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Approximate Stability Radius Analysis and Design in Linear Systems

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***Abstract*— The robustness of the stability properties of dynamical systems in the presence of unknown/adversarial perturbations to system parameters is a desirable property. In this paper, we present methods to efficiently compute and improve the approximate stability radius of linear time-invariant systems. We propose two methods to derive closed-form expressions of approximate stability radius, and use these to re-design the system matrix to increase the stability radius. Our numerical studies show that the approximations work well and are able to improve the robustness of the stability of the system.**

I. Introduction

Stability is one of the fundamental concepts in the analysis and design of dynamical systems. Earlier, the binary notion of stability was studied and the focus was on determining whether the system is stable or not. However, in practical applications, it is crucial to assess the robustness of the system towards maintaining stability when subjected to modeling/adversarial variations in its parameters.

To address this, the notion of Stability Radius (SR) was introduced for Linear Time-Invariant (LTI) system $\dot{x} = Ax$ in [1]. SR quantifies the minimum-norm perturbation Δ that a stable system can tolerate before becoming unstable, and thus, it provides a quantitative measure of system stability. The perturbation forms $A + \Delta$ and $A + B\Delta C$ are referred to as unstructured and structured, respectively. In addition, Δ may have sparsity constraints to allow only a subset of entries to change. Further, the cases when Δ is allowed to be complex and real are called complex SR and real SR, respectively. For robust system design, SR should be taken into account while designing and deploying a system. In this paper, we present techniques to find (analysis) and improve (design) the SR of an LTI system.

Related Work: Several papers have studied SR since the seminal paper [1]. Closed-form expressions of the complex (unstructured and structured) SR problems were provided in [1] and [2]. In contrast, the real SR problem is considerably difficult and no closed-form solution exists [3], [4]. Bounds on the real SR were obtained in [5], [6], and a numerical computation formula was presented in [7]. Recently, several papers have proposed numerical approaches for computing the approximate SR for Frobenius norm bounded perturbations for the non-sparse [8], [9] and sparse [10], [11] cases. A similar analysis problem has been studied in the context of controllability [12] and observability [13].

While the analysis problem of computing the SR has been studied extensively, not much focus has been given to the system design problem where the goal is to modify the system in order to improve its SR. The system design problem is considerably more difficult since it is a bi-level optimization problem where the computation of SR appears in the constraints. Note that system design problems have been studied for other problems related to dynamical systems. For instance, the design problem is studied in the context of consensus [14], [15], [16], controllability [17], and smart grid design [18]. However, to the best of our knowledge, the system design problem has not yet been studied in the context of stability radius.

The main contributions of this paper are:

1. We present two methods to approximately compute the SR based on linear approximation of eigenvalues. The approximations enable us to get closed-form solutions which can be computed easily and efficiently.
2. We use the approximate SR solutions to solve the system design problem to improve the SR. The closed-form approximate solutions allow us to solve the otherwise computationally intensive design problem efficiently.
3. We present numerical simulations to show that the approximations work well, and that the approximated and actual SR values are close.

Mathematical Notations: $(\cdot)^r$ denotes the real part of a complex number. $(\cdot)^T$ and $(\cdot)^*$ denote the transpose and complex conjugate transpose, respectively. $\text{Tr}(\cdot)$ and $\|\cdot\|$ denote the trace and Frobenius norm of a matrix, respectively. $\circ$ denotes the Hadamard product. I denotes the identity matrix. $|\cdot|$ denotes the absolute value of a scalar. $j = \sqrt{-1}$ denotes the unit imaginary number. $\alpha(\cdot)$ denotes the spectral abscissa (maximum of the real part of all eigenvalues). $\mathbf{1}_n$ and $\mathbf{1}_{n\times m}$ denote a vector and matrix of appropriate sizes whose all elements are 1, respectively.

II. Problem Formulation

We consider a perturbed continuous-time LTI system:

$$\dot{x}(t) = (A + B\Delta C)x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $A \in \mathbb{R}^{n\times n}$ is the nominal system matrix, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the structure matrices, and $\Delta \in \mathbb{R}^{m\times p}$ is the perturbation matrix. The term $B\Delta C$ captures the perturbation to A, and we denote the perturbed system as $A(\Delta) \triangleq A + B\Delta C$.

Additionally, we impose sparsity constraints on Δ as follows. Let $S \in \{0,1\}^{m\times p}$ be a binary sparsity matrix spec-

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ifying the sparsity pattern as $\Delta_{ij} = \begin{cases} * & \text{if} \quad S_{ij} = 1, \\ 0 & \text{if} \quad S_{ij} = 0, \end{cases}$ where $* \in \mathbb{R}$ is a scalar, then the sparsity constraints are

$$S^c \circ \Delta = 0, \tag{2}$$

where $S^c \triangleq 1_{m\times p} - S$ is complement of the sparsity matrix. The case where $S = 1_{m\times p}$ is the non-sparse case.

Remark 1 (Perturbation structure)**.** *The perturbation $A + B\Delta C$ can result from output feedback of linear dynamical system as: $\dot{x} = Ax + Bu, y = Cx, u = \Delta y$, where $A + B\Delta C$ is the closed-loop matrix. The perturbations can also be interpreted as attacks that modify the nominal A to disrupt the functioning of the system. The B and C matrices add structure to the perturbation (for instance, modifying certain rows or columns of A). The sparsity constraints allow us to change only certain entries of A, which is useful in networked systems where only some edge weights are allowed to change.*

Next, we define the stability radius corresponding to system (1), denoted by $SR(A)$.

Definition 1 (Stability Radius)**.**

$$SR(A) = \min\{\|\Delta\| : \alpha(A(\Delta)) = 0, S^c \circ \Delta = 0\}. \tag{3}$$

■

Note that we consider $\|\cdot\|$ as the Frobenius norm throughout this paper. Without loss of generality, we assume that A is stable ($\alpha(A) < 0$). This ensures that $SR(A)$ is strictly positive. SR is a measure of the resiliency of the system since it captures the minimum-norm perturbation, which makes the system unstable by shifting the eigenvalues of A to the right-half plane.

We propose a System Design (SD) problem where the system operator wishes to change matrix A to $A + B_o\Delta_o C_o$ in order to improve the SR. Note that $B_o \in \mathbb{R}^{n\times m_o}, C_o \in \mathbb{R}^{p_o\times n}$ and $S_o \in \mathbb{R}^{m_o\times p_o}$ impose structure and sparsity constraints on how the system operator is allowed to modify the matrix A. In contrast, B, C and S impose constraints on "potential" perturbations to a nominal matrix and are used to compute the SR. The benign perturbation Δ_o is desired to be small so that the modified system remains "close" to the original system. The system design problem is formulated as

$$\textbf{SD}: \min_{\Delta_o \in \mathbb{R}^{m_o\times p_o}} \|\Delta_o\| \tag{4}$$

$$\text{s.t.} \quad SR(A + B_o\Delta_o C_o) \geq \epsilon \tag{4a}$$

$$S_o^c \circ \Delta_o = 0, \tag{4b}$$

where $\epsilon > SR(A)$ is the desired increased level of SR.

The **SD** problem (4) is a bi-level optimization problem since constraint (4a) involves computing the SR, which itself is an optimization problem given in (3). Note that problem (3) is non-convex and does not admit a closed-form solution. Thus, computationally intensive iterative algorithms are used to obtain the solutions [10], [11], [8]. This makes the **SD** problem difficult to solve.

In order to simplify the **SD** problem, we propose to approximate the SR using approximations for $\alpha(A(\Delta))$. Specifically, we use linear and successive-linear approximations of the spectral abscissa, denoted by $\alpha_{la}(\cdot)$ and $\alpha_{sla}(\cdot)$, respectively. We explain these approximations later in Section III. The corresponding approximated SR problems are given as:

$$SR_{la}(A) = \min\{\|\Delta\| : \alpha_{la}(A(\Delta)) = 0, S^c \circ \Delta = 0\}, \tag{5}$$

$$SR_{sla}(A) = \min\{\|\Delta\| : \alpha_{sla}(A(\Delta)) = 0, S^c \circ \Delta = 0\}. \tag{6}$$

We later show that the above two problems admit closed-form solutions that can be computed easily and efficiently.

For the system design problem, we propose to use the approximations SR_{la} and SR_{sla} in constraint (4a), and denote the corresponding approximate system design problems as $\textbf{SD}_{la}$ and $\textbf{SD}_{sla}$, respectively. The approximate SR problems are analyzed in Section III and the approximate system design problems are addressed in Section IV.

III. Approximate Stability Radius Problems

In this section, we analyze the approximate SR problems given in (5) and (6). Both these problems involve linear approximation of eigenvalues, which we describe next.

A. Eigenvalue Approximation

We use the eigenvalue sensitivity analysis which dictates how eigenvalues are modified when a matrix is perturbed. In particular, we use the following result.

Lemma 1. *(Eigenvalue Sensitivity [19]) Let λ_k be a simple eigenvalue of A with corresponding left and right eigenvectors, y_k and z_k, respectively, such that $y_k^* z_k = 1$ for $k = 1, 2, \cdots, n$. Then, as A is perturbed to $A(\Delta)$, the sensitivity of λ_k with respect to parameter Δ_{ij} is given as*

$$\frac{\partial \lambda_k}{\partial \Delta_{ij}} = y_k^* \Big[\frac{\partial A(\Delta)}{\partial \Delta_{ij}}\Big] z_k = y_k^* B E_{ij} C z_k, \tag{7}$$

where E_{ij} is a matrix with $(i,j)^{th}$ entry as 1 and all other entries as 0.

The above lemma requires the eigenvalues to be simple, so we make the following assumption.

Assumption 1. *All eigenvalues of A are assumed to be simple.*

Let P_k denote the sensitivity matrix corresponding to eigenvalue λ_k, where $[P_k]_{ij} = y_k^* B E_{ij} C z_k$. Based on Lemma 1, we approximate the real part of λ_k as

$$\lambda_k^r(A + B\Delta C) \approx \hat{\lambda}_k^r(\Delta) = \lambda_k^r + \sum_{i=1}^{m}\sum_{j=1}^{p} [P_k^r]_{ij}\Delta_{ij},$$

$$\Rightarrow \hat{\lambda}_k^r(\Delta) = \lambda_k^r + \mathbf{1}_m^T (P_k^r \circ \Delta)\mathbf{1}_p, \tag{8}$$

where λ_k^r and P_k^r denote the real parts of λ_k and P_k respectively.

Remark 2 (Effect of normality of A on the approximations)**.** *When A is normal, the sensitivity values in* (7) *are small*

[19], and as a result, the approximation provided in (8) *works well. In contrast, for non-normal matrices, the approximation error may be large. We comment on this fact later in the simulation Section V.*

B. Stability Radius via Linear Approximation

Based on the linear approximation (LA) of the eigenvalues in (8), we re-write (5) as

$$SR_{la}(A) = \min\{\|\Delta\| : \max_{k=1,\cdots,n} \{\hat{\lambda}_k^r(\Delta) = 0, S^c \circ \Delta = 0\}, \quad (9)$$

$$= \min_{k=1,\cdots,n} \{\|\Delta_k^*\|\}, \text{where}$$

$$\Delta_k^* = \arg\min\{\|\Delta\| : \hat{\lambda}_k^r(\Delta) = 0, S^c \circ \Delta = 0\}. \quad (10)$$

Thus, we solve (10) individually for each eigenvalue, and then take the minimum-norm over these solutions to get $SR_{la}(A)$. Note that problem (10) is a quadratic optimization problem with linear equality constraint and admits a closed-form solution that can be computed quickly.

Next, we discuss the feasibility of the SR_{la} problem.

Lemma 2. *(Feasibility) The optimization problem in* (9) *is feasible if and only if $S \circ P_k^r \neq 0$ holds true for at least one $k \in \{1, 2, \cdots, n\}$.*

Proof. In (10), the second equality constraint $S^c \circ \Delta = 0$ is equivalent to $\Delta = S \circ \bar{\Delta}$, where $\bar{\Delta}$ is any arbitrary matrix. Substituting Δ in the first equality constraint in (10), we get

$$\mathbf{1}_m^T (P_k^r \circ S \circ \bar{\Delta}) \mathbf{1}_p = -\lambda_k^r.$$

Since $\alpha(A) < 0$, we have $\lambda_k^r \neq 0$. Thus, a solution $\bar{\Delta}$ exists for the above equation if and only if $S \circ P_k^r \neq 0$. The result then follows from (9). ■

Note that the condition $S \circ P_k^r = 0$ implies that under $S^c \circ \Delta = 0$, the term $\mathbf{1}_m^T (P_k^r \circ \Delta) \mathbf{1}_p = 0$. Therefore, the k^{th} eigenvalue of A cannot be shifted by the perturbation with the given sparsity constraints. If this holds true for all eigenvalues, then none of the eigenvalues of A can be perturbed, and $SR_{la}(A) = \infty$.

Example 1. *Consider* $A = \begin{bmatrix} -1 & 0.5 \\ -2 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0.4 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. *Here,* $\lambda = -0.4 \pm 0.8j$, $P_1^r = P_2^r = \begin{bmatrix} 0 & 0 \\ 0.7 & 1 \end{bmatrix}$. *Thus,* $S \circ P_1^r = S \circ P_2^r = 0$. *Hence, the feasibility condition is violated for both the eigenvalues.*

Next, instead of solving problem (9) directly, we present the following equivalent alternate reformulation given as

$$\|\Delta_k^*\| = \{\beta : SA_k(\beta) = 0\}, \text{ where} \quad (11)$$

$$SA_k(\beta) = \max_{\Delta} \quad \lambda_k^r + \mathbf{1}_m^T (P_k^r \circ \Delta) \mathbf{1}_p \quad (12)$$

$$\text{s.t} \quad \|\Delta\| \leq \beta \quad (12a)$$

$$S^c \circ \Delta = 0. \quad (12b)$$

The next subsection will also require this reformulation for computing $\alpha_{sla}(\cdot)$. Note that in both problems (9) and (11), we compute $\alpha(A(\Delta))$ and determine the minimum-norm perturbation that shifts it to the unstable region. Hence, the solutions to these two problems are identical.

Problem (12) is convex with linear cost and quadratic and linear constraints. Next, we present its closed-form solution.

Theorem 1. *Let the feasibility condition in Lemma 2 hold true, and define $\mathcal{K} = \{k : S \circ P_k^r \neq 0\}$. Then, the solution of the optimization problem* (11) *is given by*

$$\Delta_k^* = -\frac{\lambda_k^r (S \circ P_k^r)}{\|S \circ P_k^r\|^2}, \quad k \in \mathcal{K}. \quad (13)$$

Further, we have

$$SR_{la}(A) = \min_{k \in \mathcal{K}} \left\{ \frac{-\lambda_k^r}{\|S \circ P_k^r\|} \right\}. \quad (14)$$

Proof. We first solve optimization problem (12) for $k \in \mathcal{K}$ using the first-order KKT conditions. Using $\|A\|^2 = \text{Tr}(A^T A)$, (12a) can be rewritten as

$$\|\Delta\| \leq \beta \Leftrightarrow \text{Tr}(\Delta^T \Delta) \leq \beta^2.$$

Let $l \leq 0$ and $M \in \mathbb{R}^{m \times p}$ be the Lagrangian multipliers associated with constraints (12a) and (12b), respectively. The Lagrangian function is given by

$$\mathcal{L} = \hat{\lambda}_k^r(\Delta) + l(\|\Delta\| - \beta) + \mathbf{1}_m^T (M \circ S^c \circ \Delta) \mathbf{1}_p,$$

$$= \lambda_k^r + \text{Tr}(P_k^{rT} \Delta) + l\,(\text{Tr}(\Delta^T \Delta) - \beta^2) + \text{Tr}((M \circ S^c)^T \Delta),$$

where we use the property $\mathbf{1}_m^T (A \circ B) \mathbf{1}_p = \text{Tr}(A^T B)$. Differentiating $\mathcal{L}$ w.r.t. Δ and equating to 0, we get

$$\frac{\partial \mathcal{L}}{\partial \Delta} = P_k^r + 2l\Delta + M \circ S^c = 0. \quad (15)$$

Taking the Hadamard product of (15) with S^c, using (2) and $S^c \circ S^c = S^c$, we get

$$S^c \circ P_k^r + 2l(S^c \circ \Delta) + S^c \circ M \circ S^c = 0,$$

$$\Rightarrow M \circ S^c = -S^c \circ P_k^r.$$

Substituting the above expression in (15) and using $S = 1_{m \times p} - S^c$, we get

$$\mathbf{1}_{m \times p} \circ P_k^r + 2l\Delta - S^c \circ P_k^r = 0,$$

$$\Rightarrow 2l\Delta = -S \circ P_k^r. \quad (16)$$

Since $S \circ P_k^r \neq 0$ for $k \in \mathcal{K}$, (16) implies that $l \neq 0$, we get

$$\Delta = -(S \circ P_k^r)/2l. \quad (17)$$

By complementary slackness, $l \neq 0$ implies that constraint (12a) is active. Substituting Δ obtained in (17) in the active constraint $\|\Delta\| = \beta$, we get

$$l = -\|S \circ P_k^r\|/2\beta. \quad \text{(Since, } l \leq 0 \text{ from dual feasibility)}$$

Next, substituting l in (17), we get the solution to (12) as:

$$\Delta_k = \frac{\beta (S \circ P_k^r)}{\|S \circ P_k^r\|}, \quad (18)$$

and $SA_k(\beta) = \lambda_k^r + \beta \|S \circ P_k^r\|$. Since Problem (12) is convex, any solution satisfying the first-order KKT conditions is a global minimum. Next, we solve the equality in (11) to get

$$\|\Delta_k^*\| = \beta^* = -\frac{\lambda_k^r}{\|S \circ P_k^r\|}, \tag{19}$$

and substitute β^* in (18) to get (13), $\Delta_k^* = -\frac{\lambda_k^r (S \circ P_k^r)}{\|S \circ P_k^r\|^2}$. The result (14) then follows from (10). ■

Result (19) implies that SR_{la} is dependent on the ratio $\lambda_k^r / \|S \circ P_k^r\|$. Thus, a system with eigenvalues of large magnitude and corresponding small sensitivities requires large perturbation to shift.

Our approach of linear approximation of eigenvalues works well if the perturbations are small. Intuitively, if $SR_{la}(A)$ is small, then we expect the approximation to work well and $SR_{la}(A)$ to be close to $SR(A)$. However, when $SR_{la}(A)$ is large, the linear approximation may not be precise. To address this issue, we propose a successive-linear approximation (SLA) approach next.

C. Stability Radius via Successive-Linear Approximations

As mentioned before, we expect the linear approximation to work well if the perturbations are small. Motivated by this, we propose to decompose the perturbation Δ as

$$\Delta = \Delta^{(1)} + \Delta^{(2)} + \cdots + \Delta^{(J)}, \tag{20}$$

and ensure that each perturbation is small, that is, $\|\Delta^{(j)}\| \leq \beta << 1$. We compute the optimal value of $\Delta^{(j)}$ (denoted by $\Delta^{(j,*)}$) in a successive/iterative manner and this results in a successive approximation of the spectral abscissa.

Let $A_{j-1} \triangleq A_0 + B\Delta^{(1,*)}C + \cdots + B\Delta^{(j-1,*)}C$ with $A_0 \triangleq A$. Next, we explain the steps to obtain $\Delta^{(j,*)}$.

1. Solve problem (12) with the eigenvalues and sensitivity matrices of A_{j-1} and denote the optimal solution by $\Delta_k^{(j,*)}$.
2. Compute

$$\Delta^{(j,*)} = \underset{k=1,\cdots,n}{\arg\max} \{\alpha(A_{j-1} + B\Delta_k^{(j,*)}C)\}. \tag{21}$$

3. Update $A_j = A_{j-1} + B\Delta^{(j,*)}C$.
4. Repeat the above steps until $\alpha(A_j) < 0$.

Let J denote the number of iterations of the above algorithm. Then,

$$SR_{sla}(A) = \|\Delta^{(1,*)} + \Delta^{(2,*)} + \cdots + \Delta^{(J,*)}\|. \tag{22}$$

Algorithm 1 SR via successive-linear approximations

Require: A, B, C, S, β
  Output: $SR_{sla}(A)$
  while $\alpha(A_j) < 0$ **do** ▷ $\alpha(A_j) > 0 \Rightarrow$ unstable region
    for $k = 1, \cdots, n$ **do**
      $P_k \leftarrow$ (7) ▷ sensitivity matrix of λ_k
      $\Delta_k^{(j,*)} \leftarrow$ (18)
    end for
    $\Delta^{(j,*)} \leftarrow$ (21)
    $A_j = A_{j-1} + B\Delta^{(j,*)}C$
  end while

Several remarks are in order for Algorithm 1. First, note that although computation of $SR_{sla}(A)$ is an iterative procedure, in each iteration, we use the closed-form expression given in (18). Thus, the overall computation time is small. Second, the algorithm is greedy in nature since at each iteration in (21) we pick a perturbation that corresponds to the largest spectral abscissa. Third, since we are using the eigenvalue sensitivity result given in Lemma 1, our approach requires that all eigenvalues of A_j are simple in each iteration. Fourth, we conjecture that the $SR_{sla}(A)$ is a better approximation than $SR_{la}(A)$ since the former involves several small perturbations as compared to a single but potentially large perturbation in the latter. Our conjecture is supported by simulations presented in Section V.

Remark 3 (Implementation details of Algorithm 1)**.** *Each iteration of Algorithm 1 involves solving Problem 12 for A_j. This requires Problem 12 to be feasible as per the condition provided in Lemma 2. If this feasibility condition is violated, we slightly perturb A_j randomly such that the problem becomes feasible and continue thereafter.*

Also, for Algorithm 1 to terminate, we require that $\alpha(A_j) > \alpha(A_{j-1})$ holds true at each iteration. However, since A_j is computed based on the linear approximation of eigenvalues, this condition might be violated. In this case, then we repeat the iteration with a slightly higher value of β such that $\alpha(A_j) > \alpha(A_{j-1})$ holds, and continue thereafter.

Remark 4 (Comparison of LA and SLA based approaches)**.** *The SLA-based approach to compute SR_{sla} is more accurate than the LA-based approach to compute SR_{la}. However, the former is iterative in nature, and therefore, requires more computational time as compared to the latter (details are presented in Section V). Thus, depending on the accuracy requirements and computational resources, one can select one of these two approaches.*

Remark 5 (Approximate solutions can aid other SR algorithms)**.** *The approximate solutions provided by our algorithms can serve as a good initializing point for the iterative algorithms proposed earlier [4], [10], [11] for the SR problem. This can considerably reduce the execution time of these algorithms. We defer the demonstration of this as a future work.*

IV. System Design Problems

In this section, we study the system design problem mentioned in (4). We wish to find a benign perturbation Δ_o added by the operator that improves the stability radius of the system. In problem (4), the constraint (4a) involves computation of $SR(A + B_o\Delta_o C_o)$. As mentioned earlier, computation of the actual stability radius is computationally difficult and requires an iterative procedure. Hence, we use the approximated SR - $SR_{la}(A)$ and $SR_{sla}(A)$ presented in Section III to approximate (4a) in the system design problem.

A. System Design via Linear Approximation

We replace $SR(\cdot)$ by $SR_{la}(\cdot)$ in constraint (4a) to get

$$\min_{\Delta_o \in \mathbb{R}^{m_o \times p_o}} \|\Delta_o\| \tag{23}$$

$$\text{s.t.} \quad SR_{la}(A + B_o\Delta_o C_o) \geq \epsilon \tag{23a}$$

$$S_o^c \circ \Delta_o = 0. \tag{23b}$$

Next, we focus on the term $SR_{la}(A + B_o\Delta_o C_o)$. Note that in the SR_{la} problem, the optimal solution Δ_k^* in (10) corresponds to matrix A, where λ_k and P_k are eigenvalue and sensitivity matrix of A. Similarly, let us denote the corresponding optimal solution for the matrix $A + B_o\Delta_o C_o$ as $\Delta_{k,o}^*$. Then, by (10), we have

$$SR_{la}(A + B_o\Delta_o C_o) = \min_{k=1,\cdots,n} \{\|\Delta_{k,o}^*\|\}. \tag{24}$$

Using (24), we reformulate Problem (23) as:

$$\begin{aligned} \mathbf{SD}_{la}: \min_{\Delta_o \in \mathbb{R}^{m_o \times p_o}} \|\Delta_o\| \qquad & (25) \\ \text{s.t.} \quad \|\Delta_{1,o}^*\| \geq \epsilon, \|\Delta_{2,o}^*\| \geq \epsilon, \cdots, \|\Delta_{n,o}^*\| \geq \epsilon \\ S_o^c \circ \Delta_o = 0. \end{aligned}$$

Thus, the single constraint in (24) which involves a $\min(\cdot)$ function is converted into multiple constraints, since the $\min(\cdot)$ function can be non-smooth and can cause numerical difficulties while solving the optimization problem.

Several comments are in order. First, $\mathbf{SD}_{la}$ is not a bi-level optimization problem since $\Delta_{k,o}^*$ in the inequality constraints admit a closed-form solution. Thus, it is computationally tractable. Second, the terms $\Delta_{k,o}^*$ depend on Δ_o since their computation depends on the eigenvalues and sensitivity matrices of $A + B_o\Delta_o C_o$. This dependence makes $\mathbf{SD}_{la}$ a non-convex problem and may have multiple local minima. Third, we solve the $\mathbf{SD}_{la}$ problem using numerical solvers. More details are presented in Section V.

B. System Design via Successive-Linear Approximations

As mentioned earlier, SR_{sla} gives a better approximation of stability radius as compared to SR_{la}. Therefore, we now replace $SR(\cdot)$ by $SR_{sla}(\cdot)$ in (4) to get

$$\begin{aligned} \mathbf{SD}_{sla}: \min_{\Delta_o \in \mathbb{R}^{m_o \times p_o}} \|\Delta_o\| \qquad & (26) \\ \text{s.t.} \quad SR_{sla}(A + B_o\Delta_o C_o) \geq \epsilon \qquad & (26a) \\ S_o^c \circ \Delta_o = 0. \qquad & (26b) \end{aligned}$$

We use (22) to compute $SR_{sla}(\cdot)$ in constraint (26a) successively. Further, similar to $\mathbf{SD}_{la}$ problem, $\mathbf{SD}_{sla}$ problem is also non-convex and is solved using numerical solvers. More details are presented in Section V.

V. Numerical Simulations

In this section, we present numerical simulation results of our algorithms. We perform the simulations using MATLAB R2023a. We first consider the spectral abscissa and SR problems, and later the system design (**SD**) problems.

A. Approximations of Spectral Abscissa and SR

Our SR algorithms rely crucially on approximations of the spectral abscissa. Thus, we first analyse the quality of these approximations. Recall that $\alpha(\cdot), \alpha_{la}(\cdot)$ and $\alpha_{sla}(\cdot)$ denote the spectral abscissas without approximation, with linear approximation and with successive-linear approximations, respectively. For a given norm bound γ, we compute the spectral abscissas by solving the following problems

$$\alpha(\gamma) = \max_{\Delta} \{\alpha(A(\Delta)) : \|\Delta\| \leq \gamma, S^c \circ \Delta = 0\}, \tag{27}$$

$$\alpha_{la}(\gamma) = \max_{\Delta} \{\alpha_{la}(A(\Delta)) : \|\Delta\| \leq \gamma, S^c \circ \Delta = 0\}, \tag{28}$$

$$\alpha_{sla}(\gamma) = \max_{\Delta} \{\alpha_{sla}(A(\Delta)) : \|\Delta\| \leq \gamma, S^c \circ \Delta = 0\}. \tag{29}$$

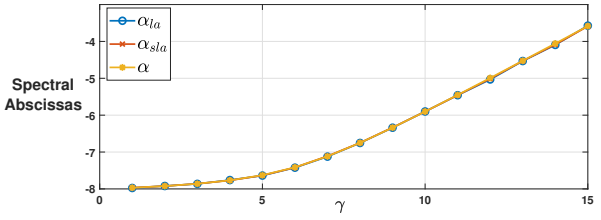
We solve Problem (27) by performing an exhaustive grid search over the set of perturbations that satisfy the constraints in (27). Problem (28) it equivalent to Problem (12) for the k^{th} eigenvalue. Thus, we have $\alpha_{la}(\gamma) = \max_{k=1,\cdots,n} \{SA_k(\gamma)\}$. The computation of $\alpha_{sla}(\gamma)$ in (29) is done in a successive manner similar to Algorithm 1. The difference is that we run the algorithm until $\|\Delta^{(1,*)} + \Delta^{(2,*)} + \cdots + \Delta^{(j,*)}\| \leq \gamma$ is not violated (see (20)). Let J_γ denote the iteration number until the above condition is not violated. Then, we get $\alpha_{sla}(\gamma) = \alpha(A + B(\Delta^{(1,*)} + \Delta^{(2,*)} + \cdots + \Delta^{(J_\gamma,*)})C)$.

We define the approximation errors of the spectral abscissas as $e_{la}(\gamma) = |\alpha(\gamma) - \alpha_{la}(\gamma)|$ and $e_{sla}(\gamma) = |\alpha(\gamma) - \alpha_{sla}(\gamma)|$. Further, we use the following non-normality measure (normality gap [20]) of a matrix A : $NG(A) = \|A^TA - AA^T\|$. To begin, we consider two cases given in Table I, one with a normal A and the other with a non-normal A.

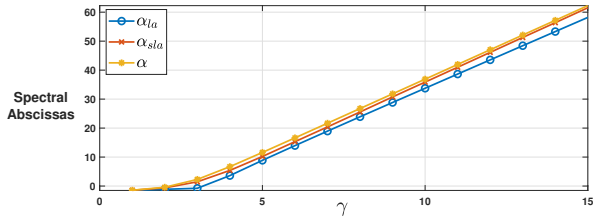
TABLE I: System specifications (A, B, C) for numerical examples

Case	$NG(A)$	A	B	C
Case I Normal	0	$\begin{bmatrix} -1.2 & -0.3 & -1 \\ -0.3 & -1.4 & -1 \\ -1 & -1 & -1.3 \end{bmatrix}$	$\begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \\ 0.4 & 0.1 \end{bmatrix}$	$\begin{bmatrix} 0.7 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$
Case II Non-normal	148.29	$\begin{bmatrix} -3 & -4 & -7 \\ -1 & -9 & -6 \\ -1 & -1 & -9 \end{bmatrix}$	$\begin{bmatrix} 1.3 & 1 \\ 1 & 0.7 \\ 0.5 & 1.4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.8 & 1.3 \\ 1.5 & 1.8 & 0.8 \end{bmatrix}$

Note: $S = I$ for both cases



(a) Case I: A is normal



(b) Case II: A is non-normal

Fig. 1: Variation of spectral abscissas as a function of γ.

Figure 1 shows the plots of $\alpha(\gamma), \alpha_{la}(\gamma)$ and $\alpha_{sla}(\gamma)$ for the above two cases. For Case I in Figure 1a, we observe that $\alpha_{la}(\gamma)$ and $\alpha_{sla}(\gamma)$ overlap with $\alpha(\gamma)$ for all values of γ. This is because A is normal, and hence, the

eigenvalue sensitivities are small ($\|P_1^r\| = 0.6063, \|P_2^r\| = 0.0666, \|P_3^r\| = 0.0399$). Thus, the approximations work well. On the other hand, for Case II in Figure 1b, we observe differences between $\alpha_{la}(\gamma), \alpha_{sla}(\gamma)$ and $\alpha(\gamma)$, especially for large values of β. This is because A is non-normal, and hence, the eigenvalue sensitivities are large ($\|P_1^r\| = 8.3881, \|P_2^r\| = 0.7848, \|P_3^r\| = 1.9765$). However, we observe that $\alpha_{sla}(\gamma)$ provides a better approximation than $\alpha_{la}(\gamma)$. Further, as γ increases, spectral abscissas increase because as the perturbation norm increases, the eigenvalue spectrum increases, hence, the spectral abscissas increase.

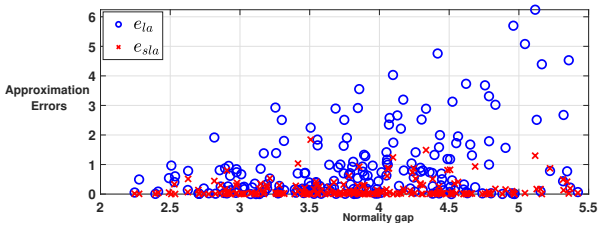


Fig. 2: Variation of approximation errors e_{la} and e_{sla} as a function of normality gap NG for 200 random triplets (A, B, C).

Next, we perform a similar comparison for a set of 200 random triplet of matrices (A, B, C) with $n = 5, m = 2$ and $p = 2$, $S = I$ and $\gamma = 10$. For each triplet, we compute normality gap $NG(A)$ and approximation errors $e_{la}(\gamma)$ and $e_{sla}(\gamma)$. Figure 2 shows that the errors $e_{sla}(\gamma)$ are considerably smaller than $e_{la}(\gamma)$, implying that the successive-linear approximation performs better than the linear approximation. Further, $e_{la}(\gamma)$ is larger for higher values of the normality gap, whereas $e_{sla}(\gamma)$ remains small. This shows that the SLA-based algorithm is more suited for non-normal problems.

These results provide an empirical evidence that our approximation-based approaches are well suited for approximate stability radius and system design problems.

Next, we use our algorithms to compute SR_{la} and SR_{sla} for a subset of test problems in the COMPl$_e$ib [21]. This library contains test problems for LTI control systems and A, B, C matrices are specified for each problem. However, the sparsity pattern is not specified for any of the test problems in the library. For some problems, we define the sparsity pattern and mark (S) against those in Table II. We compare SR_{la} and SR_{sla} with the SR values (computed using gradient based method in [10]) in Table II. We denote the computation time (measured in seconds) for computation of SR_{la} and SR_{sla} as τ_{la} and τ_{sla}, respectively. We observe that the approximation-based algorithms work well in approximating the SR. Also, the SLA-based algorithm performs better than LA-based algorithm for the majority of systems. However, the former has a larger computation time since it is iterative in nature (c.f. Remark 4).

B. System Design Problem

In this subsection, we present numerical results for the system design problems $\mathbf{SD}_{la}$ and $\mathbf{SD}_{sla}$ given in (25) and (26). We use the MATLAB function *fmincon* to solve these problems. In each iteration of *fmincon*, we use the closed-form expressions of SR_{la} and SR_{sla}.

TABLE II: SR values for different test problems

Test Problem	n	SR	SR_{la}	SR_{sla}	τ_{la}	τ_{sla}
Case I [10]	4	0.5159	0.5218	0.5140	0.0030	0.0959
Case II [10]	4	0.5653	0.6110	0.5694	0.0035	0.0929
HF2D12	5	1.4912	1.5371	1.3921	0.0017	0.1591
HF2D13 (S)	5	0.0424	0.0421	0.0422	0.0057	0.0274
TG1	10	0.0673	0.0642	0.0661	0.0014	0.0022
AGS	12	0.0688	0.0624	0.0719	0.0022	0.534
WEC2 (S)	10	0.0435	0.0420	0.0430	0.0035	0.0454
WEC3	10	0.5534	0.5221	0.5410	0.0052	0.0237
BDT1	11	0.0515	0.051	0.0514	0.0030	0.0099
MFP	4	0.7986	0.8123	0.8011	0.0020	0.1211
UWV	8	0.127	0.132	0.1239	0.0031	0.0123
EB1	10	0.0201	0.0205	0.0200	0.0137	0.1249
PSM (S)	7	0.4432	0.3508	0.4190	0.0198	0.4074
CDP (S)	120	0.0073	0.0071	0.0074	0.0649	0.3550

Note: Specified sparsity patterns specified are: $S(\text{HF2D13}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $S(\text{WEC2}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $S(\text{PSM}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $S(\text{CDP}) = I$.

We first consider Case I and Case II, which are presented in the previous subsection. We pick B_o, C_o, S_o same as B, C, S and solve $\mathbf{SD}_{la}$ and $\mathbf{SD}_{sla}$ for different values of ϵ. We denote the optimal solutions of $\mathbf{SD}_{la}$ and $\mathbf{SD}_{sla}$ as $\Delta_{o,la}^*$ and $\Delta_{o,sla}^*$, respectively. Figure 3a for Case I shows that the optimal perturbation norms $\|\Delta_{o,la}^*\|$ and $\|\Delta_{o,sla}^*\|$ are very close for all values of ϵ. This is again due to the fact that A is normal. Further, as ϵ increases, the optimal norms increase. This is because larger-norm perturbations to the system are required to achieve a larger value of SR. Further, Figure 3b verifies that the SR of the optimally perturbed system indeed matches with the desired SR value ϵ.

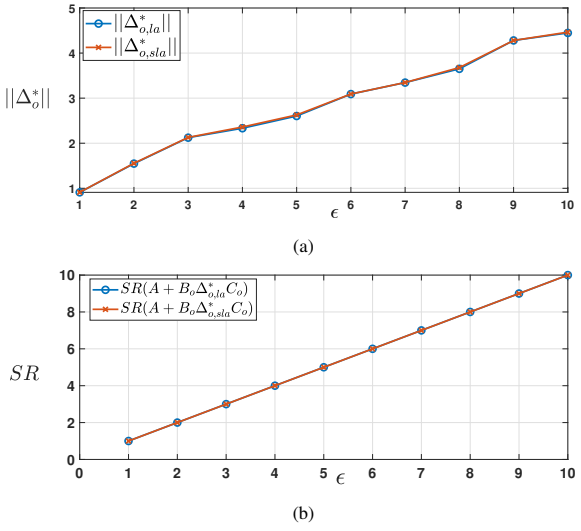


Fig. 3: Variation of (a) $\|\Delta_o^*\|$, and (b) SR as a function of ϵ for Case I.

The corresponding plots for Case II are presented in Figure 4. In contrast to Case I, we observe some difference between the optimal solutions $\Delta_{o,la}^*$ and $\Delta_{o,sla}^*$ for large values of ϵ in Figure 4a. Again, this is due to the fact that A is non-normal

in this case. In Figure 4b, we observe that the modified system achieves the desired value of SR.

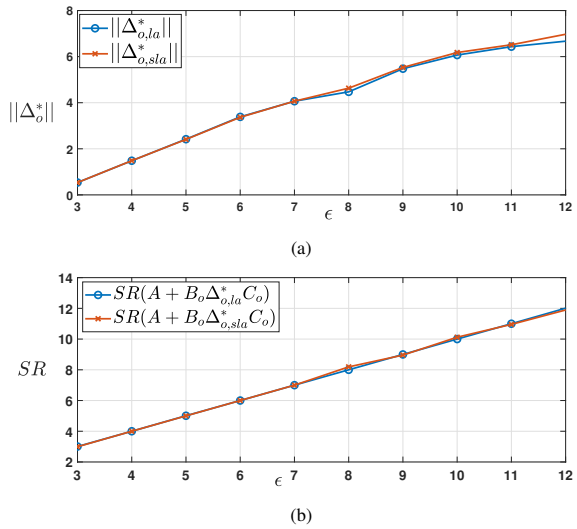


Fig. 4: Variation of (a) $\|\Delta^*_o\|$, and (b) SR as a function of ϵ for Case II.

Next, we consider the $\mathbf{SD}_{la}$ and $\mathbf{SD}_{sla}$ problems for the test problems given in Table III. For each system, we pick B_o, C_o, S_o same as B, C, S and set $\epsilon = 1.2SR(A)$ (that is, we aim to increase the SR by 20%). Table III provides the norms of the optimal solutions. We denote the computation time (measured in seconds) for $\mathbf{SD}_{la}$ and $\mathbf{SD}_{sla}$ as $\tau_{o,la}$ and $\tau_{o,sla}$, respectively. We observe that $\|\Delta^*_{o,la}\|$ and $\|\Delta^*_{o,sla}\|$ are close for majority of problems and $\|\Delta^*_{o,la}\| \geq \|\Delta^*_{o,sla}\|$ for all the cases. Further, $\tau_{o,la} < \tau_{o,sla}$ for each problem as computation of SR in $\mathbf{SD}_{sla}$ involves successive-linear approximations.

These results demonstrate that our approximated SR methods are useful for designing sparse and non-sparse systems with improved stability properties.

TABLE III: $||\Delta^*_o||$ for different test problems

Test Problem	$\|\Delta^*_{o,la}\|$	$\|\Delta^*_{o,sla}\|$	$\tau_{o,la}$	$\tau_{o,sla}$
Case I [10]	0.045	0.043	0.689	21.939
Case II [10]	0.4147	0.3631	10.23	23.025
HF2D12	0.1657	0.1649	20.74	22.1
HF2D13 (S)	0.0502	0.0467	0.6105	2.0420
TG1	0.0577	0.0520	1.591	3.579
AGS	0.1413	0.1141	5.472	31.27
WEC2 (S)	0.1015	0.1008	0.0426	1.5758
WEC3	0.5866	0.5721	12.31	77.60
BDT1	0.0033	0.0030	12.309	75.289
MFP	1.8646	1.5145	1.689	6.014
UWV	0.0944	0.0942	0.5410	4.141
EB1	0.0036	0.0034	0.4773	4.898
PSM (S)	0.4488	0.5078	1.9464	21.0752
CDP (S)	1.6542	1.5195	3.5064	153.42

VI. Conclusion

We propose approximated SR formulations based on eigenvalue sensitivity via linear and successive-linear approximations. We study these problems with sparsity constraints on perturbations and derive closed-form solutions. These results are used to develop an efficient framework to improve the stability radius. Future works include proposing a relaxed convex version of the design problem to solve it efficiently and easily. Also, we aim to extend our analysis for other system properties, like controllability, observability, detectability, etc., to design a system with improved characteristics.

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