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Data-driven event-triggering mechanism for linear systems subject to input saturations

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Abstract— **This paper focuses on designing an event-triggering mechanism aimed at reducing control updates while maintaining the stability of a saturated closed-loop system. It addresses the regional stabilization of linear systems under input saturation conditions from a data-driven perspective. To do so, we propose a systematic method to convert model-driven conditions into data-driven control design Linear Matrix Inequality (LMI) conditions, enabling the co-design of the event-triggering rule and the state feedback gain. The theoretical contribution is then applied to the control of a spacecraft rendezvous problem.**

I. INTRODUCTION

Dealing with dynamical systems presents a significant challenge, primarily in the modeling phase, as it requires the ability to develop a sufficiently precise mathematical model for control purposes, enhancing then data-driven control approaches. For a comprehensive overview of these approaches, one can refer to the recent survey in [22]. One notable aspect is the collection of a rich dataset from experiments, which helps mitigate imprecise knowledge of the plant model, as explored in [3] and [15]. Alternatively, a robust formulation can be employed to encapsulate the uncertainties affecting the model within a suitable set, as demonstrated in [4], [5], and [29]. These references provide insights into techniques derived from control theory.

In the context of data-driven approaches, significant attention has been devoted to the analysis of stability and the design of control solutions, [21], [22]. In particular, the case of linear systems has already been carefully treated; see, for example, [7], [9], [28], [29]. Still, few works guarantee stability despite control limitations [20]. A challenging issue is to account for the presence of input non-linearity such as saturation and, therefore, to revisit some well-known results dealing with regional stability of saturated systems [16], [26]. Furthermore, another direction of the paper is to address the way to implement control, differently from the classical periodic approach, through an alternative: the event-triggered control approach. This technique allows updating the system input only when a certain condition is verified, aiming to reduce transmission, communication, or energy consumption [13], [25]. Emulation and co-design are the two main approaches in the event-triggered control framework. In the first, the controller is given and only the event-triggering rule needs to be designed, while in the second, a simultaneous design of the control law and the event-triggering rules is proposed [1], [18], [17]. In this work, we mainly consider the co-design problem, by proposing a dynamic event-triggering rule inspired by [12].

Hence, we address the design of a dynamic event-triggering mechanism to reduce control updates while preserving the stability of the closed loop. The regional stabilization of linear systems subject to input saturation is tackled, with results proposed from a data-driven perspective. Leveraging a model-based solution to the problem, we propose a systematic method to transform model-driven conditions into data-driven Linear Matrix Inequality (LMI) conditions, enabling the co-design of the event-triggering rule and the state feedback gain. It is noteworthy that few works have explored the event-triggering technique in the data-driven context. Examples include [2], [10], [19]. The contribution of this paper extends these works, particularly those of [23] and [24], by considering dynamic event-triggering mechanisms and proposing theoretical conditions based on LMI formulations. These conditions enable the design of the control law and the triggering parameters, as well as the characterization of an inner approximation of the basin of attraction of the origin for the closed-loop system.

The paper is organized as follows. In Section II, preliminary results are presented as foundational elements for constructing data-driven control solutions. Section III introduces the system under consideration, along with assumptions regarding data collection, and summarizes the control objectives. The main contribution is then articulated in Section IV, along with discussions on associated optimization problems. In Section V, the results are illustrated within the context of a spacecraft rendezvous problem. Finally, concluding remarks and forthcoming issues conclude the paper.

Notation. Throughout the paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of natural numbers, the set of real numbers, the n-dimensional Euclidean space and the set of all real $n\times m$ matrices, respectively. In addition we will use notation $\mathbb{S}^n$ ($\mathbb{S}^n_+$) and $\mathbb{D}^n_+$ for the set of symmetric (positive definite) matrices in $\mathbb{R}^{n\times n}$ and the set of diagonal positive definite matrices in $\mathbb{R}^{n\times n}$. For any integers n and m in $\mathbb{N}$, matrices I_n and 0 denote the identity matrix of $\mathbb{R}^{n\times n}$ and the null matrix of appropriate dimensions, respectively. For any matrix M of $\mathbb{R}^{n\times n}$, the notation $M \succ 0$, ($M \prec 0$) means that M ($-M$) is in $\mathbb{S}^n_+$. For any matrices $A = A^\top, B, C = C^\top$ of appropriate dimensions, matrix $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ denotes the symmetric matrix $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$. For any matrix $N \in \mathbb{R}^{n\times m}$, notation $N_{(i)}$, for any $i = 1,\ldots,n$, stands for the i^{th} row of N and $N_{(ij)}$, for any $i = 1,\ldots,n$ and $j = 1,\ldots,m$, stands for the (i,j) entry of N. $\|x\|$ denotes the Euclidean norm of x. For any M

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in $\mathbb{S}^n_+$, $\lambda_m(M)$ and $\lambda_M(M)$ denote its minimal and maximal eigenvalues respectively. For a matrix $M \in \mathbb{S}^n$, we denote the ellipsoid $\mathcal{E}(M) = \{x \in \mathbb{R}^n,\ x^\top M x \leq 1\}$.

II. Preliminary results

A. Matrix-constrained Relaxation

In this section we state an instrumental result (see also [23], [28], [29]), which provides a way to write in an equivalent form a matrix inequality depending on parameters satisfying a quadratic constraint.

Lemma 1: For given positive integers n_1, n_2, n_3, consider matrices $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ in $\mathbb{S}^{n_1}_+ \times \mathbb{R}^{n_1 \times n_2} \times \mathbb{S}^{n_3}_+$ and $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)$ in $S^{n_3} \times \mathbb{R}^{n_3 \times n_2} \times \mathbb{S}^{n_3}_+$, i.e. with $\mathcal{N}_3 \succ 0$.

Then, the following statements are equivalent

(i) Inequality

$$\mathcal{M}(\mathcal{A}) = \begin{bmatrix} \mathcal{M}_1 & \mathcal{M}_2\mathcal{A} \\ * & \mathcal{M}_3 \end{bmatrix} \succ 0, \quad \forall \mathcal{A} \in \Sigma_{\mathcal{N}}, \qquad (1)$$

holds true where $\Sigma_{\mathcal{N}}$ represents the set of allowable uncertain matrices $\mathcal{A}$ characterized by

$$\Sigma_{\mathcal{N}} := \left\{ \mathcal{A} \in \mathbb{R}^{n_2 \times n_3},\ \begin{bmatrix} I_{n_3} \\ \mathcal{A} \end{bmatrix}^\top \begin{bmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ * & \mathcal{N}_3 \end{bmatrix} \begin{bmatrix} I_{n_3} \\ \mathcal{A} \end{bmatrix} \preceq 0. \right\} \quad (2)$$

(ii) There exists $\eta > 0$ such that

$$\begin{bmatrix} \mathcal{M}_1 & \mathbf{0}_{n_1,n_3} & \mathcal{M}_2 \\ * & \mathcal{M}_3 + \eta\mathcal{N}_1 & \eta\mathcal{N}_2 \\ * & * & \eta\mathcal{N}_3 \end{bmatrix} \succ 0. \qquad (3)$$

B. Preliminaries on generalized sector conditions

In this paper, we consider the classical decentralized vector-valued saturation map $\mathrm{sat}(u)$ from R^{n_u} to R^{n_u}, which is defined component-wise as follows:

$$\mathrm{sat}(u_i) = \mathrm{sgn}(u_i)\min(|u_i|, \bar{u}_i), \quad \forall i = 1, \ldots, n_u \qquad (4)$$

In (4), u_i refers to the i^{th} control input and $\bar{u}_i$ is the i^{th} entry of the level of saturation $\bar{u}$. It is well-known that the presence of saturations requires a particular care of the notion of stability [26], [30]. To handle the presence of saturations, we consider the dead-zone function $\phi(u)$ defined as

$$\phi(u) = \mathrm{sat}(u) - u. \qquad (5)$$

Following [26, Remark 7.3], we use the following *Generalized sector condition* lemma (see also [30]).

Lemma 2: ([26]) Consider a matrix $G \in \mathbb{R}^{n_u \times n_x}$. The following relation holds, for any diagonal matrix $T \in \mathbb{D}^{n_u}_+$

$$\phi(u)^\top T[\mathrm{sat}(u) + Gx] \leq 0, \quad \forall x \in \mathcal{S}(G), \qquad (6)$$

where $\mathcal{S}(G) = \{x \in \mathbb{R}^{n_x} : |G_{(i)}x| \leq \bar{u}_i, \quad \forall i = 1, \ldots, n_u\}$.

III. Problem formulation

A. System data and assumptions

The class of systems considered in this paper is that one of discrete-time linear systems subject to an input saturation, described as follows:

$$\begin{cases} x_{k+1} = \mathcal{A}x_k + \mathcal{B}\,\mathrm{sat}(u_k), & (\mathcal{A}, \mathcal{B}) \in \mathcal{Z} \\ x_0 \in \mathbb{R}^{n_x}, \end{cases} \qquad (7)$$

where, for given positive integers n_x and n_u, $x_k \in \mathbb{R}^{n_x}$ is the state vector and $u_k \in \mathbb{R}^{n_u}$ is the control input and x_0 stands for the initial condition of the system. The dynamics of the system is defined by matrices $\mathcal{A} \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{B} \in \mathbb{R}^{n_x \times n_u}$, which are subject to the following assumption.

Assumption 1: Matrices $(\mathcal{A}, \mathcal{B})$ in $\mathcal{Z}$ are assumed to be unknown and possibly time-varying.

B. Data collections and assumption

Considering the class of systems described by (7), our objective is to collect experimental data, which have been performed beforehand to compensate for the lack of knowledge of the system matrices. Based on this data collection, the main objective is to derive a data-driven control design result. To conduct this method, it is crucial to properly define the notion of data. We define the following data matrices that collect the available measurements for the control design.

$$\begin{array}{lcl} \mathcal{X}^+ & := & [\ X_1^+ \quad X_2^+ \quad \ldots \quad X_p^+\ ], \\ \mathcal{X} & := & [\ X_1 \quad X_2 \quad \ldots \quad X_p\ ], \\ \mathcal{U} & := & [\ \mathrm{sat}(U_1) \quad \mathrm{sat}(U_2) \quad \ldots \quad \mathrm{sat}(U_p)\ ]. \end{array} \qquad (8)$$

In each experiment i, a control input $U_i \in \mathbb{R}^{n_u}$ is applied when the state is located at position X_i, resulting in the system transitioning to state X_i^+.

We define the data collection $\mathcal{D}$ as follows

$$\mathcal{D} := (\mathcal{X}^+, \mathcal{X}, \mathcal{U}) \in \mathbb{R}^{n_x \times p} \times \mathbb{R}^{n_x \times p} \times \mathbb{R}^{n_u \times p}. \qquad (9)$$

and we suppose that the data collection $\mathcal{D}$ is informative, as defined below.

Definition 1: The data collection $\mathcal{D}$ in (8) is said informative if matrix $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ is full rank.

According to the system dynamics, these data verify

$$\mathcal{X}^+ = A\mathcal{X} + B\mathcal{U} + \omega \quad \in \mathbb{R}^{n_x \times p}, \qquad (10)$$

for some matrices (A, B). The noise samples $\omega = \begin{bmatrix} w(0) & w(1) & \ldots & w(p) \end{bmatrix}$ are unknown. However, the following assumption is made on the "energy" of the noise, following the presentation of [6].

Assumption 2: Assume that there exists a known parameter $\delta \geq 1$ such that the noise samples matrix ω verify

$$\omega \in \Omega := \{v \in \mathbb{R}^{n_x \times p},\ vv^\top \leq \delta\Delta_\omega\}, \qquad (11)$$

with Δ_ω given by the least square approximation error

$$\Delta_\omega = \mathcal{X}^+ \left(I_p - \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]^\top \left(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]^\top \right)^{-1} \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \right) \mathcal{X}^{+\top}.$$

This assumption is related to different interpretations of matrix Δ_ω, provided for example in terms of co-variance of the noise ω as mentioned in [6], [28], [5], where an alternative representation of Ω are proposed. Here, we emphasized the least-square interpretation.

In this paper, we consider the following set as in [6]

$$\mathcal{C} := \left\{ [A\ B],\ \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} \mathcal{X}^+\mathcal{X}^{+\top} - p\delta\Delta_\omega & -\mathcal{X}^+\mathcal{X}^\top & -\mathcal{X}^+\mathcal{U}^\top \\ * & \mathcal{X}\mathcal{X}^\top & \mathcal{X}\mathcal{U}^\top \\ * & * & \mathcal{U}\mathcal{U}^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \preceq 0 \right\} \qquad (12)$$

We can relate sets $\mathcal{Z}$ and $\mathcal{C}$ in the following assumption.

Assumption 3: The uncertainty set $\mathcal{Z}$ is included in $\mathcal{C}$.

C. Event-triggered state-feedback control law

The main objective of this paper is to design an event-triggered control described as follows:

$$u_k = \begin{cases} Kx_k, & \text{if the control law is updated,} \\ u_{k-1}, & \text{otherwise.} \end{cases} \quad (13)$$

with $K \in \mathbb{R}^{n_u \times n_x}$ is the control gain to be defined such that the trajectories of the closed-loop system (7) with (13) is (locally) asymptotically stable. To distinguish the triggered data from the current state, we introduce memory variables χ_k in $\mathbb{R}^{n_x}$ and s_k in $\mathbb{N}$ to denote the value of the state that is employed at time k to compute the control input. More precisely, it holds

$$\begin{bmatrix} \chi_k \\ s_k \end{bmatrix} = \begin{cases} \begin{bmatrix} x_k \\ k \end{bmatrix}, & \text{if the memory is updated,} \\ \begin{bmatrix} \chi_{k-1} \\ s_{k-1} \end{bmatrix}, & \text{otherwise.} \end{cases} \quad (14)$$

Hence, the control input is simply given by $u_k = K\chi_k$, for all k. Note that $\chi_0 = x_0$ and $s_0 = 0$. To determine whether the value of χ_k has to be updated or not, we introduce a dynamic discrete-time event-triggering rule, given by

$$s_{k+1} = \min_{m \in \mathbb{N}} \{m \geq s_k + 1 \mid \psi(x_m, \chi_k) \geq \rho\eta_m\}, \quad (15)$$

where the triggering condition is parameterized by:

- A quadratic triggering function, ψ given by

  $$\psi(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \ \forall (x,y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}, \quad (16)$$

  with $\Psi = \left[\begin{smallmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{smallmatrix}\right]$ being a symmetric matrix satisfying

  $$\begin{bmatrix} I \\ I \end{bmatrix}^\top \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} = \Psi_1 + \Psi_2 + \Psi_2^\top + \Psi_3 \preceq 0. \quad (17)$$

- The triggering variable η_k in $\mathbb{R}$ is driven by the following discrete-time dynamic equation:

  $$\eta_{k+1} = (\lambda + \rho)\eta_k - \psi(x_k, \chi_k), \ \forall k \in \mathbb{N}, \forall \eta_0 \geq 0, \quad (18)$$

  where λ and ρ are such that $\rho \geq 0$ and $(\lambda + \rho) \in [0, 1)$.

Constraint (17) ensures $\psi(x, x) \leq 0$ for all x in $\mathbb{R}^n$, which means that the event-triggered condition is not violated just after a control update. Note also that imposing $\rho = 0$ in (15) makes the dynamic event-triggering rule a static one. Following classical methods of the continuous-time dynamic event-triggering mechanism design [12], [11], [27], one has to guarantee that the variable η_k is non-negative for all k in $\mathbb{N}$. The following lemma addresses this issue.

Lemma 3: Consider scalar parameters (λ, ρ) such that $\rho \geq 0$ and $(\lambda + \rho) \in [0, 1)$ and assume that $\eta_0 > 0$. Then, the dynamic event-triggering variable is non-negative for any sampling instants t_k, i.e., $\eta_k \geq 0$ for all $k \in \mathbb{N}$.

Proof: Consider the dynamic of η_k in (18) together with the event-triggering rule (15). Assume that $\rho \geq 0$, $(\lambda + \rho) \in [0, 1)$ and $\eta_0 > 0$. The proof is then made by recursion.
Initialization ($k = 1$): Assume that $\eta_0 > 0$. According to the dynamic equation (18) with $k = 0$ and $x_0 = \chi_0$, we have

$$\eta_1 = (\lambda + \rho)\eta_0 - \psi(x_0, \chi_0) = (\lambda + \rho)\eta_0 - \underbrace{\psi(x_0, x_0)}_{\leq 0 \text{ from (17)}} \geq 0.$$

As $(\lambda + \rho) \in [0, 1)$, $\eta_0 > 0$ and $\psi(x_0, x_0)) \leq 0$, it is clear that $\eta_1 \geq 0$, which concludes the initialisation.
Recursion ($k \in \mathbb{N}$): Assume that, at time $k \geq 1$ the variable η_k is positive. Then, the following two situations may occur.
If $\psi(x_k, \chi_k) \geq \rho\eta_k$. Then, the value of the control law as well as χ_k are updated, i.e., $\chi_k = x_k$. In this situation, $\psi(x_k, x_k) \leq 0$ still holds since we get the same constraint (17) of the triggering function ψ as in the case $k = 0$ above. Consequently, $\eta_{k+1} \geq 0$.
If $\psi(x_k, \chi_k) \leq \rho\eta_k$, then, according to the discrete-time equation (18) and since $\lambda \geq 0$, $\eta_k \geq 0$, it holds

$$\begin{aligned} \eta_{k+1} &= (\lambda + \rho)\eta_k - \psi(x_k, \chi_k) \\ &= \underbrace{\lambda\eta_k}_{\geq 0} + \underbrace{(\rho\eta_k - \psi(x_k, \chi_k))}_{\geq 0}. \end{aligned}$$

Hence, $\eta_{k+1} \geq 0$ holds, which concludes the recursion. ■

D. Control objectives

The problem addressed in this paper concerns the co-design of the state feedback gain K and the matrix Ψ, defining the event-triggering rule (15) and (16), to minimize control law updates while maintaining stability of the origin for the closed loop. With input saturation present, it is well-known (see [16], [26], and related literature) that ensuring global asymptotic stability of the origin for saturated systems (7) is feasible only if the open loop is not exponentially unstable. This condition also holds when the matrix dynamics are uncertain. Therefore, as the basin of attraction of the origin generally cannot cover the entire state space, it becomes crucial to determine an approximation of this basin. To tackle this task, we consider an inner approximation of the basin of attraction of the origin achieved by constructing a level set associated with an appropriate Lyapunov function.

The problem to solve can be formulated as follows:

Problem 1: Design the gain K and the event-triggering parameters (Ψ, ρ, λ) such that

(P1) The regional (local) asymptotic stability is ensured for any initial condition belonging to a level set associated to the considered Lyapunov function.
(P2) The number of updates of the control law is reduced.

Let us emphasize that we want to tackle Problem 1 from a data point of view, that is using the data collection $\mathcal{D}$ defined in (8). To do this we will exploit the preliminary results proposed in Section II.

IV. Main results

A. Theoretical conditions

In this section, we exploit the material presented in Sections II-A and II-B to solve Problem 1, that is to solve the co-design problem consisting in designing the state feedback gain K and the event-triggering parameters.

Theorem 1: For given parameters λ and ρ such that $\lambda \leq \lambda + \rho < 1$, assume that there exists

$$\begin{aligned} \mathcal{D}_{\mathcal{V}}^1 &:= \{\mu, W, S, Y, Z, \bar{\Psi}_i\} \\ &\in \mathbb{R}_{>0} \times \mathbb{S}_{n_x} \times \mathbb{D}_{n_x} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_x} \end{aligned}$$

solution to the following inequalities

$$\Phi(\mathcal{D}) \succ 0, \ \begin{bmatrix} W & Z_{(i)}^\top \\ * & \bar{u}_i^2 \end{bmatrix} \succ 0, \forall i = 1, \ldots, n_u \quad (19)$$

where

$$\Phi(\mathcal{D}) = \begin{bmatrix} W+\bar\Psi_1 & \bar\Psi_2 & 0 & 0 & W & 0 \\ * & \bar\Psi_3 & Y^\top + Z^\top & 0 & 0 & Y^\top \\ * & * & 2S & 0 & 0 & S \\ * & * & * & \Phi_4 & -\mu\mathcal{X}^+\mathcal{X}^\top & -\mu\mathcal{X}^+\mathcal{U}^\top \\ * & * & * & * & \mu\mathcal{X}\mathcal{X}^\top & \mu\mathcal{X}\mathcal{U}^\top \\ * & * & * & * & * & \mu\mathcal{U}\mathcal{U}^\top \end{bmatrix},$$

with

$$\Phi_4 = W + \mu\mathcal{X}^+\mathcal{X}^{+\top} - \mu\Delta_\omega.$$

Then, the dynamic event-triggered control law (13) with the gain $K = YW^{-1}$ and the event-triggering function (18) characterized by (λ,ρ) and matrices $\Psi_i = \left(\frac{\rho}{1-\lambda}\right) W^{-1}\bar\Psi_i W^{-1}$, for $i=1,2,3$, ensures that the ellipsoid $\mathcal{E}(W^{-1})$ is an inner-approximation of the basin of attraction of the origin for the closed-loop system (7)-(13).

Proof: Consider the following candidate for the Lyapunov function

$$V(x_k,\eta_k) = x_k^\top P x_k + \eta_k. \tag{20}$$

Recall that Lemma 1 ensures that the dynamic event-triggering variable η_k is strictly positive. Hence, imposing $P \succ 0$ guarantees that the previous function is positive definite.

Computing now the forward increment of V along the trajectory of the system yields

$$\begin{array}{rcl} \Delta V(x_k,\eta_k) & := & V(x_{k+1},\eta_{k+1}) - V(x_k,\eta_k) \\ & = & x_{k+1}^\top P x_{k+1} - x_k^\top P x_k + \eta_{k+1} - \eta_k. \end{array} \tag{21}$$

The objective is to ensure that $\Delta V(x) \preceq 0$ forall (x_k,η_k) such that x in $\mathcal{E}(P)$, i.e. $x^\top P x \le 1$. The satisfaction of the last inequalities in (19) means that the ellipsoid $\mathcal{E}(W,1)$ is included in the set $\mathcal{S}(G)$, for a given matrix G. From Lemma 2, for $x \in \mathcal{E}(W,1)$, inequality (6) holds and we get:

$$2\phi^\top(K\chi_k)T\left(\mathrm{sat}(K\chi_k) + G\chi_k\right) \le 0. \tag{22}$$

Define now $\mathcal{L}$ by

$$\begin{array}{rcl} \mathcal{L}(x_k,\chi_k,\eta_k) & := & \Delta V(x_k,\chi_k,\eta_k) \\ & & -2\phi^\top(K\chi_k)T\left(\mathrm{sat}(K\chi_k)+G\chi_k\right). \end{array} \tag{23}$$

Then, the previous discussion ensures that $\Delta V(x_k,\chi_k,\eta_k) \le \mathcal{L}(x_k,\chi_k,\eta_k)$ for all χ_k in $\mathcal{S}(G)$.

Re-injecting the expression of x_{k+1} and η_{k+1}, we obtain

$$\begin{array}{rl} \mathcal{L}(x_k,\chi_k,\eta_k) = & (\mathcal{A}x_k + \mathcal{B}\,\mathrm{sat}(K\chi_k))^\top P(\mathcal{A}x_k+\mathcal{B}\,\mathrm{sat}(K\chi_k)) \\ & -x_k^\top P x_k - 2\phi^\top(K\chi_k)T\left(\mathrm{sat}(K\chi_k)+G\chi_k\right) \\ & -\psi(x_k,\chi_k) - (1-\lambda-\rho)\eta_k. \end{array} \tag{24}$$

Recalling that the event-triggering condition ensures that $\rho\eta_k \le \psi(x_k,\chi_k)$, then it holds

$$\begin{array}{rl} -\psi(x_k,\chi_k) - (1-\lambda-\rho)\eta_k = & \underbrace{\rho\eta_k - \psi(x_k,\chi_k)}_{\le 0} - (1-\lambda)\eta_k \\ \le & -(1-\lambda)\eta_k \le -\left(\dfrac{1-\lambda}{\rho}\right)\psi(x_k,\chi_k). \end{array}$$

Re-injecting the previous upper bound into the expression of $\mathcal{L}$, we get

$$\begin{array}{rl} \mathcal{L}(x_k,\chi_k,\eta_k) = & (\mathcal{A}x_k + \mathcal{B}\,\mathrm{sat}(K\chi))^\top P(\mathcal{A}x_k+\mathcal{B}\,\mathrm{sat}(K\chi)) \\ & -x_k^\top P x_k - 2\phi^\top(K\chi_k)T\left(\mathrm{sat}(K\chi_k)+G\chi_k\right) \\ & -\left(\dfrac{1-\lambda}{\rho}\right)\begin{bmatrix} x_k \\ \chi_k \end{bmatrix}^\top \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{bmatrix}\begin{bmatrix} x_k \\ \chi_k \end{bmatrix} \end{array} \tag{25}$$

Introducing the augmented vector $\xi_k = [x_k^\top \ \chi_k^\top \ \phi^\top(K\chi_k)]$, we are able to express $\mathcal{L}$ as follows

$$\mathcal{L}(x_k,\chi_k,\eta_k) = -\xi_k^\top \Theta \xi_k \tag{26}$$

where

$$\Theta := \begin{bmatrix} P + \left(\frac{1-\lambda}{\rho}\right)\Psi_1 & \left(\frac{1-\lambda}{\rho}\right)\Psi_2 & 0 \\ * & \left(\frac{1-\lambda}{\rho}\right)\Psi_3 & (K+G)^\top T \\ * & * & 2T \end{bmatrix} - \begin{bmatrix} \mathcal{A}^\top \\ K^\top\mathcal{B}^\top \\ \mathcal{B}^\top \end{bmatrix}^\top P \begin{bmatrix} \mathcal{A}^\top \\ K^\top\mathcal{B}^\top \\ B^\top \end{bmatrix}$$

Applying the Schur Complement, condition $\Theta \succ 0$ is equivalent to

$$\begin{bmatrix} P + \left(\frac{1-\lambda}{\rho}\right)\Psi_1 & \left(\frac{1-\lambda}{\rho}\right)\Psi_2 & 0 & \mathcal{A}^\top \\ * & \left(\frac{1-\lambda}{\rho}\right)\Psi_3 & (K+G)^\top T & K^\top\mathcal{B}^\top \\ * & * & 2T & \mathcal{B}^\top \\ * & * & * & P^{-1} \end{bmatrix} \succ 0$$

Matrices P and T being positive definite, their inverse exists and will be denoted as $W = P^{-1}$ and $S = T^{-1}$. Pre- and post-multiplying the previous inequality by $\mathrm{diag}(W,W,S,I_n)$, and defining $\bar\Psi_i = \left(\frac{1-\lambda}{\rho}\right) W\bar\Psi_i W$, for all $i=1,2,3$, $Y = KW$ and $Z = GW$, condition $\Theta \prec 0$ if and only if

$$\begin{bmatrix} W+\bar\Psi_1 & \bar\Psi_2 & 0 & W\mathcal{A}^\top \\ * & \bar\Psi_3 & Y^\top + Z^\top & Y^\top\mathcal{B}^\top \\ * & * & 2S & S\mathcal{B}^\top \\ * & * & * & W \end{bmatrix} \succ 0.$$

We are now in position to apply Lemma 1 with matrices

$$\mathcal{M}_1 = \begin{bmatrix} W+\bar\Psi_1 & \bar\Psi_2 & 0 \\ * & \bar\Psi_3 & Y^\top + Z^\top \\ * & * & 2S \end{bmatrix}, \mathcal{M}_2 = \begin{bmatrix} W & 0 \\ 0 & Y^\top \\ 0 & S \end{bmatrix}, \mathcal{M}_3 = W,$$

and with the uncertainty matrix $\mathcal{A} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}^\top$. ■

In the situation of emulation, i.e. when the controller gain K is fixed a priori, the previous theorem can be adapted to only deliver the parameter of the event triggering condition. This is formulated in the next corollary.

Corollary 1: Assume that there exists

$$\begin{array}{rcl} \mathcal{D}_\mathcal{V}^2 & := & \{\mu, W, S, Z, \bar\Psi_i\} \\ & \in & \mathbb{R}_{>0} \times \mathbb{S}_{n_x} \times \mathbb{D}_{n_x} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_x} \end{array}$$

solution to the following inequalities (19) with $Y = KW$.

Then, the dynamic event-triggered control law (13) with the given gain K and the event-triggering function (18) characterized by (λ,ρ) and matrices $\Psi_i = W\bar\Psi_i W$, for

$i = 1, 2, 3$, ensures that the ellipsoid $\mathcal{E}(W^{-1})$ is an inner-approximation of the basin of attraction of the origin for the closed-loop system (7)-(13).

B. Optimization problem

Let us first note that the matrix inequalities in Theorem 1 or in Corollary 1 are linear in the decision variables, as long as λ and ρ are fixed. One of the objectives behind the proposed theoretical conditions is to maximize the size of the inner-approximation of the basin of attraction (that is the size of the set $\mathcal{E}(W^{-1})$. Hence, in this context the following optimization problem can be considered

$$\begin{array}{ll} \max\limits_{\mathcal{D}_{\mathcal{V}}^{1}} & \epsilon \\ s.t. & (19), W \succeq \epsilon I_{n_x}, \epsilon > 0 \end{array} \tag{27}$$

In the paper, we focused on the maximization of the size of the inner-approximation of the basin of attraction of the origin for the closed-loop system (7)-(13), but it should be interesting to propose some criterion to reduce the amount of control updates. This will be the topic of forthcoming issue.

V. NUMERICAL APPLICATIONS

To assess the theoretical contribution of this paper, we examine the problem of spacecraft rendezvous, a topic that has been extensively studied with various mathematical models. Specifically, we concentrate on the scenario where a target evolves in a *circular* Keplerian orbit, and the approaching vehicle (chaser) is in close proximity to the target. This problem can be effectively modeled by the linear Hill-Clohessy-Wiltshire (HCW) equations, as introduced in [14] and [8], providing accurate descriptions of the relative position of the spacecraft.

The HCW model, using impulsive control, describes the relative motion of a chaser vehicle close to the target. The reduced model in the (x, y)-plan can be written as follows:

$$\ddot{r}_x - 3n^2 r_x - 2n\dot{r}_y = 0, \tag{28a}$$

$$\ddot{r}_y + 2n\dot{r}_x = 0, \tag{28b}$$

where $(r_x, r_y) \in \mathbb{R}^2$ stands for the relative position between the chaser and the target in the target reference frame. We introduce (v_x, v_y), which stands for the relative velocities between the chaser and the target in the target reference frame. The HCW model assumes that the target vehicle is passive and moving along a circular orbit of radius R. For a typical orbit at, say, an altitude of $R = 500km$, we would get $n = 0.0011$ rad/s. The control action is here performed through impulses acting only on the velocities. In order words, this can be summarized as

$$v_x^+ = v_x + u_x, \quad v_y^+ = v_y + u_y, \tag{29}$$

where the impulses are of limited amplitude.

Performing a discretization of the model with a period $T = 0.5s$, such that the sampling instants are $t_k = kT$ leads to the following discrete time dynamics

$$x_{k+1} = e^{A_0 T}(x_k + B_0 u_k), \tag{30}$$

where $x_k = [r_x(t_k)\ v_x(t_k)\ r_y(t_k)\ v_y(t_k)]$ is the state vector $u_k = [u_x(t_k)\ u_y(t_k)]$, and with the following matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & 0 & 1 \\ 0 & -2n & 0 & 0 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this context, minimizing the consumption of control action is a crucial concern for ensuring mission safety. Additionally, due to potential variations in the sampling period or deviations from a perfectly circular orbit at 500km for the target, the model may be subject to disturbances and uncertainties. Taken together, these factors motivate the application of a data-driven control design. To address these uncertainties, we collect $N = 50$ data points represented as in (8). Uncertainties have been incorporated into (30) as follows:

$$X_i^+ = (e^{A_0 T} + \Delta A)X_i + (e^{A_0 T} B_0 + \Delta B)\,\mathrm{sat}(U_i), \tag{31}$$

where ΔA and ΔB are random matrices such that $|\Delta A_{(ij)}| \leq \delta$, $|\Delta B_{(ij)}| \leq \delta$, with $\delta = 10^{-3}$, for all appropriate integers i, j, and where X_i and U_i have also been selected randomly. In particular, some values of U_i are such that $\mathrm{sat}(U_i)_{(j)} = \pm \bar{u}_j$. To get a better understanding of the effect of these random matrices and of the saturation, the optimal least squares estimate of $\mathbb{A}$ and $\mathbb{B}$, using a set of data are given by

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \end{bmatrix} = \begin{bmatrix} 1.0211 & 0.4694 & -0.0318 & 0.0241 & 1.9107 & 0.2610 \\ 0.0423 & 0.9389 & -0.0636 & 0.0487 & 3.8214 & 0.5239 \\ -0.0147 & 0.0090 & 1.0061 & 0.4658 & 0.0378 & 1.8873 \\ -0.0294 & 0.0176 & 0.0123 & 0.9316 & 0.0737 & 3.7744 \end{bmatrix},$$

which differs notably from the original system

$$\begin{bmatrix} e^{A_0 T} & e^{A_0 T} B_0 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0 & 0.002 & 0.5 & 0.0002 \\ 1.5.10^6 & 1 & 0 & 0.001 & 1 & 0.001 \\ 0 & -0.002 & 1 & 0.5 & -0.0002 & 0.5 \\ 0 & -0,001 & 0 & 1 & -0.001 & 1 \end{bmatrix}.$$

While the matrix $\mathbb{A}$ is quite similar to $e^{A_0 T}$, the values of $\mathbb{B}$ present great differences with respect to $e^{A_0 T} B_0$, which is due to the presence of the input saturation. The matrix Δ_ω associated with the set of data is given by

$$\Delta_\omega = \begin{bmatrix} 28.28 & 56.56 & 0.99 & 1.97 \\ 56.56 & 113.11 & 2.03 & 4.00 \\ 0.99 & 2.03 & 32.38 & 64.75 \\ 1.97 & 4.00 & 64.75 & 129.51 \end{bmatrix},$$

and we have selected parameter $\delta = 1.005$. Solving the optimization problem (27), with (19), $\lambda = 0.8$ and $\rho = 0.1$ leads to the following controller gain K and matrix W:

$$\begin{array}{rcl} K & = & \begin{bmatrix} -0.1335 & -0.2819 & 0.0626 & 0.0215 \\ 0.0117 & -0.0015 & -0.1327 & -0.2801 \end{bmatrix}, \\ W & = & \begin{bmatrix} 608.56 & -249.62 & -24.40 & 29.84 \\ -249.62 & 249.36 & 63.36 & -19.94 \\ -24.40 & 63.36 & 358.52 & -138.25 \\ 29.84 & -19.94 & -138.25 & 188.32 \end{bmatrix}, \end{array}$$

so that both $\mathbb{A} + \mathbb{B}K$ and $e^{A_0 T} + e^{A_0 T} B_0 K$ are Schur stable.

The simulation results are illustrated in Figure 1, depicting the trajectories of the system with disturbances δA and δB, adhering to the same constraints as those during data generation. The initial condition $x_0 = \begin{bmatrix} 12.03 & -20.89 & 2.63 & 4.28 \end{bmatrix}^\top$ is situated at the boundary of the ellipsoid $\mathcal{E}(W^{-1})$. Figure 1 demonstrates that despite the saturation of both control inputs, the resulting trajectory converges asymptotically to the origin. Additionally, it shows a reduction in the number of control updates, highlighting the efficiency of the method.

VI. CONCLUSIONS AND PERSPECTIVES

This paper addressed the design of a dynamic triggering mechanism within a data-driven framework, incorporating limitations on input magnitude modeled using a saturation map. The primary objectives were to jointly design the triggering parameters and the state-feedback gain while characterizing an inner approximation of the basin of attraction of

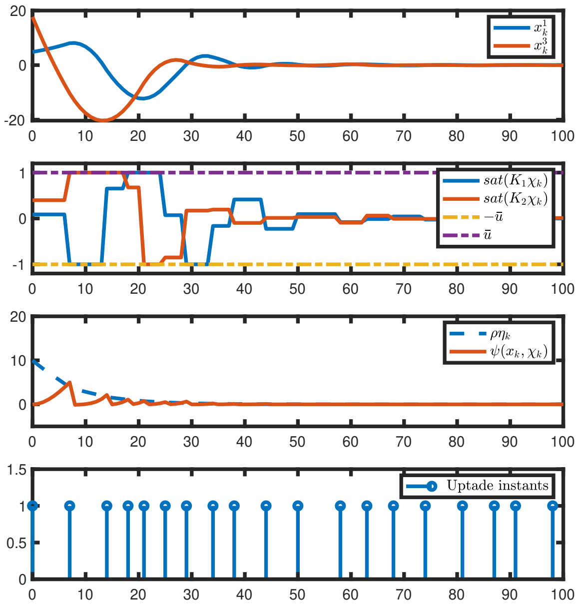

Fig. 1. Graphs showing the trajectories of r_x, r_y, the two saturated control inputs u_x, u_y, the triggering functions $\rho\eta_k$ and $\psi(x_k, \chi_k)$ and the triggering instants.

the origin for saturated closed-loop systems. Achieving these goals relied on a collection of informative data experiments, with theoretical conditions formulated based on quadratic Lyapunov criteria, leading to convex optimization problems.

The obtained results provide valuable insights for future research. Particularly, the optimization problem introduced in this paper emphasizes maximizing the inner-approximation of the basin of attraction of the origin, leaving the reduction of the number of control updates as a topic for further investigation. Moreover, a promising avenue for future work involves developing a more systematic approach to design the event-triggering mechanism. This could entail exploring decentralized triggering rules, each tailored to specific control channels.

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