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Dynamic Average Consensus as Distributed PDE-Based Control for Multi-Agent Systems

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Abstract— **This paper delves into the distributed estimator-based Dynamic Average Consensus (DAC) control problem within multi-agent systems (MASs) modeled by Partial Differential Equations (PDEs). The objective of DAC is for agents to converge to time-varying average profiles, referred to as dynamic average reference signals. Unlike prior research, this study will use a distributed estimator to recover the average reference signals for agents to track, which converges to the desired average reference in finite time. The reference signal with compact support employed in this study represents a more generalized signal type compared to previous works. Based on the distributed estimator, this research explores the DAC problem within in-domain PDE control. In-domain control is where input control acts within the governing equation. To assess the stability of the closed-loop system, we employ the Lyapunov technique for analysis. Finally, the proposed control designs' effectiveness in each section of the paper is demonstrated through simulation examples.**

Index Terms— **Multi-Agent Systems, Distributed PDE Control, Dynamic Average Consensus, Distributed Estimator.**

I. INTRODUCTION

An MAS refers to a group of autonomous agents that interact with each other to achieve a common goal. Control of MASs involves the coordination and regulation of the behavior of these autonomous agents. The behavior of each agent can be affected by the actions of other agents in the system and this will make the control of the system very complex [1]. There are several approaches to controlling MASs, including centralized [2], decentralized [3], and distributed control. Distributed control involves agents making decisions and coordinating their behavior with other agents based on local information, without a central controller [4]-[5].

The consensus-based control is a fundamental problem in MASs, where a group of agents must reach an agreement on a common value [6]. Various algorithms have been proposed in the literature to address the consensus problem, each with its advantages and limitations and one may refer to different order of model such as first-order [7], second-order [8], fractional-order [9], higher-order models [10] etc. The average consensus problem is a fundamental problem in MASs where the goal is for a group of agents to converge to a consensus value or average of a given set of values [11]. Average consensus has been investigated in two types of static and dynamic. In static average consensus, the common value is a constant where in most cases it is the average of initial conditions of agents' states [12].

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The DAC problem is an extension of the average consensus problem, where the values to be averaged are not static but rather vary over time [13]. Solving the DAC problem is of significant interest in applications such as distributed control of unmanned aerial vehicles (UAVs) [14] and coordinated control of autonomous vehicles [15]. Several approaches have been proposed to solve the DAC problem, including event-triggered algorithms [16] and robust algorithms [17]. In the existing literature on DAC, most studies have focused on systems described by Ordinary Differential Equation (ODE) models [18]. This paper considers a PDE model which describes MASs in a generalized way [19].

The control of PDEs is a challenging problem due to the infinite-dimensional nature of the PDE models [20]. For example, [5] investigates the problem of formation tracking which is often formulated as determining a coordinated control law that keeps the MASs maintain a desired, possibly time-varying, formation while tracking a target or following a reference orbit. In their research, [12] delved into the PDE control of static average consensus where agents will converge to the average of their initial condition.

Moreover, prior works typically consider a limited range of reference signals that the system needs to track [13]. Also, a significant limitation of prior research in the field of DAC is the assumption that agents possess direct access to the average of reference signals [13]. In practice, this is unattainable as agents lack the capacity for direct communication and data sharing among themselves. This paper consider a distributed estimator to address the limitations of previous works in this area and the idea of solving this issue has been borrowed from [21]. The distributed dynamic encirclement control for MASs has been studied in [21].

- To the authors' best knowledge, this work is the first control design results on in-domain control of a PDE-based model for MASs studying DAC problem.
- Compared with [13] and [18] in which studied DAC problem for stable linear ODE model of MASs, the main contribution of this paper is in considering general PDE model for MASs.
- Compared to previous works such as [13] and [18], our paper contributes significantly by introducing two key elements. First, we employ a distributed estimator for the average of reference signals to enhance system stability. Second, we extend our focus to general signals with compact Fourier transforms as reference signals, allowing for more versatile applications of DAC.
- In contrast to [12], which primarily addressed the static average consensus problem in PDE models of MASs,

our main contribution lies in our dynamic approach to average consensus.

The subsequent sections are organized as follows: Section II introduces reference signals and the problem formulation. Section III discusses the estimated average reference signal estimator. The dynamics of the distributed PDE system as an in-domain problem is detailed in Section IV. Section V explores a simulation example, and finally, Section VI concludes the paper and outlines future research directions.

II. Problem Formulation

Consider each agent as a distinct node and each interaction between agents as a single edge, forming an undirected simple graph $G(\Psi, E)$ where $\Psi = \{1, 2, ..., n\}$ denotes the set of nodes and E denotes the set of edges. A path in this context refers to a sequence of unique edges and nodes between two distinct nodes, ensuring no duplicated edges or nodes within the sequence. If a path exists between all distinct nodes, the graph is considered connected. For any simple connected graph with n nodes, there exists a unique adjacency matrix $\mathbf{A}(G) = [a_{ij}] \in \Re^{n\times n}$, where $a_{ij} = 1$ if nodes i and j share an edge in between, and $a_{ij} = 0$ otherwise. The graph Laplacian L associated with an undirected graph G is symmetric and positive semi-definite matrix for each connected graph.

A. System Formulation

Consider a group of n agents communicating with each other under an undirected simple graph. The MAS is modeled by a set of reaction-advection-diffusion PDE as

$$\boldsymbol{\theta}_t(t,x) = A\boldsymbol{\theta}_{xx}(t,x) + B\boldsymbol{\theta}_x(t,x) + \Gamma\boldsymbol{\theta}(t,x) + \boldsymbol{u}(t,x), \tag{1a}$$

$$\boldsymbol{\theta}_x(t,0) = 0, \quad \boldsymbol{\theta}_x(t,1) = 0, \tag{1b}$$

$$\boldsymbol{\theta}(0,x) = \boldsymbol{\theta}_0(x). \tag{1c}$$

In the context of $(t,x) \in \Re^+ \times \Omega$ and $\Omega = \{x : 0 < x < 1\}$ and $\boldsymbol{\theta}(x,t) \in \Re^n$ is the collecting vector of states of agents as

$$\boldsymbol{\theta}(t,x) = [\theta_1(t,x), \theta_2(t,x), \cdots, \theta_n(t,x)]^T, \tag{2}$$

and $\theta_i(t,x) \in \Re$ is the state of i-th agent for $i \in \Psi$. $\boldsymbol{u}(t,x) = [u_1(t,x), u_2(t,x), \cdots, u_n(t,x)]^T \in \Re^n$ is the collecting vector of control inputs of agents. Note $A = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $B = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n)$, and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n)$ where $\forall i \in \Psi, \alpha_i$, β_i, and γ_i are diffusivity, advectivity, and reactivity coefficients respectively. Consider that A is positive definite. The boundary conditions are considered to be of Neumann type.

***Definition** 1:* ([13]) $\boldsymbol{u}(t,x) \in C[0,\infty) \times [0,1]$ is a **Dynamic Average Consensus** control for system modeled by (1) such that $\forall x \in (0,1)$ the following holds

$$\lim_{t\to\infty} \boldsymbol{\theta}(t,x) = \boldsymbol{v}^{\text{avg}}(t), \tag{3}$$

where $\boldsymbol{v}^{\text{avg}}(t)$ is the collecting vector of average reference signal defined as

$$\boldsymbol{v}^{\text{avg}}(t) = v^{\text{avg}}(t)\mathbf{1}_n,$$

where $v^{\text{avg}}(t)$ is defined in (5).

B. Reference Signal

Reference signals that are with compact support and their Fourier transform exist are considered in this paper. Also, sinusoidal signals are considered as a periodic group of reference signals. Let

$$\boldsymbol{v}^{\text{ref}}(t) = [v_1^{\text{ref}}(t), v_2^{\text{ref}}(t), \cdots, v_n^{\text{ref}}(t)]^T \in \Re^n, \tag{4}$$

be the vector of all reference signals where $v_i^{\text{ref}}(t) \in \Re, t \geqslant 0$ is the reference signal of agent i. Let

$$v^{\text{avg}}(t) = \frac{1}{n}\mathbf{1}_n^T.\boldsymbol{v}^{\text{ref}}(t), \tag{5}$$

be the dynamic average reference signal.

***Assumption** 1:* Assume that the Fourier transform of chosen reference signal $v_i^{\text{ref}}(t)$ exist as

$$\bar{v}_i^{\text{ref}}(\xi) = \int_{-\infty}^{\infty} v_i^{\text{ref}}(t)e^{-i2\pi\xi t}\mathrm{d}t := F(v_i^{\text{ref}}(t)). \tag{6}$$

Let $\bar{v}^{\text{avg}}(\xi)$ be the Fourier transform of the average reference signal as

$$\bar{v}^{\text{avg}}(\xi) = F(v^{\text{avg}}(t)). \tag{7}$$

***Lemma** 1:* Consider the Fourier transform of reference signal and dynamic average reference signal proposed as (6) and (7) respectively. $v^{\text{avg}}(t)$ is compactly supported on a bounded set Υ. The integral is bounded and the measure of the set is $\mu(\Upsilon) > 0$. The Fourier transform of average reference signal and its first derivative have finite upper bounds η_1 and η_2 respectively as

$$|\bar{v}^{\text{avg}}(\xi)| \leqslant \mu(\Upsilon) \cdot \|v^{\text{avg}}(t)\| := \eta_1, \tag{8}$$

$$|\dot{\bar{v}}^{\text{avg}}(\xi)| \leqslant \mu(\Upsilon) \cdot \|\frac{\mathrm{d}}{\mathrm{d}t}v^{\text{avg}}(t)\| := \eta_2, \tag{9}$$

where $\|\cdot\|$ denotes the ∞-norm.

***Remark** 1:* Periodic signals with bounded amplitude like sinusoidal signal

$$v_i^{\text{ref}}(t) = b_i \sin(\omega_i t + \phi_i), \tag{10}$$

doesn't need to have compact support and their upper bound is fixed at any time as

$$|v^{\text{avg}}(t)| \leqslant \frac{1}{n}\sum_{i=1}^{n}|b_i| = \eta_1, \tag{11}$$

and

$$|\dot{v}^{\text{avg}}(t)| \leqslant \frac{1}{n}\sum_{i=1}^{n}|b_i| \cdot |\omega_i| = \eta_2. \tag{12}$$

III. Distributed Average Reference Signal Estimator

Given that each agent communicates exclusively with its local neighbors and lacks knowledge of the exact value of $v^{\text{avg}}(t)$, the initial task for agents is to estimate the average of reference signals. To tackle this challenge, each agent is equipped with an average reference estimator designed to compute the estimated average of reference signals denoted as $\hat{v}_i^{\text{avg}}(t) \in \Re$. The dynamics governing this average reference estimator is

$$\hat{v}_i^{\text{avg}}(t) = \delta_i(t) + v_i^{\text{ref}}(t), \tag{13a}$$

$$\dot{\delta}_i(t) = \begin{cases} \kappa \sum_{j=1}^{n} a_{ij} \frac{\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)}{\|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|}, \\ \qquad \text{if} \quad \hat{v}_j^{\text{avg}}(t) \neq \hat{v}_i^{\text{avg}}(t), \quad \forall t > 0, \\ 0 \qquad \text{if} \quad \hat{v}_j^{\text{avg}}(t) = \hat{v}_i^{\text{avg}}(t), \quad \exists t > 0, \end{cases} \tag{13b}$$

in which $\delta_i(t)$ is the intermediate state of estimator and $\delta_i(0) = 0, \quad \forall i \in \Psi$. $\kappa > 0$ is a design parameter. a_{ij} is the relevant element of adjacency matrix of network topology.

Theorem 1: Consider the average reference estimator dynamic in equation (13), by designing $\kappa > \eta_2(n-1)$. The estimator (13) is convergent to the average of reference signals in a finite and desirable time T as

$$\lim_{t \to T} \hat{v}_i^{\text{avg}}(t) = v_i^{\text{avg}}(t), \quad i \in \Psi, \tag{14}$$

where

$$T \leqslant \frac{2\sqrt{\frac{1}{2}\sum_{i=1}^{n} e_i^2(0)}}{n^{1.5}(\kappa - \eta_2(n-1))}, \tag{15}$$

and $e_i(0) = \hat{v}_i^{\text{avg}}(0) - \frac{1}{n}\sum_{i=1}^{n} \hat{v}_i^{\text{avg}}(0)$.

Proof: Define the error signal as

$$e_i(t) = \hat{v}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{i=1}^{n} \hat{v}_i^{\text{avg}}(t). \tag{16}$$

Choose the Lyapunov candidate as follows

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{n} e_i^2(t).$$

The time derivative of $V_1(t)$ is

$$\begin{aligned}
\dot{V}_1(t) =& \sum_{i=1}^{n} e_i(t) \left(\dot{\hat{v}}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \dot{\hat{v}}_k^{\text{avg}}(t) \right) \\
=& \sum_{i=1}^{n} e_i(t) \left(\kappa \sum_{j=1}^{n} a_{ij} \frac{\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)}{\|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|} \right. \\
& \left. + \dot{\hat{v}}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \dot{\hat{v}}_k^{\text{avg}}(t) \right) \\
=& \sum_{i=1}^{n} e_i(t) \left(\kappa \sum_{j=1}^{n} a_{ij} \frac{\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)}{\|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|} \right) \\
& + \sum_{i=1}^{n} e_i(t) \dot{\hat{v}}_i^{\text{avg}}(t) - \sum_{i=1}^{n} e_i(t) \left(\frac{1}{n}\sum_{k=1}^{n} \dot{\hat{v}}_k^{\text{avg}}(t) \right).
\end{aligned} \tag{17}$$

Define

$$\hat{f}_{ji}(t) := \frac{\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)}{\|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|}. \tag{18}$$

For the first term in (17), we have

$$\begin{aligned}
\kappa & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} e_i(t) \frac{\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)}{\|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|} \\
=& \kappa \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \big(\hat{v}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \big) \hat{f}_{ji}(t) \\
=& \frac{\kappa}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \big(\hat{v}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \big) \hat{f}_{ji}(t) \\
& + \frac{\kappa}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \big(\hat{v}_j^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \big) \hat{f}_{ij}(t) \\
=& \frac{\kappa}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \big(\hat{v}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \big) \hat{f}_{ji}(t) \\
& - \frac{\kappa}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \big(\hat{v}_j^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \big) \hat{f}_{ji}(t) \\
=& - \frac{\kappa}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \big(\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t) \big) \hat{f}_{ji}(t) \\
=& - \frac{\kappa}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|.
\end{aligned} \tag{19}$$

In the third row of (19), the order of summations has been switched from index i to j, and from $\hat{f}_{ji}(t)$ to $\hat{f}_{ij}(t)$. It's important to note that $a_{ij} = a_{ji}$. In the subsequent row, the sign has been inverted due to the substitution of $\hat{f}_{ij}(t)$ with $\hat{f}_{ji}(t)$, utilizing the fact that $\hat{f}_{ij}(t) = -\hat{f}_{ji}(t)$. Regard of the second term in (17), based on 12 we have

$$\begin{aligned}
\sum_{i=1}^{n} e_i(t) \dot{\hat{v}}_i^{\text{avg}}(t) \leqslant& \eta_2 \sum_{i=1}^{n} \|\hat{v}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t)\| \\
\leqslant& \frac{\eta_2}{n} \sum_{i=1}^{n} \left\| n\hat{v}_i^{\text{avg}}(t) - \sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \right\| \\
\leq& \eta_2 \sum_{j=1, j\neq i}^{n} \max_{i=1,2\ldots,n} \big(\|\hat{v}_i^{\text{avg}}(t) - \hat{v}_j^{\text{avg}}(t)\| \big) \\
\leq& (n-1)\eta_2 \max_{i,j=1,2\ldots,n} \|\hat{v}_i^{\text{avg}}(t) - \hat{v}_j^{\text{avg}}(t)\| \\
\leq& \frac{\eta_2(n-1)}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \|\hat{v}_j^{\text{avg}}(t) - \hat{v}_i^{\text{avg}}(t)\|.
\end{aligned} \tag{20}$$

For the third in (17) we have

$$\begin{aligned}
& \sum_{i=1}^{n} e_i(t) \left(-\frac{1}{n}\sum_{k=1}^{n} \dot{\hat{v}}_k^{\text{avg}}(t) \right) \\
&= \sum_{i=1}^{n} \left(\hat{v}_i^{\text{avg}}(t) - \frac{1}{n}\sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \right) \left(-\frac{1}{n}\sum_{k=1}^{n} \dot{\hat{v}}_k^{\text{avg}}(t) \right) \\
&= \left(\sum_{i=1}^{n} \hat{v}_i^{\text{avg}}(t) - \sum_{k=1}^{n} \hat{v}_k^{\text{avg}}(t) \right) \left(-\frac{1}{n}\sum_{k=1}^{n} \dot{\hat{v}}_k^{\text{avg}}(t) \right)
\end{aligned}$$

$$=\left(\frac{1}{n}\sum_{i=1}^{n}\hat{v}_i^{\text{avg}}(t)-\frac{1}{n}\sum_{k=1}^{n}\hat{v}_k^{\text{avg}}(t)\right)\left(-\sum_{k=1}^{n}\dot{\hat{v}}_k^{\text{avg}}(t)\right)$$
$$=\left(\frac{1}{n}\sum_{i=1}^{n}\hat{v}_i^{\text{avg}}(t)-\frac{1}{n}\sum_{k=1}^{n}\hat{v}_k^{\text{avg}}(t)\right)\left(-\sum_{k=1}^{n}\dot{\hat{v}}_k^{\text{avg}}(t)\right)=0. \tag{21}$$

By adding (19), (20), and (21) we have

$$\begin{aligned}\dot{V}_1(t)\leqslant&-\frac{\kappa}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\|\\&+\frac{\eta_2(n-1)}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\|\\\leqslant&-\big(\frac{\kappa-\eta_2(n-1)}{2}\big)\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\|\\\leqslant&-\big(\frac{\kappa-\eta_2(n-1)}{2}\big)n^2\max_{i,j\in\Psi}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|,\end{aligned} \tag{22}$$

then $\kappa>\eta_2(n-1)$ results in $\dot{V}_1(t)\leqslant 0$. On the other hand we have

$$\begin{aligned}\left\|\hat{v}_i^{\text{avg}}(t)-\frac{1}{n}\sum_{k=1}^{n}\hat{v}_i^{\text{avg}}(t)\right\|&\leqslant\frac{1}{n}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|\\&\leqslant\max_{i,j\in\Psi}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|,\end{aligned}$$

which yields

$$V_1(t)\leqslant\frac{n}{2}\big(\max_{i,j\in\Psi}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|\big)^2,$$

and the following is derived

$$\begin{aligned}&n^2\sqrt{\frac{2}{n}}\big(\frac{\kappa-\eta_2(n-1)}{2}\big)V_1^{\frac{1}{2}}(t)\\&\leqslant n^2\big(\frac{\kappa-\eta_2(n-1)}{2}\big)\max_{i,j\in\Psi}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|.\end{aligned} \tag{23}$$

By adding (23) to (22), we have

$$\begin{aligned}&\dot{V}_1(t)+n^2\sqrt{\frac{2}{n}}\big(\frac{\kappa-\eta_2(n-1)}{2}\big)V_1^{\frac{1}{2}}(t)\\&\leqslant-\big(\frac{\kappa-\eta_2(n-1)}{2}\big)n^2\max_{i,j\in\Psi}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|\\&\quad+n^2\big(\frac{\kappa-\eta_2(n-1)}{2}\big)\max_{i,j\in\Psi}\left\|\hat{v}_j^{\text{avg}}(t)-\hat{v}_i^{\text{avg}}(t)\right\|\\&\leqslant 0.\end{aligned} \tag{24}$$

Based on Finite-Time Stability Theorem [22], estimator (13) is convergent to the average of reference signals in a finite time T as

$$\lim_{t\to T}\hat{v}_i^{\text{avg}}(t)=v_i^{\text{avg}}(t),\quad i\in\Psi, \tag{25}$$

then we have

$$T\leqslant\frac{2\sqrt{\frac{1}{2}\sum_{i=1}^{n}e_i^2(0)}}{n^{\frac{3}{2}}(\kappa-\eta_2(n-1))}. \tag{26}$$

■

IV. In-domain Control

In some applications, input control is within the system's domain, known as ”in-domain control” where the system's governing PDE is involved as in (1). DAC is categorized as a tracking problem where the aim is for the agents' states to follow a time-varying function referred to as the average reference signal. To elaborate, denote $e_i(t,x)\in\Re$ as the error signal for agent i, which is defined as

$$e_i(t,x):=\theta_i(t,x)-v^{\text{avg}}(t), \tag{27}$$

and

$$\boldsymbol{e}(t,x)=[e_1(t,x),e_2(t,x),\cdots,e_n(t,x)]^T\in\Re^n, \tag{28}$$

is the collective vector of error signals. Based on the error signal definition (27), we have the following error dynamic

$$\begin{aligned}\boldsymbol{e}_t(t,x)=&A\boldsymbol{e}_{xx}(t,x)+B\boldsymbol{e}_x(t,x)+\Gamma\boldsymbol{e}(t,x)\\&+\Gamma\boldsymbol{v}^{\text{avg}}(t)-\dot{\boldsymbol{v}}^{\text{avg}}(t)+\boldsymbol{u}(t,x),\end{aligned} \tag{29a}$$
$$\boldsymbol{e}_x(t,0)=0,\boldsymbol{e}_x(t,1)=0, \tag{29b}$$
$$\boldsymbol{e}(0,x)=\boldsymbol{e}_0(x), \tag{29c}$$

where $\dot{\boldsymbol{v}}^{\text{avg}}(t)=\dot{v}^{\text{avg}}(t)\mathbf{1}_{2n}$. To achieve the DAC introduced in Definition 1, the distributed input control

$$\begin{aligned}\boldsymbol{u}(t,x)=&-LK\boldsymbol{e}(t,x)-\Gamma\hat{\boldsymbol{v}}^{\text{avg}}(t)+\dot{\hat{\boldsymbol{v}}}^{\text{avg}}(t)\\&-B\boldsymbol{e}_x(t,x),\end{aligned} \tag{30}$$

is proposed where we have the control input as

$$\begin{aligned}\boldsymbol{u}(t,x)=&-LK\boldsymbol{\theta}(t,x)-B\boldsymbol{\theta}_x(t,x)-(\Gamma-LK)\hat{\boldsymbol{v}}^{\text{avg}}(t)\\&+\dot{\hat{\boldsymbol{v}}}^{\text{avg}}(t),\end{aligned} \tag{31}$$

where $\hat{\boldsymbol{v}}^{\text{avg}}(t)$ is the estimated average of reference signals from (13) originated from the reference signal $\boldsymbol{v}^{\text{ref}}(t)$. L is the Laplacian Matrix representing the communication topology in the system. $K=\text{diag}(k_1,k_2,\cdots,k_n)$ and $k_i,\forall i\in\Psi$ are design parameters.

Theorem 2: Consider the system modeled by (1) in conjunction with the input control proposed in (31) which results in the following closed-loop system

$$\begin{aligned}\boldsymbol{\theta}_t(t,x)=&A\boldsymbol{\theta}_{xx}(t,x)+(\Gamma-LK)\boldsymbol{\theta}(t,x)\\&-(\Gamma-LK)\boldsymbol{v}^{\text{avg}}(t)+\dot{\boldsymbol{v}}^{\text{avg}}(t),\end{aligned} \tag{32a}$$
$$\boldsymbol{\theta}_x(t,0)=0,\quad\boldsymbol{\theta}_x(t,1)=0, \tag{32b}$$
$$\boldsymbol{\theta}(0,x)=\boldsymbol{\theta}_0(x). \tag{32c}$$

By designing K in a way that $\Gamma-LK<0$ then the states of system are asymptotically convergent to dynamic average reference signal in the sense of L_2 norm and we have

$$\lim_{t\to\infty}\boldsymbol{\theta}(t,x)=\boldsymbol{v}^{\text{avg}}(t). \tag{33}$$

Proof: Consider error signal definition in (27) and (28). Choose the Lyapunov candidate as

$$V_2(t)=\frac{1}{2}\int_0^1\boldsymbol{e}^T(t,x)\boldsymbol{e}(t,x)\mathrm{d}x.$$

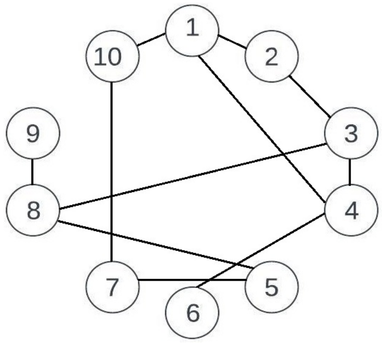

Fig. 1: Communication topology between agents

The time derivative of $V_2(t)$ is

$$
\begin{aligned}
\dot{V}_2(t) &= \int_0^1 \boldsymbol{e}^T(t,x)\boldsymbol{e}_t(t,x)\mathrm{d}x \\
&= \int_0^1 \big(\boldsymbol{e}^T(t,x)A\boldsymbol{e}_{xx}(t,x) \\
&\quad + \boldsymbol{e}^T(t,x)(\Gamma - LK)\boldsymbol{e}(t,x)\big)\mathrm{d}x \\
&= -\int_0^1 \boldsymbol{e}_x^T(t,x)A\boldsymbol{e}_x(t,x)\mathrm{d}x \\
&\quad + \int_0^1 \boldsymbol{e}^T(t,x)(\Gamma - LK)\boldsymbol{e}(t,x)\mathrm{d}x \\
&\leqslant -\lambda_{\min}(A)\|\boldsymbol{e}_x(t,x)\|^2 + \lambda_{\max}(\Gamma - LK)\|\boldsymbol{e}(t,x)\|^2 \\
&\leqslant -\min_{\forall i\in\Psi}(\alpha_i)\|\boldsymbol{e}_x(t,x)\|^2 + \lambda_{\max}(\Gamma - LK)\|\boldsymbol{e}(t,x)\|^2,
\end{aligned} \tag{34}
$$

where for functions of (t,x), $\|\cdot\|$ denotes L_2 norm, and integration by parts is used in the third row. By designing K in a way that $\Gamma - LK < 0$ then the system state converges to the dynamic average reference signal. ■

V. Simulation

Consider a group of $n = 10$ agents with the network communication topology shown in Fig. 1 as an undirected connected graph. System (1) coupled with the dynamic average consensus control in (31) is under investigation with the system parameters $A = 10I_{10}, B = I_{10}$, and $\Gamma = -I_{10}$. $K = I_{10}$ is considered in this section and two types of reference signals will be considered. Theorem 2 is hold. Note that $\theta_i(t,x)$ maps into $\mathbb{R}$. Assume that the initial conditions of all agents $\boldsymbol{\theta}_0(x) = 30\sin(2\pi x)$ in all different parts of example.

1) Sinusoidal Reference Signals: In this part, an arbitrary sinusoidal reference signal $v_i^{\text{ref}}(t) = (2i-1)\sin(0.2\pi t), \quad \forall i \in \Psi$ is considered and the average of reference signals is $v^{\text{avg}}(t) = 10\sin(0.2\pi t)$. The calculated upper bounds for this example, are $\eta_1 = 10$ and $\eta_2 = 2\pi$. Fig. 2, shows that the calculated upper bound η_1 precisely matches the signal's amplitude in the figure.

***Remark** 2:* Extension to a group of sinusoidal reference signal with different frequencies is possible, but a single frequency sinusoidal signal is considered here in order to observe that tracking is occurring with easy observation.

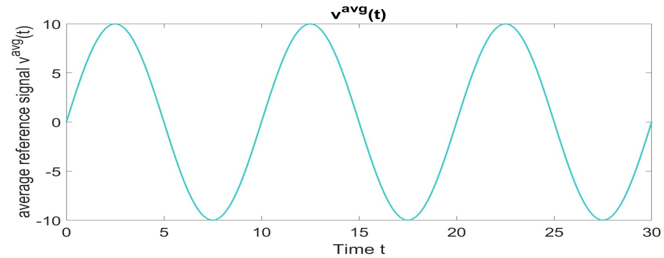

Fig. 2: The average of sinusoidal reference signals

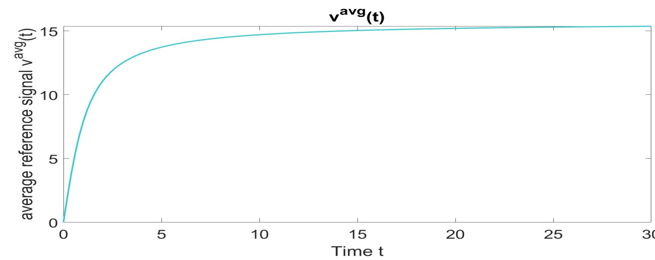

Fig. 3: The average of saturation-like reference signals

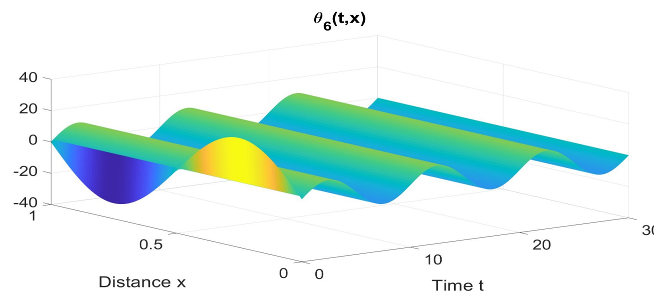

Fig. 4: $\theta_6(x,t)$ with in-domain control with sinusoidal reference signal.

2) Saturation-like Reference Signals: An arbitrary saturation-like reference signal is considered as $v_i^{\text{ref}}(t) = (2i-1)\arctan(t), \quad \forall i \in \Psi$ and the average of reference signals is $v^{\text{avg}}(t) = 10\arctan(t)$. Here, the upper bounds are $\eta_1 = 5\pi$ and $\eta_2 = 10$.

Fig. 3, visually confirms that the calculated upper bound η_1 perfectly aligns with the amplitude of the signal depicted in the figure.

Figs 4 and 5, along with each other illustrate that the state $\theta_6(x,t)$ precisely converges to the average of the sinusoidal reference signal. Initially, the sinusoidal behavior of $\boldsymbol{\theta}_0(x)$ is apparent, and after some time, the system's state effectively tracks the reference signal. Figs 6 and 7 along with each other illustrate that the state $\theta_6(x,t)$ precisely converges to the average of the saturation-like reference signal. At first sinusoidal behavior of $\boldsymbol{\theta}_0(x)$ is evident and later the system's state effectively pursue the saturation-like reference signal. Figs 5 and 7 are right side view of Figs 4 and 6 respectively. A video illustrating the behavior of states 2, 6, and 10 is available in [23] to confirm that all of 10 agents closely resemble that of agent 6. Due to space limitations, we only show simulation results of agent 6.

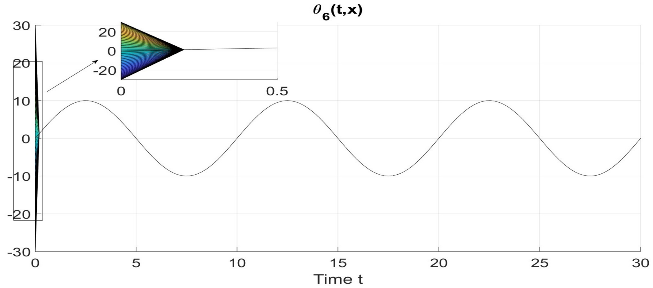

Fig. 5: Side view of $\theta_6(x,t)$ with in-domain control with sinusoidal reference signal.

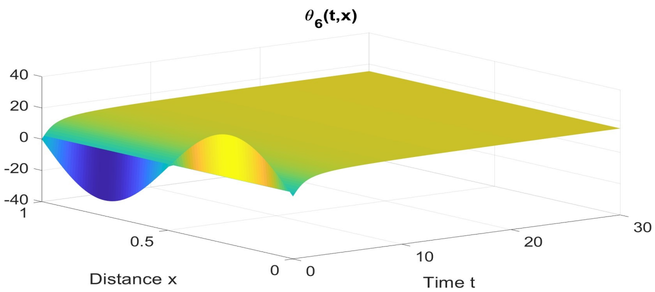

Fig. 6: $\theta_6(x,t)$ with in-domain control with saturation-like reference signal.

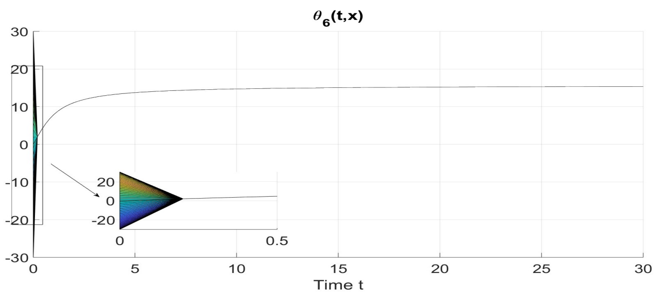

Fig. 7: Side view of $\theta_6(x,t)$ with in-domain control with saturation-like reference signal.

VI. Conclusion and Future works

This paper has explored the distributed estimator-based DAC control problem within multi-agent systems modeled by reaction-advection-diffusion PDEs. Unlike previous research, it considered generalized reference signals, extending the scope of the DAC problem. This paper introduced a distributed estimator to overcome limitations. The paper investigated in-domain control category, providing in-depth analysis, a first in the context of DAC. The stability of the closed-loop system is assessed using the Lyapunov technique. Simulation examples demonstrated the effectiveness of the proposed control designs.

Future research will focus on boundary control of PDE systems in context of DAC and addressing the DAC problem of PDE systems that feature spatially varying coefficients.

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