Enforcing the CBFs to be supermartingale leads us to an inequality that involves the predicate functions of the STL specification. Once the corresponding CBF parameters are obtained, the supermartingale inequality can be treated as a constraint in a one-step optimization along with minimizing a performance function, such as the control effort. For linear and quadratic predicates, we study this robust optimization and provide tractable formulations. The contributions of this work are thus summarized as follows. 1) Introducing a novel stochastic CBF based on the concept of supermartingales that guarantee STL specifications with a desired probability threshold. 2) Providing a robust optimization to determine the parameters of the CBF. 3) Presenting tractable formulations to solve the robust optimization in the case of linear and quadratic predicate functions.

Related work. A receding horizon approach was introduced in [7] to solve the control synthesis problem for STL specifications by decomposing the STL specification into a series of formulas over each time horizon and solving mixed-integer programming for each subformula. A robust Model Predictive Control (MPC) problem was utilized in [15] to address STL tasks for uncertain dynamics using mixed-integer linear programming techniques. Shrinking horizon MPC with STL constraints and stochastic dynamics was addressed in [16] utilizing the structure of the formula and appropriate concentration inequalities.

In the context of continuous-time deterministic systems with STL specifications, a novel time-varying control barrier function is proposed in [13]. An MPC based on time-varying CBFs with a recursive feasibility guarantee was proposed in [14] and was extended to multi-agent systems in [17]. Mixed-integer quadratic programming was proposed in [18] to satisfy STL formulas for continuous-time linear systems.

In [19], the authors proposed synthesizing control policies for discrete-time stochastic systems under linear temporal logic specifications using CBFs. The authors in [20] developed a framework to interpret STL formulas over discrete-time stochastic processes in terms of the induced risk using STL robustness risk. In [21], partially unknown dynamics with STL specifications were considered, and the STL satisfaction was studied using trajectories of the system and checking whether the probability of satisfying the specification is at least a given threshold. Verification and synthesis problems for safety specifications have been studied in [22] using barrier certificate as a robust convex program.

The closest work to our approach is the time-varying CBF presented in [13], which is then utilized in an MPC setting [14] and in a multi-agent system context [17]. These works focus on continuous-time deterministic systems and provide a CBF based on a log-sum-exp function of the predicate functions that results in quadratic optimizations. In contrast, our approach addresses discrete-time stochastic systems and develops CBFs that depend linearly on the predicate functions. We then employ robust programming techniques for computing a lower bound on that satisfaction probability of the specification.

Outline. The remainder of this paper is organized as follows. Section II presents a preliminary overview of the problem setting and formulates the problem. The time-varying CBFs for the STL specifications are designed in Section III. Section IV introduces a robust optimization problem to maximize the probability of STL satisfaction while enforcing the CBFs to be supermartingale. Moreover, we synthesize the control input while optimizing a performance criterion. We discuss the tractability of the robust optimization problem for linear and control-affine nonlinear systems. Section V provides case studies to illustrate the efficiency of the proposed framework, and Section VI gives a conclusion.

Notation. We use normal font for scalars and bold font for vectors. We denote the interior points of a closed set $\mathbb{X}$ by $\mathrm{Int}(\mathbb{X})$. We denote the set of real numbers and non-negative integers by $\mathbb{R}$ and $\mathbb{N}$, respectively.

II. Preliminaries and Problem Formulation

This section describes the model of the dynamical system used in the paper, STL specifications, and the problem statement.

A. Dynamics

We consider a nonlinear stochastic discrete-time system with a compact state space $\mathbb{X} \subset \mathbb{R}^n$, input space $\mathbb{U} \subseteq \mathbb{R}^m$, and disturbance set $\mathbb{W} \subseteq \mathbb{R}^n$. The dynamics of the system are as

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k) + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{u}_k + \boldsymbol{w}_k, \tag{1}$$

where k is the time index, $\boldsymbol{x}_k \in \mathbb{X}$ is the state, $\boldsymbol{u}_k \in \mathbb{U}$ is the control input, $\boldsymbol{w}_k \in \mathbb{W}$ is the stochastic disturbance, and $\boldsymbol{f}: \mathbb{X} \to \mathbb{R}^n$ and $\boldsymbol{g}: \mathbb{X} \to \mathbb{R}^{n\times m}$ are the functions describing the system dynamics. Note that due to the potential unbounded stochastic disturbance $\boldsymbol{w}_k$, the states of the process $\boldsymbol{x}_k$ may depart from the state space $\mathbb{X}$, which leads us to define the stopped process.

Definition 1. *Suppose that $\kappa > 0$ is the first time of exit of the states from the open set $\mathrm{Int}(\mathbb{X})$. The stopped process $\tilde{\boldsymbol{x}}$ is then defined, as follows:*

$$\tilde{\boldsymbol{x}}_k = \begin{cases} \boldsymbol{x}_k & \text{for } k < \kappa \\ \boldsymbol{x}_{\kappa-1} & \text{for } k \geq \kappa\,. \end{cases}$$

For the sake of notational simplicity, in this paper, we consider the stochastic process in (1) as a stopped process, and we denote it simply by $\boldsymbol{x}$ instead of $\tilde{\boldsymbol{x}}$. Without loss of generality, we assume $\mathbb{E}[\boldsymbol{w}] = 0$ and $\mathbb{E}[\boldsymbol{w}\boldsymbol{w}^\top] = \Sigma$. Moreover, we make the following assumption on the function $\boldsymbol{g}$.

Assumption 1. *Function $\boldsymbol{g}$ is such that $\boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}^\top(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in \mathbb{X}$.*

Assumption 1 enables us to make the system (1) feedback equivalent to $\boldsymbol{x}_{k+1} = \boldsymbol{u}_k + \boldsymbol{w}_k$ which will be used in Section IV-C to solve the proposed robust optimization. A similar assumption is also adopted in prior works such as [13].

B. Stochastic Control Barrier Function (CBF)

We now review the concepts from probability theory and martingales that will allow us to construct a notion of safety with probabilistic guarantees. We will use this framework to generate time-varying CBFs for STL specifications. We start by recalling the definition of nonnegative supermartingales provided, e.g., in [23].

Definition 2. *Let $\boldsymbol{x}_k$ be the trajectory of the system in (1), $B: \mathbb{X}\times\mathbb{N} \to \mathbb{R}$, and suppose that $\mathbb{E}\left[|B(\boldsymbol{x}_k, k)|\right] < \infty$ for all $k \in \mathbb{N}$. The process $B_k := B(\boldsymbol{x}_k, k)$ is a supermartingale for the system (1) if:*

$$\mathbb{E}\left[B_{k+1}|\boldsymbol{x}_{0:k}\right] \leq B_k, \quad \text{almost surely for all } k \in \mathbb{N},$$

where $\boldsymbol{x}_{0:k}$ indicates $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k\}$. If, additionally, $B_k \geq 0$ for all $k \in \mathbb{N}$, B_k is a nonnegative supermartingale.

Roughly speaking, function B_k is non-increasing in expectation along the system trajectories for supermartingale functions. The next Lemma is an important consequence of nonnegative supermartingales.

Lemma 1. *Let B_k be a nonnegative supermartingale as given in definition 2, then for all $\lambda \in \mathbb{R}_{>0}$, the following holds:*

$$\mathbb{P}\left\{\sup_{k\in\mathbb{N}} B_k > \lambda\right\} \leq \frac{B_0}{\lambda}\,.$$

Proof. The proof is based on Villes's inequality and can be found, e.g., in [24]. ∎

The above inequality can also be expressed equivalently as follows

$$\mathbb{P}\left\{\sup_{k\in\mathbb{N}} B_k \leq \lambda\right\} \geq 1 - \frac{B_0}{\lambda}\,,$$

which we are more interested in. Note that the term B_0/λ represents the probability of escaping the set $\mathbb{S} := \bigcap_{k=0}^{N}\{\boldsymbol{x}_k \in \mathbb{X} \,|\, B(x_k, k) \leq \lambda\}$ before a given time instant $N \in \mathbb{N}$.

C. Signal Temporal Logic (STL)

Signal Temporal Logic (STL) is a predicate logic composed of atomic predicates μ that may be false $\perp$ or true $\top$ [5], [6]. The assessment of the validity of each atomic predicate μ relies on the sign of a continuously differentiable predicate function $h : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\mu = \begin{cases} \top, & \text{if } h(\boldsymbol{x}) \geq 0 \\ \perp, & \text{if } h(\boldsymbol{x}) < 0 \,. \end{cases}$$

The syntax of the STL formulae is defined recursively according to the following grammar:

$$\phi ::= \top \,|\, \mu \,|\, \neg\phi \,|\, \phi \wedge \psi \,|\, \phi\, \mathcal{U}_{[a,b]}\, \psi \,,$$

where ϕ, ψ are STL formulas, notations $\neg$ and $\wedge$ denote *negation* and *conjunction* of formulas and $\mathcal{U}_{[a,b]}$ denotes the *until* operator.

The *eventually* and the *always* operators are defined as $\mathcal{F}_{[a,b]}\phi := \top\, \mathcal{U}_{[a,b]}\, \phi$ and $\mathcal{G}_{[a,b]}\phi := \neg\mathcal{F}_{[a,b]}\neg\phi$, respectively. Note that since we are focusing on discrete-time systems in this paper, the intervals that appear as subscripts of formulas comprise integers between, and including, two integers, $a, b \in \mathbb{N}$ where $0 \leq a \leq b < +\infty$. The satisfaction relation $(\boldsymbol{\xi}, k) \models \phi$ denotes if the sequence $\boldsymbol{\xi}$ satisfies ϕ at time k, where $(\boldsymbol{\xi}, k) := \{\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots\}$ is a trajectory of a system starting at time k and e.g., obtained from system (1) for a given input sequence $\{\boldsymbol{u}_k, \boldsymbol{u}_{k-1}, \ldots\}$.

Semantics: The satisfaction of an STL formula ϕ by a trajectory $\boldsymbol{\xi}$ at time k is defined recursively as follows:

$$\begin{aligned}
(\boldsymbol{\xi}, k) \models \mu \quad &\Leftrightarrow \quad h(\boldsymbol{x}_k) \geq 0 \\
(\boldsymbol{\xi}, k) \models \neg\phi \quad &\Leftrightarrow \quad \neg((\boldsymbol{\xi}, k) \models \phi) \\
(\boldsymbol{\xi}, k) \models \phi \wedge \psi \quad &\Leftrightarrow \quad (\boldsymbol{\xi}, k) \models \phi \wedge (\boldsymbol{\xi}, k) \models \psi \\
(\boldsymbol{\xi}, k) \models \phi\mathcal{U}_{[a,b]}\, \psi &\Leftrightarrow \exists k' \in \{k+a, \ldots, k+b\}, (\boldsymbol{\xi}, k') \models \psi \\
&\qquad \wedge \forall k'' \in \{k, \ldots, k'\}, (\boldsymbol{\xi}, k'') \models \phi \,.
\end{aligned}$$

Robustness measure: Quantitative or robust semantics for an STL formula ϕ are defined by providing a real-valued function ρ^ϕ of the signal $\boldsymbol{\xi}$ at time k such that $\rho^\phi(\boldsymbol{\xi}, k) > 0$ implies that $(\boldsymbol{\xi}, k) \models \phi$. Such functions are defined recursively as follows:

$$\begin{aligned}
&\rho^\top(\boldsymbol{\xi}, k) = +\infty \\
&\rho^\mu(\boldsymbol{\xi}, k) = h(\boldsymbol{x}_k) \\
&\rho^{\neg\phi}(\boldsymbol{\xi}, k) = -\rho^\phi(\boldsymbol{\xi}, k) \\
&\rho^{\phi\wedge\psi}(\boldsymbol{\xi}, k) = \min(\rho^\phi(\boldsymbol{\xi}, k), \rho^\psi(\boldsymbol{\xi}, k)) \\
&\rho^{\phi\,\mathcal{U}_{[a,b]}\,\psi}(\boldsymbol{\xi}, k) = \max_{k' \in \{k+a, \ldots, k+b\}} \Big(\min\big(\rho^\psi(\boldsymbol{\xi}, k'), \\
&\qquad\qquad \min_{k'' \in \{k, \ldots, k'\}} \rho^\phi(\boldsymbol{\xi}, k'')\big)\Big) \,.
\end{aligned}$$

The robustness function $\rho^\phi(\boldsymbol{\xi}, k)$ can be interpreted as how much $\boldsymbol{\xi}$ satisfies ϕ.

D. Problem formulation

We consider in this paper a fragment of STL as:

$$\psi = \top \,|\, \mu \,|\, \neg\mu, \tag{2a}$$

$$\bar{\psi} = \mathcal{G}_{[a,b]}\psi \,|\, \mathcal{F}_{[a,b]}\psi \,|\, \psi_1\, \mathcal{U}_{[a,b]}\, \psi_2 \,, \tag{2b}$$

$$\phi = \bigwedge_{j=1}^{n_\phi} \bar{\psi}_j \,, \tag{2c}$$

where ψ_1, ψ_2 are STL formulas of the form (2a), $\bar{\psi}_j$s for $j = 1, \ldots, n_\phi$ are STL formulas of the form (2b) for some n_ϕ. Any ϕ in (2c) can be fulfilled by the following formula:

$$\phi = \bigwedge_{i=1}^{M} \phi_i, \tag{3}$$

where $M = n_\phi + n_u$ and n_u is the number of until operators. The sub-formula ϕ_i is either an eventually formula or an always formula with the corresponding time interval $[a_i, b_i]$ and predicate function h_i. Note that this is true since the satisfaction of any until formula $\psi_1\, \mathcal{U}_{[a,b]}\, \psi_2$ in (2c) can be ensured by the satisfaction of a formula written in conjunction of an always and an eventually formula, i.e., $\mathcal{G}_{[a,b]}\psi_1 \wedge \mathcal{F}_{[b,b]}\psi_2$.

Since the system discussed in this paper is stochastic and may involve unbounded disturbances, a challenge in dealing with such systems is to satisfy the constraints probabilistically. Therefore, the objective of this paper is to synthesize a control strategy that fulfills the provided STL formulas with a desired confidence level.

Problem 1. *For a given robustness level $r > 0$ and probability threshold ϵ, design state feedback control inputs $\boldsymbol{u}_k : \mathbb{X} \to \mathbb{U}, k \in \mathbb{N}$, for the system in (1), such that the probability of the system's trajectories satisfying the STL task in (3) with robustness level r is at least $(1-\epsilon)$:*

$$\mathbb{P}\{\rho^\phi(\boldsymbol{\xi}, 0) \geq r\} \geq 1 - \epsilon.$$

In the next section, we detail the design of CBFs for stochastic systems to satisfy STL tasks in probability and solve problem 1.

III. CBF design for STL specifications

In the following, we propose a novel time-varying CBF as a nonnegative supermartingale that fulfills the STL specifications, defined in Section II-D, with a desired confidence level. We first define a time-varying CBF, denoted by B_k^i, corresponding to the STL sub-formula ϕ_i, defined in (3), as follows:

$$B_k^i(\boldsymbol{x}_k, k) := h_{\max}^i - \gamma_0^i - h_i(\boldsymbol{x}_k) + \gamma_k^i \,, \tag{4}$$

for all $1 \leq i \leq M$ and $k \in \mathbb{N}$, where γ_k^i is a time-varying parameter describing a desired temporal behavior for the system and $h_{\max}^i := \sup_{\boldsymbol{x}\in\mathbb{X}} h_i(\boldsymbol{x}) < \infty$. Note that we assume that $h_{\max}^i$ (or an upper bound) is provided as in [13], [14]. Inspired by [17], we design the time-varying parameters γ_k^i as

$$\gamma_k^i = \begin{cases} \frac{\gamma_\infty^i - \gamma_0^i}{k^{\star,i}} k + \gamma_0^i & \text{if } k < k^{\star,i} \\ \gamma_\infty^i & \text{if } k \geq k^{\star,i} \,, \end{cases} \tag{5}$$

where γ_0^i, γ_∞^i and $k^{\star,i}$ are the design parameters selected as

$$\gamma_0^i \in (-\infty, h_i(\boldsymbol{x}_0))\,, \tag{6a}$$
$$\gamma_\infty^i \in (\max(r, \gamma_0^i), h_{\max}^i)\,, \tag{6b}$$
$$k^{\star,i} \in \begin{cases} \{a_i, \ldots, b_i\} & \text{if } \phi_i = \mathcal{F}_{[a_i,b_i]}\psi \\ \{a_i\} & \text{if } \phi_i = \mathcal{G}_{[a_i,b_i]}\psi\,, \end{cases} \tag{6c}$$

for a robustness parameter $r > 0$, satisfying

$$r \in \begin{cases} (0, h_i(\boldsymbol{x}_0)) & \text{if } k^{\star,i} = 0 \\ (0, h_{\max}^i) & \text{if } k^{\star,i} > 0\,. \end{cases}$$

Note that the justification for piece-wise linearly considering the time-varying parameter γ_k^i can be found in [25]. From (5) and (6b), it can be seen that for $k \geq k^{\star,i}$, $r - h_i(\boldsymbol{x}) \leq \gamma_k^i - h_i(\boldsymbol{x})$ since $r \leq \gamma_\infty^i$. Subsequently

$$\gamma_k^i - h_i(\boldsymbol{x}) \leq 0 \Rightarrow r - h_i(\boldsymbol{x}) \leq 0 \tag{7}$$

for all time $k \geq k^{\star,i}$. Another consequence of (6a) is $\gamma_0^i < h_{\max}^i$. From (6b), one can see $\gamma_0^i \leq \gamma_\infty^i$, therefore, $\gamma_k^i \leq \gamma_{k+1}^i$ from (5). Note that here, we only provide an upper bound on γ_0^i, and this parameter will be used as a decision variable to maximize the satisfaction probability. We will detail on this in the next section. For the always formula, $k^{\star,i}$ is at the beginning of the interval to guarantee the satisfaction of the formula for the entire interval, and for the eventually formula, this time can be any value in the given time interval. This choice of $k^{\star,i}$, depending on the type of the STL formula and its time interval, is made similarly to the choice of t_i^* in [13] to guarantee that the desired STL property is satisfied at the desired time instance.

In the next section, we provide an optimization problem to maximize the probability of STL satisfaction and synthesize the control input.

IV. Control Synthesis

This section presents an optimization based on the CBFs, designed in the previous section, to maximize the probability of STL satisfaction and extract the control input. To this end, we require the functions B_k^i to be nonnegative supermartingale functions for all $i \in \{1, \ldots, M\}$, provided that certain inequalities are satisfied in the predicate function for all states. These conditions are addressed as constraints in a robust optimization problem.

A. Maximizing the probability of satisfaction

We now consider a robust optimization problem that maximizes the probability of STL satisfaction while making the functions B_k^i nonnegative supermartingales for all $i \in \{1, \ldots, M\}$. Based on B_k^is, the maximum satisfaction probability can be obtained from the following robust optimization:

$$\min_{\epsilon, \gamma_0^i} \quad \epsilon \tag{8a}$$
$$\text{s.t.} \quad \forall i \in \{1, \ldots, M\}:$$
$$\frac{h_{\max}^i - h_i(\boldsymbol{x}_0)}{h_{\max}^i - \gamma_0^i} \leq \epsilon, \tag{8b}$$
$$\gamma_0^i \leq h_i(\boldsymbol{x}_0), \tag{8c}$$
$$\forall k \in \mathbb{N},\, \forall \boldsymbol{x}_k \in \mathbb{X},\, \exists \boldsymbol{u}_k \in \mathbb{U}:$$
$$\mathbb{E}\left[h_i(\boldsymbol{x}_{k+1}) | \boldsymbol{x}_k\right] \geq h_i(\boldsymbol{x}_k) + \delta_k^i, \tag{8d}$$

where

$$\delta_k^i = \gamma_{k+1}^i - \gamma_k^i = \begin{cases} \frac{\gamma_\infty^i - \gamma_0^i}{k^{\star,i}} & \text{if } k < k^{\star,i} \\ 0 & \text{if } k \geq k^{\star,i}\,, \end{cases} \tag{9}$$

and ϵ is the STL violation probability. We denote the optimal solution of (8) by $\epsilon^\star$. In (8), the objective is to minimize the probability of the violation, the constraint (8b) provides a tight upper-bound on the probability of the violation of all sub-formulas ϕ_i for all $i \in \{1, \ldots, M\}$, which is minimized in the objective function and the constraint (8d) ensures that B_k^is are supermartingale functions. Since δ_k^i in constraint (8d) depends on time k and appears as a piecewise constant function in (9), the condition "$\forall k \in \mathbb{N}$" can be omitted. Thus, (8d) can be solved without time indices with condition: "$\forall \boldsymbol{x} \in \mathbb{X}$, $\exists \boldsymbol{u} \in \mathbb{U}$". Consequently, it is sufficient to solve (8d) only for the constants $\delta_k^i = \frac{\gamma_\infty^i - \gamma_0^i}{k^{\star,i}}$ and $\delta_k^i = 0$ which is also fulfilled by solving only for $\delta_k^i = \frac{\gamma_\infty^i - \gamma_0^i}{k^{\star,i}}$ because $\frac{\gamma_\infty^i - \gamma_0^i}{k^{\star,i}} \geq 0$. It is worth noting that, for the sake of simplicity, we treat γ_0^i as a decision variable to shape B_k^i, whereas γ_∞^i and $k^{\star,i}$ are considered known parameters and are selected from (6b)-(6c). These statements will be detailed in the following theorem.

Theorem 1. *The following statements hold for the solution of* (8)*:*

1) *The functions B_k^i are non-negative supermartingale for all $i \in \{1, \ldots, M\}$.*
2) *For the given STL formula ϕ in* (3)*:*

$$\mathbb{P}\{\rho^\phi(\boldsymbol{\xi}, 0) \geq r\} \geq 1 - M\epsilon^\star\,,$$

where $\epsilon^\star$ is an optimal solution of (8) *and $\boldsymbol{\xi}$ is the system trajectory when the control input $\boldsymbol{u}_k$ obtained from the constraint* (8d) *is applied to the system at the state $\boldsymbol{x}_k$. Therefore, we obtain a solution for problem 1 with any $\epsilon \geq M\epsilon^*$.*

Proof. First, we show that B_k^i is a nonnegative supermartingale function. Considering that $\gamma_0^i \leq \gamma_\infty^i$, then according to (5), $\gamma_0^i \leq \gamma_k^i$. It results in $B_k^i \geq 0$ due to the fact that $h_{\max}^i \geq h_i(\boldsymbol{x}_k)$. Considering the Markovian property of the system (1), $\mathbb{E}\left[B_{k+1}^i | \boldsymbol{x}_{0:k}\right] = \mathbb{E}\left[B_{k+1}^i | \boldsymbol{x}_k\right]$ for all $i \in \{1, \ldots, M\}$ holds because $\boldsymbol{x}_{k+1}$ (and B_{k+1}^i) only depends on the state, input and disturbance at time k. From constraint (8d), we have:

$$\mathbb{E}\left[h_i(\boldsymbol{x}_{k+1}) | \boldsymbol{x}_k\right] \geq h_i(\boldsymbol{x}_k) + \gamma_{k+1}^i - \gamma_k^i,$$
$$\Rightarrow \mathbb{E}\left[B_{k+1}^i | \boldsymbol{x}_k\right] \leq B_k^i,$$

for all $\boldsymbol{x}_k \in \mathbb{X}$ and $i \in \{1, \ldots, M\}$. Thus, the requirement of inequality (8d) for the functions h_i ensures that functions B_k^i are supermartingales. Applying lemma 1 for B_k^i, $\forall i \in \{1, \ldots, M\}$, and using the first statement of the theorem 1, $\lambda = h_{\max}^i - \gamma_0^i > 0$ for all $i \in \{1, \ldots, M\}$, yields:

$$\mathbb{P}\left\{\sup_{k \in \mathbb{N}} B_k^i \leq h_{\max}^i - \gamma_0^i\right\} \geq 1 - \frac{h_{\max}^i - h_i(\boldsymbol{x}_0)}{h_{\max}^i - \gamma_0^i} \overset{(8b)}{\geq} 1 - \epsilon^\star, \tag{11}$$

for all $i \in \{1, \ldots, M\}$. Moreover, we have:

$$\sup_{k \in \mathbb{N}} B_k^i \leq h_{\max}^i - \gamma_0^i \Rightarrow \forall k \in \mathbb{N}: \; B_k^i \leq h_{\max}^i - \gamma_0^i$$
$$\Rightarrow \forall k \in \mathbb{N}: \; \gamma_k^i \leq h_i(\boldsymbol{x}_k) \overset{(7)}{\Rightarrow} \forall k \geq k^{\star,i}: \; r \leq h_i(\boldsymbol{x}_k), \tag{12}$$

for all $i \in \{1, \ldots, M\}$. If (12) holds, for the always formula ϕ_i, we have $k^{\star,i} = a_i$ and

$$\rho^{\phi_i}(\boldsymbol{\xi}, 0) = \min_{k \in \{a_i, \ldots, b_i\}} h_i(\boldsymbol{x}_k) \geq r,$$

and for the eventually formula ϕ_i, $k^{\star,i} \in \{a_i, \ldots, b_i\}$ and

$$\rho^{\phi_i}(\boldsymbol{\xi}, 0) = \max_{k \in \{a_i, \ldots, b_i\}} h_i(\boldsymbol{x}_k) \geq h_i(\boldsymbol{x}_{k^{\star,i}}) \geq r,$$

Therefore (11) results in $\mathbb{P}\{\rho^{\phi_i}(\boldsymbol{\xi}, 0) \geq r\} > 1 - \epsilon^\star$ for all $i \in \{1, \ldots, M\}$. Then, since $\rho^{\phi}(\boldsymbol{\xi}, 0) = \min_{1 \leq i \leq M} \rho^{\phi_i}(\boldsymbol{\xi}, 0)$, we have:

$$\begin{aligned}
\mathbb{P}\{\rho^{\phi}(\boldsymbol{\xi}, 0) \geq r\} &= 1 - \mathbb{P}\{\min_{1 \leq i \leq M} \rho^{\phi_i}(\boldsymbol{\xi}, 0) < r\} \\
&\geq 1 - \sum_{i=1}^{M} \mathbb{P}\{\rho^{\phi_i}(\boldsymbol{\xi}, 0) < r\} \\
&= 1 - \sum_{i=1}^{M} \big(1 - \mathbb{P}\{\rho^{\phi_i}(\boldsymbol{\xi}, 0) \geq r\}\big) \\
&\geq 1 - \sum_{i=1}^{M} \epsilon^\star = 1 - M\epsilon^\star, \qquad \blacksquare
\end{aligned}$$

B. Control synthesis using an optimization problem

In the robust optimization problem (8), our objective is to determine a lower bound on the violation probability of the STL specifications and evaluate the corresponding CBFs offline. However, the control input $\boldsymbol{u}_k$ obtained from this optimization may not be optimal in terms of performance criteria. After obtaining the optimal values for ϵ and γ_0^i, we can aim to minimize control efforts, for instance, in terms of 2-norm of the input, using the following online optimization problem:

$$\begin{aligned}
\boldsymbol{u}_k^\star(\boldsymbol{x}_k) \in \arg\min_{\boldsymbol{u}_k \in \mathbb{U}} \quad & \boldsymbol{u}_k^\top \boldsymbol{u}_k \\
\text{s.t.} \quad & \mathbb{E}\left[h_i(\boldsymbol{x}_{k+1}) | \boldsymbol{x}_k\right] \geq h_i(\boldsymbol{x}_k) + \delta_k^{\star,i},
\end{aligned} \tag{13}$$

which is solved at every time instance k, where $\delta_k^{\star,i}$ is evaluated given by (9) based on the optimal CBF parameters obtained from (8). It is worth noting that alternative objective functions such as minimizing a nominal control with the STL control or other tracking-type objectives can be considered. The feasibility of (13) is ensured for all $\boldsymbol{x}_k \in \mathbb{X}$ as long as (8) is feasible.

C. Tractable formulation for robust optimization (8)

In this section, we provide a tractable optimization problem, equivalent to the robust optimization in (8), for cases of the linear and quadratic predicate functions using the concept of duality in optimization. Through this section, we assume $\mathbb{U} := \mathbb{R}^m$ for the sake of simplicity and clarity. However, it is important to note that this assumption is not limiting; a control set can be incorporated as an additional robust inequality, expanding the applicability of our approach.

Linear Predicate: Suppose that the predicate functions are linear in the form of $h_i(\boldsymbol{x}) = \boldsymbol{\alpha}_i^\top \boldsymbol{x} + \beta_i$ and the state set is a polyhedron in the form of $\mathbb{X} := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, D_x \boldsymbol{x} \leq \boldsymbol{d}_x\}$ with $\boldsymbol{d}_x \in \mathbb{R}^p$. Then the robust inequality in (8d) is written as

$$\forall \boldsymbol{x} \in \mathbb{X}, \; \exists \boldsymbol{u} \in \mathbb{R}^m: \; \boldsymbol{\alpha}_i^\top(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} - \boldsymbol{x}) \geq \delta_k^i, \tag{14}$$

Considering a control input $\boldsymbol{u}$ in the form of

$$\boldsymbol{u} = \boldsymbol{g}^\top(\boldsymbol{x})(\boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}^\top(\boldsymbol{x}))^{-1}(-\boldsymbol{f}(\boldsymbol{x}) + \bar{\boldsymbol{u}}) \in \mathbb{R}^m, \tag{15}$$

results in $\bar{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u}$ for all $\boldsymbol{x} \in \mathbb{X}$ and an auxiliary variable $\bar{\boldsymbol{u}} \in \mathbb{R}^m$. Then (14) can be written as

$$\forall \boldsymbol{x} \in \mathbb{X}, \; \exists \bar{\boldsymbol{u}} \in \mathbb{R}^m: \; \boldsymbol{\alpha}_i^\top(\bar{\boldsymbol{u}} - \boldsymbol{x}) \geq \delta_k^i, \tag{16}$$

Note that the control input $\boldsymbol{u}$ in (15) exists for all $\boldsymbol{x} \in \mathbb{X}$ from assumption 1. In order to solve (16), it is sufficient to find a $\bar{\boldsymbol{u}} \in \mathbb{R}^m$ such that the inequality holds with respect to the worst-case $\boldsymbol{x} \in \mathbb{X}$, i.e.,:

$$\exists \bar{\boldsymbol{u}} \in \mathbb{R}^m: \; \boldsymbol{\alpha}_i^\top \bar{\boldsymbol{u}} + \min_{\boldsymbol{x} \in \mathbb{X}}(-\boldsymbol{\alpha}_i^\top \boldsymbol{x}) \geq \delta_k^i, \tag{17}$$

gives a solution for $\bar{\boldsymbol{u}}$ that satisfies (16) for all $\boldsymbol{x} \in \mathbb{X}$ regardless of the value $\boldsymbol{x}$. Note that the control input $\boldsymbol{u}$ still depends on the state $\boldsymbol{x}$ based on the transformation in (15). We now consider the duality of the minimization in (17), as follows:

$$\begin{cases} \min_{\boldsymbol{x}} & -\boldsymbol{\alpha}_i^\top \boldsymbol{x}, \\ \text{s.t.} & D_x \boldsymbol{x} \leq \boldsymbol{d}_x \end{cases} = \begin{cases} \max_{\boldsymbol{\lambda}_i} & -\boldsymbol{d}_x^\top \boldsymbol{\lambda}_i, \\ \text{s.t.} & D_x^\top \boldsymbol{\lambda}_i = \boldsymbol{\alpha}_i, \; \boldsymbol{\lambda}_i \geq 0 \end{cases}$$

It should be emphasized that the strong duality holds for linear programming [26] assuming that $\mathbb{X}$ is a non-empty set and a feasible solution exists. Therefore, the optimal values of the above optimizations are identical. Consequently, (17) can be written as the following feasibility problem:

$$\begin{aligned}
&\exists \, \bar{\boldsymbol{u}} \in \mathbb{R}^m, \boldsymbol{\lambda}_i \in \mathbb{R}^p: \\
&\qquad \boldsymbol{\alpha}_i^\top \bar{\boldsymbol{u}} - \boldsymbol{d}_x^\top \boldsymbol{\lambda}_i \geq \delta_k^i, \; D_x^\top \boldsymbol{\lambda}_i = \boldsymbol{\alpha}_i, \; \text{and} \, \boldsymbol{\lambda}_i \geq 0
\end{aligned}$$

and (8) can be written as follows:

$$\begin{aligned}
\min_{\bar{\boldsymbol{u}}, \epsilon, \gamma_0^i, \boldsymbol{\lambda}_i} \quad & \epsilon \\
\text{s.t.} \quad & \forall i \in \{1, \ldots, M\}: \\
& \frac{h_{\max}^i - h_i(\boldsymbol{x}_0)}{h_{\max}^i - \gamma_0^i} \leq \epsilon, \quad \gamma_0^i \leq h_i(\boldsymbol{x}_0), \\
& \alpha_i^\top \bar{\boldsymbol{u}} - \boldsymbol{d}_x^\top \boldsymbol{\lambda}_i \geq \delta_k^i, \quad D_x^\top \boldsymbol{\lambda}_i = \boldsymbol{\alpha}_i \, .
\end{aligned}$$

Quadratic Predicate: We now consider the case where the predicate function is quadratic in the form of

$$h_i(\boldsymbol{x}) = \boldsymbol{x}^\top \Gamma_i \boldsymbol{x} + 2\boldsymbol{\alpha}_i^\top \boldsymbol{x} + \beta_i$$

and the state set $\mathbb{X} = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{x}^\top \boldsymbol{x} \leq d_x\}$. From (15), the inequality (8d) is written as follows:

$$\begin{aligned} &\forall \boldsymbol{x} \in \mathbb{X},\ \exists \bar{\boldsymbol{u}} \in \mathbb{R}^m : \\ &\bar{\boldsymbol{u}}^\top \Gamma_i \bar{\boldsymbol{u}} + 2\boldsymbol{\alpha}_i^\top \bar{\boldsymbol{u}} + \mathrm{tr}(\Gamma_i \Sigma) \geq \boldsymbol{x}^\top \Gamma_i \boldsymbol{x} + 2\boldsymbol{\alpha}_i^\top \boldsymbol{x} + \delta_k^i . \end{aligned}$$

In the STL formulations, the predicate functions are not necessarily convex. Accordingly, we investigate both concave and convex quadratic functions. Based on a procedure similar to the linear predicate case, and from where in [26], the following strong duality holds for indefinite Γ_i:

$$\begin{cases} \max_{\boldsymbol{x}} & -\boldsymbol{x}^\top \Gamma_i \boldsymbol{x} - 2\boldsymbol{\alpha}_i^\top \boldsymbol{x}, \\ \text{s.t.} & \boldsymbol{x}^\top \boldsymbol{x} \leq d_x \end{cases} =$$

$$\begin{cases} \min_{\lambda_i, \mu_i} & \mu_i, \\ \text{s.t.} & \begin{bmatrix} \lambda_i I + \Gamma_i & \boldsymbol{\alpha}_i^\top \\ \boldsymbol{\alpha}_i^\top & -\lambda_i d_x + \mu_i \end{bmatrix} \succeq 0,\ \lambda_i \geq 0 . \end{cases}$$

Hence, similar to the linear predicate, we can extract the following optimization:

$$\begin{aligned} \min_{\bar{\boldsymbol{u}}, \epsilon, \gamma_0^i, \lambda_i, \mu_i} \quad & \epsilon \\ \text{s.t.} \quad & \forall i \in \{1, \dots, M\} : \\ & \frac{h_{\max}^i - h_i(\boldsymbol{x}_0)}{h_{\max}^i - \gamma_0^i} \leq \epsilon, \quad \gamma_0^i \leq h_i(\boldsymbol{x}_0), \\ & \bar{\boldsymbol{u}}^\top \Gamma_i \bar{\boldsymbol{u}} + 2\boldsymbol{\alpha}_i^\top \bar{\boldsymbol{u}} + \mathrm{tr}(\Gamma_i \Sigma) \geq \mu_i + \delta_k^i, \\ & \begin{bmatrix} \lambda_i I + \Gamma_i & \boldsymbol{\alpha}_i^\top \\ \boldsymbol{\alpha}_i^\top & -\lambda_i d_x + \mu_i \end{bmatrix} \succeq 0,\ \lambda_i \geq 0 . \end{aligned}$$

In order to apply the method to control restricted problems, it is sufficient to confirm whether $\boldsymbol{u}$ belongs to the set $\mathbb{U}$ for all $\boldsymbol{x} \in \mathbb{X}$. This verification can be performed for specific cases, such as linear and quadratic systems with polyhedral and quadratic control sets. In such situations, an additional robust inequality must be satisfied and can be treated similarly to the approach we have proposed. For instance, this issue is addressed in case study V-A.

V. Case Studies

In this section, we present two case studies to demonstrate the efficiency of the proposed method for both linear and quadratic predicates.

A. Linear dynamics

We implement our method on a simple linear system with a bounded control set. The dynamics are described by $x_{k+1} = x_k + u_k + w_k$, where $-10 \leq x_k \leq 10$ is the state, $-d_u \leq u_k \leq d_u$ is the input with a parametric set, and w_k is a zero mean random variable. Moreover, we consider an eventually formula $\mathcal{F}_{[0,10]}\phi_1$ with predicate function $h_1(x) = x - 5$. Figure 1 illustrates the maximum probability of satisfaction as a function of the control set size. As it can be seen, the probability is zero for a very small control set, and it increases as the control set size grows.

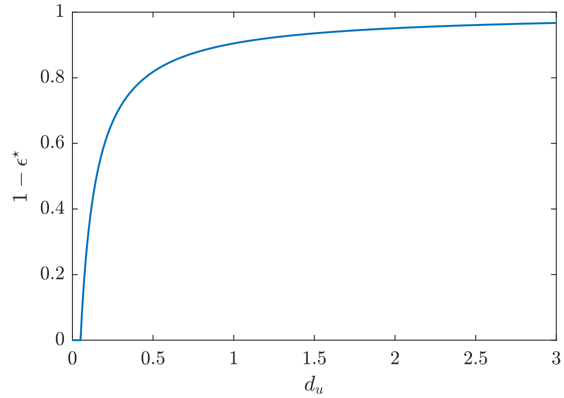

Fig. 1. The probability of the STL satisfaction as a function of the control set size.

B. Robot path planning

Consider the Wheeled Mobile Robot (WMR) path planning problem with the following dynamics:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \begin{bmatrix} \cos(z_k) & -\sin(z_k) & 0 \\ \sin(z_k) & \cos(z_k) & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{u}_k + \boldsymbol{w}_k ,$$

where $\boldsymbol{x}_k = [x_k, y_k, z_k]^\top$, $\boldsymbol{u}_k = [u_k^1, u_k^2, u_k^3]^\top$ and $\boldsymbol{w}_k \sim \mathcal{N}(\mathbf{0}, \mathrm{diag}(0.025, 0.025, 0.005))$ are the system state, input, and disturbance, respectively, and where (x_k, y_k) is the position of the robot in $\mathbb{R}^2$ and z_k is the orientation angle. Based on the quadratic predicates, we define the following sets:

$$\begin{aligned} O &= \{x, y \,|\, h_1(\boldsymbol{x}) := \|[x-4, y-4]\|^2 - 4 \leq 0\} , \\ A &= \{x, y \,|\, h_2(\boldsymbol{x}) := 2.25 - \|[x-1, y-8]\|^2 \geq 0\} , \\ T &= \{x, y \,|\, h_3(\boldsymbol{x}) := 1 - \|[x-9, y-9]\|^2 \geq 0\} . \end{aligned}$$

The sets A, O, and T are shown in green, gray, and purple colors, respectively, in figure 2. The objective is to control the WMR to visit set A within 30 steps and eventually visit set T within 40 steps from the beginning while always avoiding set O with the level of robustness $r = 0.1$. The task can be written as $\phi = \bigwedge_{l=1}^{3} \bar{\psi}_l$, where $\bar{\psi}_1 = \mathcal{G}_{[0,40]}\,\psi_1$, $\bar{\psi}_2 = \neg\psi_3\,\mathcal{U}_{[0,30]}\,\psi_2$ and $\bar{\psi}_3 = \mathcal{F}_{[30,40]}\,\psi_3$. The corresponding predicate functions for ψ_is are h_is. We get the maximum satisfaction probability of $\epsilon^\star \approx 0.12$. Figure 2 shows 100 trajectories of the WMR for the different realizations of the disturbance. As it can be seen, most of the trajectories (96 of them) avoid the obstacle set $\mathbb{O}$, and all the trajectories reach the target set $\mathbb{T}$ while passing from the set $\mathbb{A}$ within the desired deadlines.

Figure 3 illustrates the histogram of the time taken to reach sets A and T. As it can be seen, both respect the required deadline for the reaching time from the STL specification.

VI. Conclusion

The paper addresses the control synthesis problem for discrete-time stochastic systems that are subject to STL specifications. A time-varying Control Barrier Function (CBF) was constructed to ensure a desired probability of the

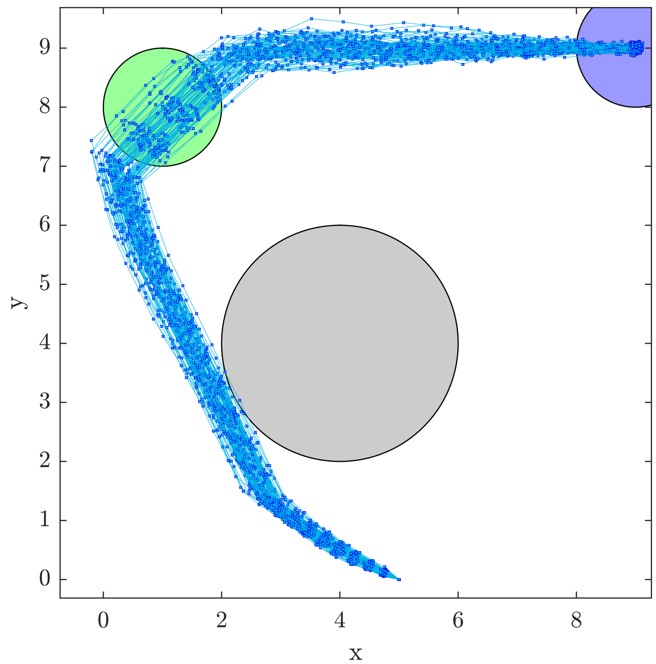

Fig. 2. The simulation result for the system trajectory.

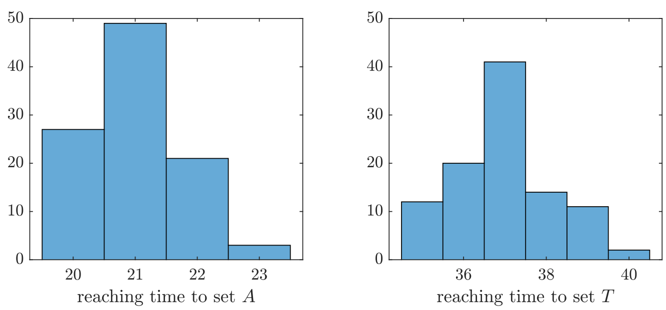

Fig. 3. Histogram of the reaching time to the sets A and T.

STL satisfaction with a given user-defined robustness value. The paper thus provides a probabilistic guarantee of the satisfaction of STL formulas by employing the concepts of stochastic CBFs and supermartingales. The efficacy of the proposed controller is verified in a simple linear case and a robot motion planning scenario. The extension of these results to multi-agent systems and investigating data-driven methods for unknown dynamics constitute future research directions.

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