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# Network Small Gain Theorem

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***Abstract*—The network stabilization problem in a heterogeneous multi-agent system with diffusive connections is investigated in this paper. It is demonstrated that, under the assumption that the agents are finite-gain $\mathcal{L}_2$–stable and zero state observable, the interconnection is zero input asymptotically stable, and all agents' dynamics converge to the origin. The stability analysis is based on small-gain theory. We provide the maximum bound on the $\mathcal{L}_2$–gain of the controller using various optimization methods when the $\mathcal{L}_2$–gain of each agent is known. A study on the variation of $\mathcal{L}_2$–gain bound of the controller with the Laplacian eigenvalues of the underlying graph is provided. Numerical examples, which support and illustrate the analytical results, are also included.**

***Index Terms*—Diffusively connected network, small-gain theorem, interconnected system, edge dynamics, Networked dynamical system (NDS).**

## I. Introduction

A multi-agent system (MAS) is a group of autonomous systems (agents) which collaborate to achieve a common shared objective. Collaboration usually happens by sharing their information with the other agents, e.g., position, velocity, etc., through a communication channel. However, the information exchange between agents is costly while working in large-scale MASs. Therefore, it is preferable to often restrict this communication among a smaller group of agents, such as neighbors. Control techniques that rely on the information received by an agent from its neighboring agents are referred to as distributed methods. In this work, we study a class of distributed control laws in which only the exchange of relative measurements among neighbors is permissible. In other words, each agent can only access the difference between its output and the output of its neighboring agents. Such control laws are referred to as diffusive, and the systems they regulate are referred to as *diffusively connected*.

Control of diffusively connected systems has been frequently studied in the context of many MAS applications, such as distributed remote sensing, aerial exploration, sensor localization, etc. The application of these techniques becomes crucial when the absolute measurements are unavailable or difficult to obtain [1]–[3]. Collectively, such systems can also be described as networked dynamical systems (NDS). The most effective way to model these systems is as a graph, where each node represents a separate dynamical system, and the edges of the graph reflect the communication topology between them. Recent advancements in the theory of NDS have led to the emergence of several applications, some of which may require even disparate dynamical systems to work in tandem in order to successfully complete a mission. Therefore, the stability of such an interconnection of possibly heterogeneous dynamical systems is a crucial problem that needs to be addressed.

In the last decade, there has been significant progress in extending the passivity framework as a powerful tool for assessing the stability of coupled dynamical systems. The notion of equilibrium-independent passivity (EIP), which includes nontrivial equilibria that are often desired in interconnected systems [4], [5], has proved to be useful in the analysis of multi-agent systems. In [6], the stabilization of an NDS, that constitutes passivity-short systems, was shown to be related to a pair of dual network optimization problems that were not necessarily convex. It was demonstrated in [7] that network-only passivation of passivity short systems was comparable to a convex optimization problem, given that the sum of the passivity indices over the nodes was positive, by using only network-level variables to regularize the network optimization issue. The authors in [8] showed the global asymptotic convergence to desired outputs for NDS constituting multi-input multi-output (MIMO) node dynamics, and a connection between the global convergence and network optimization problem was established. Authors in [9] studied the stability issue of network control systems using the small-gain theorem. To check the stability, they used a scalar called the *network gain*. Some of the related results that have looked beyond passivity or have considered a dissipativity-based paradigm for NDS are [10], [11]. The authors in [12] generalized the nonlinear small-gain theorem for a network of coupled input-to-state stable systems, while in [13], the authors addressed the problem in a more generalized dissipativity framework and provided a geometrical approach to construct ISS Lyapunov functions. In [14], the authors have shown the stability of a linear interconnected system in the presence of disturbance, using $\mathcal{H}_\infty$ control method. Authors in [15] have studied the internal stability of heterogeneous linear networked systems, in the presence of unstable poles, via edge analysis. However, these results did not provide any algorithm for designing networked controllers that ensured stability.

Despite the proliferation of the NDS paradigm along several directions over the last few years, the theory of NDS still has several open questions that have remained unresolved to

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this day. Stability in the presence of heterogeneous dynamical systems in the network is one such question that will be addressed in this paper. In light of the above discussions, the contributions of this work may be summarized thus:

- First, we formulate the network stabilization problems for a class of heterogeneous multi-agent systems, which differs from existing approaches.
- We present a small-gain-based approach towards distributed control of diffusively connected NDS.
- We provide a sufficient criterion for the asymptotic stability of the diffusively connected system.

## II. Mathematical Preliminaries and Problem Statement

This section presents some basic concepts of graph theory and diffusively connected networked dynamical systems.

### A. Notation

$\mathbb{Z}$ and $\mathbb{R}$ represent the sets of integer and real numbers, respectively, with subset $\mathbb{N}_\mathrm{n} := \{i \in \mathbb{Z} \mid 1 \le i \le \mathrm{n}\}$. Using $I_\mathrm{n}$ and $1_\mathrm{n}$, we denote the $\mathrm{n} \times \mathrm{n}$ identity matrix and the n–dimensional vector of all ones, respectively. We use $I$ and 1 when the dimension is clear from the context. The transpose of a matrix $A$ is denoted by $A^T$. The maximum eigenvalue for a symmetric positive semi-definite matrix, $A$, is denoted by $\|A\|_2^2$. The notation $\mathrm{diag}(A_i)$ represents a block-diagonal matrix with block diagonal elements being $A_i$.

### B. Algebraic graph theory

The interaction topology of a networked dynamical system is modeled via graphs. Throughout this work, we will assume that the graphs are connected and undirected.

*Definition* 1. A graph is an ordered pair $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ denotes the node set of $\mathcal{G}$ with $\mathbb{V} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, ..., \mathrm{v}_\mathrm{n}\}$, and $\mathbb{E} \subseteq \mathbb{V}\times\mathbb{V}$ denotes the edge set of $\mathcal{G}$ with $\mathbb{E} \subset \{\{\mathrm{v_i}, \mathrm{v_j}\} | \mathrm{v_i}, \mathrm{v_j} \in \mathbb{V} \text{ and } \mathrm{i} \neq \mathrm{j}\}$, corresponding to the communication links among the agents [16].

*Definition* 2. The incidence matrix for a graph with rows and columns indexed by the vertices and the edges, respectively is given as

$$\mathcal{E}(\mathcal{G})_{ik} := \begin{cases} 1 & \text{if an edge } e_k \text{ emanates from node } \mathrm{v_i}; \\ -1 & \text{if an edge } e_k \text{ terminates at node } \mathrm{v_i}; \\ 0 & \text{otherwise} \end{cases}$$

### C. Diffusively connected NDS

An NDS is described by a $q$-tuple of dynamical systems (plant and controller ensemble) interconnected with each other via a network. This work describes this interconnection network by an (undirected) graph. An NDS with distributed control inputs can be described as follows: for all $i = 1, 2, 3, ...., \mathrm{n}$

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(x_i)u_i; \qquad y_i = h_i(x_i), \tag{1a}$$

$$u_i = \sum_{\mathrm{v_j} \in \mathcal{N}(\mathrm{v_i})} \mathcal{K}_{ij}(y_i - y_j), \tag{1b}$$

where $y_i$ denotes the output of each node $\mathrm{v_i}$, and $\mathcal{N}(\mathrm{v_i}) \subseteq \mathbb{V}$ denotes the set of neighboring nodes of $\mathrm{v_i}$, $u_i$ is the disturbance/reference input to node $\mathrm{v_i}$, and $\mathcal{K}_{ij}$ are coupling coefficients. To put it more specifically, we consider n

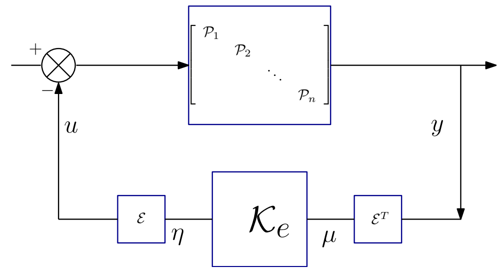



Fig. 1: Diffusively connected feedback setup.

continuous-time linear or non-linear time-invariant agents $\mathcal{P}_i$, who communicate over a network, $\mathcal{G}$, with n nodes and m edges. In this structure, agents $i$ and $j$ are neighbors if they share an edge. A general diffusively connected MAS, originating in [17], is shown in Fig. 1. It comprises the block-diagonal aggregate plant $P := \mathrm{diag}\,(\mathcal{P}_i)$ of n blocks, a block-diagonal static edge controller $\mathcal{K}_e := \mathrm{diag}\,(\mathcal{K}_{ij})$ with $\mu$ blocks, and pre- and post-processing blocks based on the incidence matrix $\mathcal{E}$ associated with $\mathcal{G}$. The overall feedback path $\hat{\mathcal{K}} : y \mapsto u$ is thus defined as,

$$\hat{\mathcal{K}} := \mathcal{E}\mathcal{K}_e\mathcal{E}^T.$$

However, with a slight abuse of notations, we shall use (1b) to denote dynamic controller as well, where each $\mathcal{K}_{ij}$ needs to be viewed as an operator on the suitable $\mathcal{L}$-spaces. In such case, one may write the $\nu_i'$–order dynamical controller $\mathcal{K}_e$ as

$$\dot{z}_i = \phi_i(z_i) + \Gamma_i(z_i)\mu_i, \tag{2a}$$

$$\eta_i = \psi_i(z_i), \tag{2b}$$

where $\phi_i$ is locally Lipschitz and $\psi_i$ is continuous for all $z_i \in \mathbb{R}^{\nu_i'}$, and $\mu_i \in \mathbb{R}^\mathrm{m}$; $z_i$, $\mu_i$, and $\eta_i$ are the state, input, and output of the controller, respectively with $z_i = 0$ as an equilibrium point.

*Remark* 1. The analysis that follows is predicated on the NDS, being well-posed, i.e., both the plants and the controller have some underlying state-space descriptions with inputs $u_i$ and $\mu_i$ and outputs $y_i$ and $\eta_i$, respectively and the internal signals of the feedback loop, namely $u_i$ and $\mu_i$, are uniquely defined for every choice of the system state variables $x_i$ and $z_i$. That is, the interconnection does not bring in singularity of solutions. See [18] for more on well-posedness. Note that due to the absence of a direct feedthrough term in the output equations of the plant and the controller, the interconnection is guaranteed to be well-posed, (see Ex. 6.12 of [18]).

The general question of interest in this paper is: under what conditions on the agents $\mathcal{P}_i$, the edge controllers $\mathcal{K}_e$ stabilize the diffusively connected system shown in Fig. 1. We now present a motivating example for the current work.

*D. A motivating example*

We consider a cycle of three heterogeneous agents. The agents evolve according to

$$\mathcal{P}_1 := \dot{x}_1 = -x_1 + u,\;\; y = 0.04x_1,$$
$$\mathcal{P}_2 := \dot{x}_1 = x_2; \dot{x}_2 = -x_1 - 2x_2 + u,\;\; y = 0.06x_1 + 0.07x_2,$$
$$\mathcal{P}_3 := \dot{x}_1 = x_2; \dot{x}_2 = x_3; \dot{x}_3 = -x_1 - 4x_2 - 5x_3 + u, y = 0.04x_1,$$

where $x_i, y, u \in \mathbb{R}$ represent the state, output, and input of the concerned agent. The goal is to ensure the asymptotic stability of the interconnected system formed by these agents. However, it is assumed that each agent has only information about its neighbors, that is the agents whose indices belong to a set $\mathcal{N}(\mathrm{v_i})$.

Towards that, we choose a controller as $u_i = -\mathcal{E}\mathcal{K}_e\mathcal{E}^T y_i$, where $\mathcal{K}_e$ is a diagonal matrix that needs to be designed. If we choose $\mathcal{K}_e = \mathrm{diag}(100\ 190\ 290)$, then it does not ensure stability of the interconnected system. Despite the fact that each system was individually stable, stability of the interconnected system was not guaranteed. One can readily verify this by calculating the eigenvalues $A - B\mathcal{E}\mathcal{K}_e\mathcal{E}^T C$, where, $A$, $B$, $C$ are system, input, and output matrix of the aggregate system.

In this paper, we investigate this phenomenon of instability arising due to the interconnection of individual stable plants over a network, where the communication links are possibly dynamic in nature. We show that the generalization of the small-gain theorem provides sufficient conditions for stable interconnection.

**Problem.** *Consider block-diagonal aggregate plant $P := \mathrm{diag}(\mathcal{P}_i)$ with $\mathrm{n}$ blocks, and a diagonal edge controller $\mathcal{K}_e := \mathrm{diag}(\mathcal{K}_{ij})$, as shown in Fig. 1. Formulate all the admissible controllers, $\mathcal{K}_e$ such that the network interconnection is asymptotically stable.*

## III. Network small gain theorem

Consider the $\nu_i$–order time-invariant system

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \tag{3a}$$
$$y_i = h_i(x_i), \tag{3b}$$
$$u_i = \sum_{\mathrm{v_j}\in\mathcal{N}(\mathrm{v_i})} \mathcal{K}_{ij}(y_i - y_j) \tag{3c}$$

where $f_i$ is locally Lipschitz and $h_i$ is continuous for all $x_i \in \mathbb{R}^{\nu_i}$, and $u_i \in \mathbb{R}^{\mathrm{n}}$ with $x_i = 0$ as the equilibrium point.

**Assumption 1.** *There exist constant signals $u$, $y$, $\mu$, $\eta$ such that $u = \mathcal{E}\eta$, and $\mu = \mathcal{E}^T y$ and each node (dynamical system) (3) is finite gain $\mathcal{L}_2$–stable with $\mathcal{L}_2$–gain $\leq \gamma_i$. Also, let the controller $\mathcal{K}_{ij}$ be finite gain $\mathcal{L}_2$–stable with $\mathcal{L}_2$–gain $\leq k_i$, Hence, define $\mathcal{K}_e := \mathrm{diag}(k_i) \in \mathbb{R}^{\mathrm{m}\times\mathrm{m}}$.*

**Assumption 2.** *Suppose that both the plant and the edge controllers in (3) and (2) are zero-state observable. That is, $h_i(x_i) \equiv 0 \Leftrightarrow x_i \equiv 0$ and $\psi_i(z_i) \equiv 0 \Leftrightarrow z_i \equiv 0$.*

**Theorem 1.** *Consider a dynamical network depicted in Fig. 1, and suppose Assumptions 1 and 2 hold, then the interconnection is zero input asymptotically stable if*

$$\frac{I_{\mathrm{n}}}{\gamma^2\|\mathcal{E}\|_2^2} - \mathcal{E}\mathcal{K}_e\mathcal{E}^T \geq 0, \tag{4}$$

*where, $\mathcal{E}$ is node-edge incidence matrix and $\gamma := \max(\gamma_i)$ for $i = 1, \cdots, \mathrm{n}$.*

*Proof:* By Assumption 1, for all $1 \leq i \leq \mathrm{n}$ there exists $\mathcal{V}_i : \mathbb{R}^{\nu_i} \to \mathbb{R}$, positive semi-definite storage function corresponding to each node such that

$$\dot{\mathcal{V}}_i(x_i) \leq \gamma_i^2 u_i^T u_i - y_i^T y_i. \tag{5}$$

Selection of positive semi-definite, $\mathcal{V}_i$, together with Assumption 2, implies $\mathcal{V}_i$ positive definite. Since $\gamma := \max(\gamma_i)$ for $1 \leq i \leq \mathrm{n}$, (5) can be rewritten as

$$\dot{\mathcal{V}}_i(x_i) \leq \gamma^2 u_i^T u_i - y_i^T y_i. \tag{6}$$

Define $\mathcal{V}(x) = \sum_{i\in\mathcal{N}} \mathcal{V}_i(x_i)$, where $x = [x_1, \cdots, x_{\mathrm{n}}]^T$, $u = [u_1, \cdots, u_{\mathrm{n}}]^T$, and $y = [y_1, \cdots, y_{\mathrm{n}}]^T$. Therefore,

$$\dot{\mathcal{V}}(x) \leq \gamma^2 u^T u - y^T y. \tag{7}$$

Also, as the controller, $\mathcal{K}_e$, is $\mathcal{L}_2$ stable with $\mathcal{L}_2$–gain $\leq k_i$, therefore for each $\mathcal{K}_e : \mu \longmapsto \eta$, $\exists$ a positive semi-definite storage function $\mathcal{W}_i : \mathbb{R}^{\nu_i'} \to \mathbb{R}$, such that,

$$\dot{\mathcal{W}}_i(z_i) \leq k_i^2 \mu_i^T \mu_i - \eta_i^T \eta_i. \tag{8}$$

Positive semi-definiteness of $\mathcal{W}_i$, together with Assumption 2, makes $\mathcal{W}_i$ positive definite. Next we define $\mu = [\mu_1, \mu_2, \cdots, \mu_{\mathrm{m}}]^T$ and $\eta = [\eta_1, \eta_2, \cdots, \eta_{\mathrm{m}}]^T$ where m is the number of edges in the network. Therefore, (8) can be rewritten as

$$\dot{\mathcal{W}}(z) \leq k_i^2 \mu^T \mu - \eta^T \eta. \tag{9}$$

Using the relation $\mu = \mathcal{E}^T y$, (9) is further written as

$$\dot{\mathcal{W}}(z) \leq y^T \mathcal{E}\mathcal{K}_e\mathcal{E}^T y - \eta^T \eta. \tag{10}$$

By taking the Euclidean norm of $u = \mathcal{E}\eta$, one may obtain $\|u\|_2^2 \leq \|\mathcal{E}\|_2^2 \|\eta\|_2^2$, which is equivalent to

$$-\eta^T \eta \leq -\frac{1}{\|\mathcal{E}\|_2^2} u^T u. \tag{11}$$

Substituting (11) in (10) one may obtain

$$\dot{\mathcal{W}}(z) \leq y^T \mathcal{E}\mathcal{K}_e\mathcal{E}^T y - \frac{1}{\|\mathcal{E}\|_2^2} u^T u. \tag{12}$$

On multiplying (6) by $1/\|\mathcal{E}\|_2^2$ and (12) by $\gamma^2$ and adding, we get

$$\frac{1}{\|\mathcal{E}\|_2^2}\dot{\mathcal{V}}(x) + \gamma^2\dot{\mathcal{W}}(z) \leq \gamma^2 y^T \mathcal{E}\mathcal{K}_e\mathcal{E}^T y - \frac{1}{\|\mathcal{E}\|_2^2} y^T y. \tag{13}$$

Define $\mathcal{S}(x, z)$ to be a storage function such that

$$\mathcal{S}(x, z) := \frac{1}{\|\mathcal{E}\|_2^2}\mathcal{V}(x) + \gamma^2\mathcal{W}(z). \tag{14}$$

Note that, due to Assumption 2 and positive definiteness of $\mathcal{V}(x)$ and $\mathcal{W}(z)$, $\mathcal{S}(x,z)$ is also positive definite. Therefore, $\dot{\mathcal{S}}(x,z) = \left(1/\|\mathcal{E}\|_2^2\right)\dot{\mathcal{V}}(x) + \gamma^2\dot{\mathcal{W}}(z)$, which implies

$$\dot{\mathcal{S}} \leq -y^T\left[\frac{I_{\mathrm{n}}}{\|\mathcal{E}\|_2^2} - \gamma^2\mathcal{E}\mathcal{K}_e\mathcal{E}^T\right]y. \tag{15}$$

Using the relation in (4), we get $\dot{\mathcal{S}} \leq 0$. Here, $\dot{\mathcal{S}}$ is only negative semi-definite and $\dot{\mathcal{S}} = 0$, implies $y = 0$. Following the relation in (14), one may write

$$\dot{\mathcal{S}} \leq \frac{1}{\|\mathcal{E}\|_2^2}\gamma^2u^Tu - \frac{1}{\|\mathcal{E}\|_2^2}y^Ty + \gamma^2k_i^2\mu^T\mu - \gamma^2\eta^T\eta. \tag{16}$$

Using the zero input condition and substituting $u = 0$ and $\mu = 0$, in (16), we get

$$\dot{\mathcal{S}} \leq -\frac{1}{\|\mathcal{E}\|_2^2}y^Ty - \gamma^2\eta^T\eta. \tag{17}$$

By Assumption 2 we consider the set $E = \{(x,z)|\dot{\mathcal{S}} = 0\}$. Now let $M \subseteq E$ be the largest invariant set, then $M = 0$ by Assumption 2. From Lasalle's Invariance Principle [18], all solutions of the dynamical system in (3) and (2) converge to $M$ as $t \to \infty$, therefore the origin is asymptotically stable. This concludes the proof. ∎

## IV. Edge Controller Design: LMI Formulation

In this section, we will use Theorem 1 to formulate the $\mathcal{L}_2$–gain of the edge controller, $\mathcal{K}_e$. More specifically, we exploit the LMI structure of the sufficient condition given in Theorem 1 to pose several optimization problems that will instill the edge controllers with certain desirable properties in addition to ensuring stability of NDS.

**Optimization 1.** *Consider that Assumption 1 holds. Then, solve the following:*

$$\text{maximize trace}(\mathcal{K}_e) \text{ s.t. } \frac{I_{\mathrm{n}}}{\|\mathcal{E}\|_2^2} - \gamma^2\mathcal{E}\mathcal{K}_e\mathcal{E}^T \geq 0,\ \mathcal{K}_e \geq 0.$$

In order to get the maximum value of controller gains we attempt to maximize the feedback gain $\mathcal{K}_e$ for a given value of $\gamma$, which is obtained from the plant specifications. This is motivated by the desirability of the high gain controllers that instill the closed loop system with properties like high speed, and low error. etc. Alternatively, to obtain the maximum value of the controller gains, one may choose to maximize the Frobenious norm of $\mathcal{K}_e$ subjected to the constraint (4). This is presented as the next optimization problem.

**Optimization 2.** *Consider that Assumption 1 holds. Then solve the following:*

$$\text{maximize} \sum_{i=1}^{\mathrm{m}} k_i^2 \text{ s.t. } \frac{I_{\mathrm{n}}}{\|\mathcal{E}\|_2^2} - \gamma^2\mathcal{E}\mathcal{K}_e\mathcal{E}^T \geq 0,\ \mathcal{K}_e \geq 0$$

The knowledge of $\gamma$ is necessary in order to get a $\mathcal{K}_e$ that is positive definite. Also, the LMI condition in (4) implies that the knowledge of $\gamma$ is beneficial for robustness against non-linearity and disturbance. Next, we endeavor to maximize $\gamma$ by designing the graph using a convex optimization problem.

**Optimization 3.** *Consider that Assumption 1 holds. Then solve the following:*

$$\text{maximize } \gamma \text{ s.t. } \frac{\mathrm{I_n}}{\gamma^2\|\mathcal{E}\|_2^2} - \mathcal{E}\mathcal{K}_{\mathrm{e}}\mathcal{E}^{\mathrm{T}} \geq 0,\quad \mathcal{K}_{\mathrm{e}} \geq 0$$

The above optimization problem is solved in a similar manner to the eigenvalue minimization problem given in Sec. 2.2.2 of [19]. The solution to this problem provides the largest value of $\gamma$ for which a valid $\mathcal{K}_e$ exists.

The edge controller can also be characterized by optimizing it over different network conditions for a known value of $\gamma$. This can be achieved by fixing $\gamma$ in Optimization 3 and maximizing trace$(\mathcal{K}_e)$ or $\sum_{i=1}^{\mathrm{m}} k_i^2$ for different standard graphs. One can solve this problem by varying $\|\mathcal{E}\|_2^2$ in Optimization 1 and Optimization 2.

The optimization problems can be efficiently solved using numerical methods. The convex optimization problems, Optimization 1, Optimization 2, and Optimization 3 return the maximum values for $\mathcal{K}_e$ and $\gamma$, respectively.

To solve the combinatorial problem of optimal edge control design via selection of the most suitable network, the following proposition provides a possible tool.

**Proposition 1.** *If the dynamics in (2) are heterogeneous and finite gain $\mathcal{L}_2$–stable, then (4) can be written as*

$$\frac{I_n}{\gamma^2\|\mathcal{E}\|_2^2} - k_1\mathcal{E}_1\mathcal{E}_1^T - k_2\mathcal{E}_2\mathcal{E}_2^T - \cdots - k_{\mathrm{m}}\mathcal{E}_{\mathrm{m}}\mathcal{E}_{\mathrm{m}}^T \geq 0, \tag{18}$$

*where, $\mathcal{E}_i$ are the columns of the incidence matrix. Therefore, the objective of obtaining the $\mathcal{L}_2$–gain, $k_i$, of $\mathcal{K}_e$, translates to that of finding the elements of a set $\mathbb{S} := \{k := (k_1 \cdots k_{\mathrm{m}}) \in \mathbb{R}^{\mathrm{m}} \mid (4)$ is satisfied$\}$*

*Proof:* Consider a weighted undirected graph $\mathcal{G} = (\mathbb{V},\mathbb{E})$, of n nodes and m edges. Let $\mathcal{E}$ be the node-edge incidence matrix and $\mathcal{K}_e$ be a diagonal matrix of order m. Define $\mathcal{E} := [\mathcal{E}_i]$, where $i = (1,2,\cdots \mathrm{m})$ and $\mathcal{K}_e := \mathrm{diag}(k_i)$ for $i = (1,2,\cdots \mathrm{m})$. Using Theorem 1 one can write

$$\frac{I_{\mathrm{n}}}{\gamma^2\|\mathcal{E}\|_2^2} - [\mathcal{E}_i]\,\mathrm{diag}(k_i)\,[\mathcal{E}_i]^T \geq 0.$$

$$\implies \frac{I_{\mathrm{n}}}{\gamma^2\|\mathcal{E}\|_2^2} - k_1\mathcal{E}_1\mathcal{E}_1^T - k_2\mathcal{E}_2\mathcal{E}_2^T - \cdots - k_{\mathrm{m}}\mathcal{E}_{\mathrm{m}}\mathcal{E}_{\mathrm{m}}^T \geq 0.$$

This completes the proof. ∎

### *A. Variation of $\mathcal{L}_2$–gain of edge controller for different standard graphs*

In what follows, we show the variation of the normalized value of trace$(\mathcal{K}_e)$ with the number of nodes in the graph, keeping the value of $\gamma$ fixed. The parameter trace$(\mathcal{K}_e)/\mathrm{n}$ was obtained for different graphs, namely path, star, and cycle graph while keeping the number of nodes, n, as a variable for each graph. The results are shown in Fig. 2. It may be seen

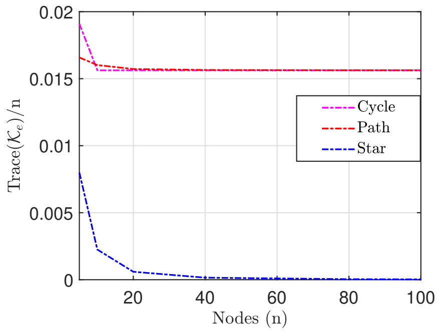



Fig. 2: Variation of trace $(\mathcal{K}_e)$/n.

that with the increase in the value of $\|\mathcal{E}\|_2^2$ and n the value of $\mathcal{L}_2$–gain of the controller, $\mathcal{K}_e$, decreases. For a smaller value of n the gain is less in the case of cycle graph. Even, though the values of the trace $(\mathcal{K}_e)$/n were comparable in all the cases, the highest value of trace $(\mathcal{K}_e)$/n was obtained in the case of a cycle which implies that for a smaller n, the value of trace $(\mathcal{K}_e)$/n will be more for a graph with more edges but the individual gains will be less.

*Remark* 2. The main results and the associated optimization processes provide bounds on the gains of the edge controller to guarantee stability. These bounds thus provide "safety regions" while an edge controller is being designed to possibly meet different network goals, e.g., formation. However, our optimization algorithms can also be used for designing stabilizing controllers, assuming a specific structure for these controllers while keeping their gains as variable parameters that eventually get fixed via the optimization algorithms.

## V. Worked out Examples

In this section, we present a simulation study to illustrate the performance of the proposed controller dynamics. Towards that, we consider five dynamical systems (agents) each being finite-gain $\mathcal{L}_2$–stable connected in a diffusive manner with dynamic edge controller $\mathcal{K}_e$ as shown in Fig. 1. Furthermore, for the controller dynamics, a standard 1st order transfer function is considered. Therefore, The controller dynamics are given as

$$\mathcal{K}_e := \text{diag}(\mathcal{K}_{ij}) = \text{diag}\left(\frac{k'}{(s+b_i)}\right), \tag{19}$$

where $k_i = k'/b_i$ and $-b_i$ are the dc-gain (also the $\mathcal{L}_2$–gains) and pole locations of the respective controllers. The poles and the $k'$ are chosen so as to satisfy the loop-gain criterion on $k_i = k'/b_i$, where $k_i$ is obtained by solving (4). The agents' dynamics evolve according to

$$\begin{aligned}
\mathcal{P}_1 &: = \dot{x}_1 = x_2;\ \dot{x}_2 = -2x_1^3 - 0.2x_2 + u,\ \ y = x_2,\\
\mathcal{P}_2 &: = \dot{x}_1 = x_2;\ \dot{x}_2 = -0.9x_1 - 2x_2^3 + u,\ \ y = x_2,\\
\mathcal{P}_3 &: = \dot{x}_1 = x_2;\ \dot{x}_2 = -0.96x_1^3 - 1x_2 + u,\ \ y = x_2,\\
\mathcal{P}_4 &: = \dot{x}_1 = x_2;\ \dot{x}_2 = -0.5x_1^3 - 0.5x_2^3 + u,\ \ y = x_2,\\
\mathcal{P}_5 &: = \dot{x}_1 = x_2;\ \dot{x}_2 = -0.08x_1 - 0.02x_2^3 + u,\ \ y = x_2.
\end{aligned}$$

The initial condition of the plants are given as $[-2\ -3\ 4\ -1.9\ 1.4\ -1.6\ 1.5\ -3.2\ 3.5\ -4.3]$. The $\mathcal{L}_2$–gains of agents are $\gamma_1 = 5$, $\gamma_2 = 0.5$, $\gamma_3 = 1$, $\gamma_4 = 2$, and $\gamma_5 = 50$, hence $\gamma = \max(\gamma_i) = 50$ for $1 \le i \le 5$. Therefore, one needs to choose a controller of $\mathcal{L}_2$–gain $\le 1/50$ to satisfy the loop gain criterion. For our examples, we have chosen plant gain as 60 leading to an upper bound on $\mathcal{L}_2$–gain equal to $1/60$. The performance of controllers has been studied for different graphs, namely cycle, path, and star graphs. The simulation results are discussed as follows:

### A. Cycle

Consider a network made of 5 nodes connected in a cycle fashion, the node-edge incidence matrix is

$$\mathcal{E} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1\\ 1 & -1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

The $\|\mathcal{E}\|_2^2$ is obtained as 3.9021 while the $\mathcal{L}_2$–gains of the controller obtained by solving (4) are $k_i = [2.123\ 2.122\ 2.124\ 2.132\ 2.112] \times 10^{-5}$. As there are 5 edges, therefore, 5 controllers are required. The controller dynamics are chosen as per (19), with $[-0.02\ 0.04\ 0.03\ -0.1\ 0.43]$ as the initial conditions. The poles in (19), obtained from relation $b_i = k'/k_i$ for $k' = 3$ are given as $b_i = [1.4131\ 1.4137\ 1.4124\ 1.4107\ 1.4209] \times 10^5$. The performance of the interconnected network is shown in Fig. 3a. It is observed that the edge controller is able to stabilize the interconnection, and both the positions and velocities approach zero asymptotically. It is also evident from Fig. 4a, that the controller states, $z_i$, converge to zero asymptotically, which supports the claims made in Theorem 1.

### B. Path

For the network comprising 5 nodes connected in a path, the node-edge incidence matrix is

$$\mathcal{E} = \begin{bmatrix} -1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

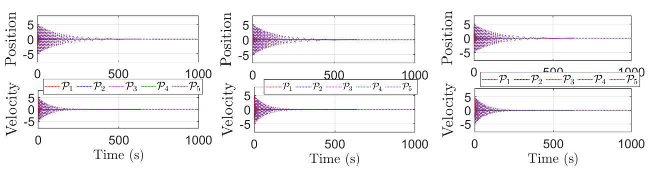
Since there are fewer edges than nodes, we would need only 4 controllers in this case. The $\|\mathcal{E}\|_2^2$ for a path is obtained as 3.90, while the $\mathcal{L}_2$–gains of the controller obtained by solving (4) are $k_i = [3.012\ 1.522\ 1.534\ 3.132] \times 10^{-5}$.The controller dynamics are chosen as per (19), with $[-0.02\ 0.04\ 0.03\ -0.01]$ as the initial conditions. The poles in (19), obtained from relation $b_i = k'/k_i$ for $k' = 3$ are given as $b_i = [9.4131\ 1.9137\ 1.9524\ 9.5107] \times 10^5$. The performance of the network is shown in Fig. 3b. It is observed that the edge controller is able to stabilize the interconnection, and all the agent states approach zero asymptotically. It is also seen in Fig. 4b, that the controller states, $z_i$, converge to zero asymptotically, as before.

*C. Star*

Consider a network of 5 nodes connected in a star with its center at node 2, the node-edge incidence matrix is obtained as

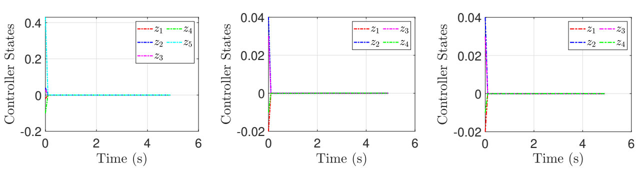
$$\mathcal{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

following which $\|\mathcal{E}\|_2^2$ is obtained as 5 while the $\mathcal{L}_2$–gains of the controller obtained by solving (4) are $k_i = [1.123\ 1.112\ 1.114\ 1.132] \times 10^{-5}$. The controller dynamics are chosen as per (19), with $[-0.02\ 0.04\ 0.03\ -0.01]$ as the initial conditions. The poles in (19), obtained from relation $b_i = k'/k_i$ for $k' = 3$ are obtained as $b_i = [2.7131\ 2.4137\ 2.6124\ 2.6507] \times 10^5$. The performance of the network is shown in Fig. 3c. It is observed that the edge controller is able to stabilize the interconnection, and both the positions and velocities of individual plants approach zero asymptotically, as well. As seen in Fig. 4c, the controller states converge to zero asymptotically as well.




(a) Cycle. (b) Path. (c) Star.

Fig. 3: Node dynamics.




(a) Cycle. (b) Path. (c) Star.

Fig. 4: Edge controller dynamics.

*Remark* 3. It may be observed that for a fixed value $\mathcal{L}_2$–gain $\gamma$, with an increase in the number of edges for a graph containing the same number of nodes, the value of $\mathcal{L}_2$–gain of the controller decreases. One should note that any value of gain $k'$ that satisfies the relation $k_i = k'/b_i$ can be chosen as a controller parameter to further satisfy the loop gain criterion according to (4).

## VI. Conclusion

This paper proposed small-gain methods to derive a sufficient condition on the asymptotic stability of networks of diffusively connected finite-gain $\mathcal{L}_2$–stable systems, even when each individual system/node admitted heterogeneous nonlinear dynamics. Our findings show that, under the assumption that the agents are finite-gain $\mathcal{L}_2$–stable and zero-state observable, the interconnection is zero input asymptotically stable, and all agents dynamics converge to origin. Furthermore, the $\mathcal{L}_2$–gain of the edge controllers was obtained by solving a proposed LMI condition. The optimization problems posed in this work provide an upper bound on the family of edge controllers that can be used for dynamic coupling. Extensive simulation results are provided that demonstrate the effectiveness of our theoretical results. Generalization of the NDS problem in dissipativity framework and the problem of network optimization over graphs are some of the interesting future research directions.

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