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On Variation Bounding System Operators

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***Abstract*— Bounding or diminishing the number of sign changes and local extrema in a signal is an intrinsic system property in, e.g., low-pass filtering or the over- and undershooting behaviour in the step-response of controlled systems. This work shows how to verify these properties for the observability/controllability operator of a linear time-invariant system under strict external positivity of a set of compound systems, which relaxes/generalizes the standard external positivity notion. In contrast to earlier work, the presented approach is significantly less dependent on a particular realization. The results are demonstrated by bounding the number of sign changes in an impulse response and, thus, the number of local extrema in the step response.**

I. Introduction

Linear time-invariant (LTI) systems

$$\begin{aligned} x(t+1) &= Ax(t) + bu(t) \\ y(t) &= cx(t), \end{aligned} \tag{1}$$

$A \in \mathbb{R}^{n\times n}$, $b, c^\mathsf{T} \in \mathbb{R}^n$, that map nonnegative inputs u to nonnegative outputs y are characterized by a *nonnegative impulse* response $g(t) := cA^{t-1}b \geq 0,\ t \geq 1$ and are referred to as *externally positive* [1]. A particular property of such systems is the monotonicity of their step response, which motivated several studies on the avoidance of over- and undershooting in closed-loop design [2], [3], [4], [5]. Indeed, the number of sign changes of g, denoted by $S^-(g)$, also known as the *variation* of g, equals the number of local extrema in the step response.

The main application addressed in this work is the extension of the positivity framework towards establishing upper bounds on the number of over- and undershoots in the step response of non-externally positive single-input-single-output (SISO) discrete-time systems of the form (1). There exist many lower bounds for this problem [6], [7], [8], but only few upper bounds [8], [9]. In our approach, we generalize the recent advances in [9], where the impulse response of (1) is expressed by the *observability operator*

$$(\mathcal{O}(A,c)x_0)(t) := cA^t x_0,\ x_0 \in \mathbb{R}^n,\ t \geq 0 \tag{2}$$

as $g(t) = (\mathcal{O}(A,c)b)(t)$. For brevity we write $\mathcal{O}(A,c)$ as $\mathcal{O}$. The idea in [9] is then to derive a computationally tractable certificate for the largest integer k such that $S^-(g) = S^-(\mathcal{O}b) \leq S^-(b)$ for all $\{b : S^-(b) \leq k\}$. An (observability) operator with this property is said to be k-*variation diminishing* (VD_k). If additionally, the first non-zero elements in b and $\mathcal{O}b$ share the same sign whenever $S^-(b) = S^-(\mathcal{O}b)$, the operator is called *order-preserving* VD_k (OVD_k). In [9], a certificate for OVD_k $\mathcal{O}$ has been derived under the assumption that also A is OVD_k.

This work generalizes the approach mentioned above in two significant aspects: (i) we are able to verify whenever $\mathcal{O}$ is *strictly* k-*variation bounding* (SVB_k), i.e., $S^+(g) = S^+(\mathcal{O}b) \leq k$ for all $\{b : S^-(b) \leq k\}$. Thus expanding the existing analysis beyond VD_k. If also applied to the *controllability operator* $\mathcal{C} = \mathcal{C}(A,b) = \mathcal{O}(A^\mathsf{T}, b^\mathsf{T})^\mathsf{T}$, this work presents a first certificate for a Hankel operator $H_g = \mathcal{O}\mathcal{C}$ to be VB_k. Therefore, our results also lead to a generalization of earlier work on VD_k systems [10]; (ii) in contrast to [9], our approach allows to verify OVD_k of $\mathcal{O}$ without requiring A to be OVD_k. This is important as finding a realization with such an A may be difficult or does not even exist.

Our main utility is [11], which in conjunction with our characterization of SVB_{k-1} matrices (see Proposition 3) allows to certify SVB_{k-1} via the strict sign-consistency of a subset of k-th order minors (see Proposition 4 and Corollary 1). A main contribution of this work is to show that these minors of $\mathcal{O}$ correspond to impulse responses of related LTI systems. Thus, checking of $\mathcal{O}$ being SVB_k turns into the verification of strict external positivity of a set of LTI systems. Numerically, this can be done efficiently by using, e.g., [2]. Our characterizations also enable further analytic investigations. Under the assumption that A is diagonalizable, we will show that SVB_{k-1} requires that the k largest poles (in modulus) have to be positive. This is the same pole constraint that had been observed for OVD_{k-1} Hankel and Toeplitz operators [10]. Surprisingly, if $k = n$, this means that there exists a simple state-space transformation $T \in \mathbb{R}^{n\times n}$ such that $\mathcal{O}(T^{-1}AT, cT)$ is OVD_{n-1}. That is, the relaxation from variation-diminishment to variation-bounding does not directly translate into fewer restrictions on (A,c). In future work, we hope to further derive characteristics for the locations of zeros.

The remainder of the paper is organized as follows. We begin with some extensive preliminaries in Section II that will enable us to present our main results in Section III. In Section IV, these result are illustrated by examples and a conclusion is drawn in Section V.

II. Preliminaries

In this section, we briefly introduce notations and concepts that are essential for our results.

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A. Notations

We write $\mathbb{Z}$ for the set of integers and $\mathbb{R}$ for the set of reals, with $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ standing for the respective subsets of nonnegative elements – the corresponding notations are also used for subsets starting from non-zero values, strict inequality as well as reversed inequality signs. The set of real sequences with indices in $\mathbb{Z}$ is denoted by $\mathbb{R}^{\mathbb{Z}}$. For matrices $X = (x_{ij}) \in \mathbb{R}^{n\times m}$, we say that X is *nonnegative*, $X \geq 0$ or $X \in \mathbb{R}^{n\times m}_{\geq 0}$ if all elements $x_{ij} \in \mathbb{R}_{\geq 0}$ – corresponding notations are used for matrices with strictly positive entries and reversed inequality signs. These notations are also used for sequences $x = (x_i) \in \mathbb{R}^{\mathbb{Z}}$. For $k, l \in \mathbb{Z}$, we write $(k : l) := \{k, k+1, \dots, l\}$, $k \leq l$. In the case $k > l$, the notation represents the empty set. If $X \in \mathbb{R}^{n\times n}$, then $\sigma(X) = \{\lambda_1(X), \dots, \lambda_n(X)\}$ denotes its *spectrum*, where the eigenvalues are ordered by descending absolute value, i.e., $\lambda_1(X)$ is the eigenvalue with the largest magnitude, counting multiplicity. If the magnitude of two eigenvalues coincides, we sub-sort them by decreasing real part. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n or if the dimensions are obvious, simply by I. A *(consecutive) j-minor* of X in $\mathbb{R}^{n\times m}$ is a minor which is constructed of j columns and j rows of X (with (consecutive indices). The submatrix with rows $I \subset (1 : n)$ and columns $J \subset (1 : m)$ is written as $X_{I,J}$. In case of subvectors, we simply write x_I. With slight abuse of notation, we also use this to denote subsets of ordered index sets $x \in \mathcal{I}_{n,r}$, where

$$\mathcal{I}_{n,r} := \{v = \{v_1, \dots, v_r\} \subset \mathbb{N} : 1 \leq v_1 < \cdots < v_r \leq n\}.$$

B. Variation diminishing maps

The *variation* of a sequence or vector u is defined as the number of sign-changes in u. We employ two versions that only differ in the treatment of zero entries.

$$S^-(u) := \sum_{i\geq 1} \mathbb{1}_{\mathbb{R}_{<0}}(\tilde{u}_i \tilde{u}_{i+1}), \quad S^-(0) := -1$$

where $\tilde{u}$ is the vector resulting from deleting all zeros in u and $\mathbb{1}_{\mathbb{A}}(x)$ is the indicator function with subset $\mathbb{A}$, i.e., $\mathbb{1}_{\mathbb{A}}(x) = 1$ if $x \in \mathbb{A}$ and zero else. Further,

$$S^+(u) := \sum_{i\geq 1} \mathbb{1}_{\mathbb{R}_{<0}}(\bar{u}_i \bar{u}_{i+1}),$$

where $\bar{u}$ is the vector resulting from replacing zeros by elements that maximize the resulting sum. Obviously, $S^-(u) \leq S^+(u)$.

Definition 1. A linear map $u \mapsto Xu$ is said to be *order-preserving k-variation diminishing* (OVD_k), $k \in \mathbb{Z}_{\geq 0}$, if for all u with $S^-(u) \leq k$ it holds that

i. $S^-(Xu) \leq S^-(u)$.
ii. The sign of the first non-zero elements in u and Xu coincide whenever $S^-(u) = S^-(Xu)$.

If the second item is dropped, then $u \mapsto Xu$ is called *k-variation diminishing* (VD_k). For brevity, we simply say X is (O)VD_k.

Definition 2. A linear map $u \mapsto Xu$ is said to be *strictly k-variation bounding* (SVB_k), $k \in \mathbb{Z}_{\geq 0}$, if for all $u \neq 0$ with $S^-(u) \leq k$ it holds that

$$S^+(Xu) \leq k.$$

C. Total Positivity Theory

Total positivity theory [12] provides algebraic conditions for the OVD_k and SVB_k property by means of compound matrices. To define these, let the i-th elements of the r-tuples in $\mathcal{I}_{n,r}$ be defined by *lexicographic ordering*. Then, the (i, j)-th entry of the *r-th multiplicative compound matrix* $X_{[r]} \in \mathbb{R}^{\binom{n}{r}\times\binom{m}{r}}$ of $X \in \mathbb{R}^{n\times m}$ is defined by $\det(X_{I,J})$, I is the i-th and J is the j-th element in $\mathcal{I}_{n,r}$ and $\mathcal{I}_{m,r}$, respectively. For example, if $X \in \mathbb{R}^{3\times 3}$, then

$$\begin{pmatrix} \det(X_{\{1,2\},\{1,2\}}) & \det(X_{\{1,2\},\{1,3\}}) & \det(X_{\{1,2\},\{2,3\}}) \\ \det(X_{\{1,3\},\{1,2\}}) & \det(X_{\{1,3\},\{1,3\}}) & \det(X_{\{1,3\},\{2,3\}}) \\ \det(X_{\{2,3\},\{1,2\}}) & \det(X_{\{2,3\},\{1,3\}}) & \det(X_{\{2,3\},\{2,3\}}) \end{pmatrix}$$

equals $X_{[2]}$. Consider OVD_0, a matrix with this property has to be nonnegative, which expressed in terms of the compound matrix reads $X = X_{[1]} \geq 0$. This equivalence can be generalized to higher orders [10, Prop. 7].

Definition 3. Let $X \in \mathbb{R}^{n\times m}$ and $k \leq \min\{m, n\}$. X is called *k-positive* if $X_{[j]} \geq 0$ for $1 \leq j \leq k$, and *strictly k-positive* if $X_{[j]} > 0$ for $1 \leq j \leq k$. In case $k = \min\{m, n\}$, X is called *(strictly) totally positive*.

Proposition 1. *Let $X \in \mathbb{R}^{n\times m}$ with $n \geq m$. Then, X is k-positive with $k \in (1 : m)$ if and only if X is OVD_{k-1}.*

Proposition 1 can be verified by the following sufficient condition [10, Proposition 8].

Proposition 2. *Let $X \in \mathbb{R}^{n\times m}$, $k \leq \min\{n, m\}$, be such that*

i. all consecutive r-minors of X are positive, $r \in (1 : k-1)$,
ii. all consecutive k-minors of X are nonnegative (positive).

Then, X is (strictly) k-positive.

Analogously, we define the notion of *(strict) k-sign consistency* to characterize SVB_k.

Definition 4. Let $X \in \mathbb{R}^{n\times m}$ and $k \leq \min\{m, n\}$. X is called *k-sign consistent* (SC_k) if $X_{[k]} \geq 0$ or $X_{[k]} \leq 0$, and *strictly k-sign consistent* (SSC_k) if $X_{[k]} > 0$ or $X_{[k]} < 0$.

A well-known characterization for SVB_m matrices $X \in \mathbb{R}^{n\times m}$ [12, Chapter 5 Theorem 1.1] is the following.

Proposition 3. *$X \in \mathbb{R}^{n\times m}$, $n > m$, is SSC_m if and only if X is SVB_{m-1}.*

Note that one is only interested in cases, where $n > m$, because any $X \in \mathbb{R}^{n\times n}$ is SVB_{n-1}. Integral to our results is the following characterization for SSC_m [11, Theorem 2.2] and its application to SSC_k case.

Proposition 4. *$X \in \mathbb{R}^{n\times m}$, $n > m$, is SSC_m if and only if the minors $\det(X_{\alpha,(1:m)})$ have the same strict sign for all $\alpha = \{(1 : m-r), (t : t+r-1)\}$ with $r \in (1 : m)$ and $m-r+1 \le t \le n-r+1$.*

For example, checking whether

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \in \mathbb{R}^{4\times 2}$$

is SSC_2, it is sufficient (and necessary) to verify that

$$\begin{aligned} r = 1 : &\quad \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \det\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \det\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \\ r = 2 : &\quad \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \det\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}. \end{aligned}$$

have the same strict sign. This characterization can also be extended to the general SSC_k case.

Corollary 1. *$X \in \mathbb{R}^{n\times m}$, $n > m$, is SSC_k, $k \le m$ if and only if the minors $\det(X_{\alpha,\beta})$ have the same strict sign for all $\alpha = \{(1 : k-r), (t : t+r-1)\}$ with $r \in (1 : k)$ and $t \in (k-r+1 : n-r+1)$ and all $\beta = \{(1 : k-\bar{r}), (\bar{t} : \bar{t}+\bar{r}-1)\}$ with $\bar{r} \in (1 : k)$ and $\bar{t} \in (k-\bar{r}+1 : m-\bar{r}+1)$.*

To illustrate Corollary 1, let us verify that

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix} \in \mathbb{R}^{5\times 3}$$

is SSC_2. Corollary 1 simply requires to apply Proposition 4 to each of the submatrices

$$\begin{aligned} \bar{r} = 1 : X_{(1:5),\{1,2\}} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}^{\mathsf{T}} \\ X_{(1:5),\{1,3\}} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}^{\mathsf{T}} \\ \bar{r} = 2 : X_{(1:5),\{2,3\}} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}^{\mathsf{T}} \\ X_{(1:5),\{3,4\}} &= \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \end{pmatrix}^{\mathsf{T}} \end{aligned}$$

where the strict sign of the $2-$minors across all submatrices has to coincide. Note that Corollary 1 turns the combinatorial complexity of verifying SSC_k via $X_{[k]}$ into a polynomial complexity.

In our derivations, the following properties of the multiplicative compound matrix will be elementary (see, e.g., [13, Section 6] and [14, Subsection 0.8.1]).

Lemma 1. *Let $X \in \mathbb{R}^{n\times p}$ and $Y \in \mathbb{R}^{p\times m}$.*

i) $(XY)_{[r]} = X_{[r]}Y_{[r]}$ (Cauchy-Binet formula).
ii) For $p = n$: $\sigma(X_{[r]}) = \{\prod_{i\in I} \lambda_i(X) : I \in \mathcal{I}_{n,r}\}$.
iii) For $p = n$ and $\det(X) \neq 0$: $(X^{-1})_{[r]} = X_{[r]}^{-1}$.

D. Variation diminishing observability operators

As discussed in [9, Lemma 3.4], the OVD$_k$ property of the observability operator (2) is equivalent to

$$\mathcal{O}^t(A, c)^{\mathsf{T}} := \begin{pmatrix} c^{\mathsf{T}} & A^{\mathsf{T}}c^{\mathsf{T}} & \dots & A^{t-1^{\mathsf{T}}}c^{\mathsf{T}} \end{pmatrix}$$

being OVD$_k$ for all $t \ge k$. For brevity we write $\mathcal{O}^t$ instead of $\mathcal{O}^t(A, c)$. The property is verified as follows [9, Theorem 3.5].

Proposition 5. *If A is k-positive and $\mathcal{O}^j{}_{[j]} \ge 0$ for $1 \le j \le k$, then $\mathcal{O}^t$ is k-positive for all $t \ge k$.*

Finally, we will abbreviate a discrete-time LTI system (1) by the triple (A, b, c) and say that it is *strictly externally positive* (1) if $g(t) = cA^{t-1}b > 0$ for all $t \ge 1$. Correspondingly, we use the term *strictly externally negative* if the inequality is reversed.

III. Main Results

The main goal of this work is to verify the general case of $\mathcal{O}$ being SVB$_{k-1}$ using SSC_k. This will enable us to numerically check this property as well as to derive necessary conditions in terms of the system poles. We will use these results to verify OVD$_k$ of $\mathcal{O}$ without requiring k-positivity of A as in Proposition 5. Note that by substitution of (A, c) with $(A^{\mathsf{T}}, b^{\mathsf{T}})$, these results can also be used to checked if the *controllability operator* of (1) is SVB$_k$.

A. The SVB$_{k-1}$ case

Analogously to the OVD$_{k-1}$ case, it is readily verified that $\mathcal{O}$ is SVB$_{k-1}$ if and only if $\mathcal{O}^t$ is SVB$_{k-1}$ for all $t \ge k-1$. By Proposition 3, it suffices to check SSC_{n-1} of $\mathcal{O}^t$ for all $t \ge n-1$, which using Proposition 4 is equivalent to the strict positivity/negativity of certain sequences of $n-1$-minors. Next, we will show that these sequences of $n-1$-minors correspond to impulse responses of related LTI system. Checking SSC_n is then equivalent to checking strict external positivity of n LTI systems.

Theorem 1. *Let (A, c) be observable. Then, $\mathcal{O}$ is SSC_n if and only if $(\tilde{A}_r, \tilde{b}_r, \tilde{c}_r)$ is strictly externally positive for all $r \in (1 : n)$ with $\tilde{A}_r = A_{[r]}$, $\tilde{c}_r = (\mathcal{O}^r)_{[r]}$ and*

$$\tilde{b}_r = \left(A^{n-r}\left(\mathcal{O}^n\right)^{-1}\begin{pmatrix} 0 \\ I_r \end{pmatrix}\right)_{[r]}.$$

A sufficient and numerically efficient certificate to verify the strict external positivity of these so-called *compound systems* can be found, e.g., in [2]. Interestingly, we need to check external positivity for the same number of compound systems as required for verifying n-positivity of Hankel/Toeplitz operators [10]. In order to proceed with the general SVB$_{k-1}$ case, we derive the following generalization of Proposition 3 for linking SSC_k to SVB$_{k-1}$.

Proposition 6. *Let $X \in \mathbb{R}^{n\times m}$ and $k < \min(m+1, n)$. Then, X is SSC_k if and only if X is SVB$_{k-1}$.*

While many related versions of Proposition 6 exist (see, e.g., [12], [15]), to the best of our knowledge, Proposition 6 has not appeared elsewhere. With the next Theorem we achieve a method of checking if $\mathcal{O}$ is SSC_k for given (A, c).

Theorem 2. *Let (A, c) be observable, $k \in (1:n)$, $r \in (1:k)$ and $\beta \in \mathcal{I}_{n,k}$. Then, for $N \in \mathbb{N}_{\geq 1}$, $t \in (1:N-k+1)$ and $\alpha = \{(1:k-r), (k-r+t:k+t-1)\}$, it holds that*

$$\det(O^N_{\alpha,\beta}) = \tilde{c}_r \tilde{A}_r^{t-1} \tilde{b}_{k,r,\beta},$$

where $\tilde{A}_r := A_{[r]}$, $\tilde{c}_r := \mathcal{O}^r{}_{[r]}$ and

$$\tilde{b}_{k,r,\beta} := \left(\left(A^{k-r} (\mathcal{O}^n)^{-1} \begin{pmatrix} 0 \\ I_{n-k+r} \end{pmatrix} \right)_{[r]} \quad 0 \right) (\mathcal{O}^n P_\beta)_{[k]}$$

with $P_\beta := I_{(1:n),\beta}$. Therefore, $\mathcal{O}$ is SSC_k if and only if $(\tilde{A}_r, \tilde{b}_{k,r,\beta}, \tilde{c}_r)$ is strictly externally positive/negative for all (r, β) with $r \in (1:k)$ and β as in Corollary 1.

As before, SVB_{k-1} of $\mathcal{O}$ can, thus, be checked by verifying external positivity/negativity of the compound systems defined in Theorem 2. Since strict external positivity requires a dominant positive pole (see, e.g., [16], [17], [18]), the following eigenvalue constraint can be shown as a consequence of item ii in Lemma 1.

Corollary 2. *Let (A, c) be such that*
i) A is diagonalizable.
ii) $\mathcal{O}$ is observable and SSC_k.
Then, the dominant eigenvalues $\lambda_1(A), \ldots, \lambda_k(A) \in \mathbb{R}_{>0}$.

Interestingly, this is the same necessary condition as in Proposition 5 as well as for the k-positivity of the Hankel and Toeplitz operators [10]. In particular, if $k = n$, there exists a $T \in \mathbb{R}^{n \times n}$ such that $T^{-1}AT = \text{diag}(\lambda_1(A), \ldots, \lambda_n(A))$ and $cT = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$, which as shown in [9] fulfills the requirements of Proposition 5.

B. The OVD_k case

Theorem 2 can also be used for checking OVD_k. The systems in Theorem 2 that represents the k-consecutive minors of $\mathcal{O}$ are the pairs $r = k$. In conjunction with Proposition 2, this provides the following corollary for k-positive $\mathcal{O}$.

Corollary 3. *Let (A, c) be observable. Using the notation of Theorem 2, it holds that if*
i. $(\tilde{A}_r, b_{r,r,\beta}, \tilde{c}_r)$ is strictly externally positive, $r \in (1:k-1)$
ii. $(\tilde{A}_r, b_{k,k,\beta}, \tilde{c}_r)$ is (strictly) externally positive,
then $\mathcal{O}$ is (strictly) k-positive.

It is important to note that unlike Proposition 5, Corollary 3 does not require A to be k-positive.

IV. Illustrative Examples

In this section, we want to illustrate our results based on two examples with third-order systems, where (i) $\mathcal{O}$ is OVD_1 but A is not 2-positive, i.e., Proposition 5 does not apply; (ii) an SVB_1 $\mathcal{O}$, which is not 2-positive, i.e., variation diminishing results cannot be used, but the new results can be used to show the variation bounding property.

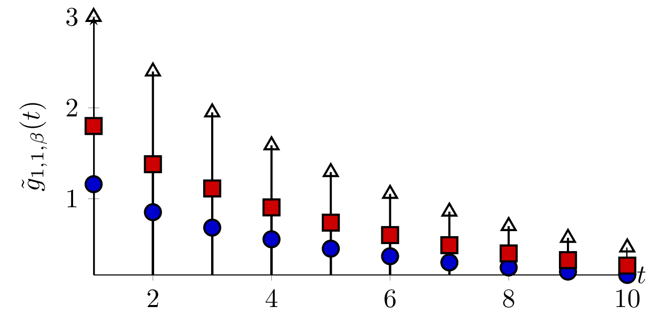

Fig. 1: Impulse responses $\tilde{g}_{1,1,\beta}(t)$ of $(\tilde{A}_1, b_{1,1,\beta}, \tilde{c}_1)$ in Theorem 2: ● $\tilde{g}_{1,1,\{1\}}(t)$, ■ $\tilde{g}_{1,1,\{2\}}(t)$ and △ $\tilde{g}_{1,1,\{3\}}(t)$ are strictly positive.

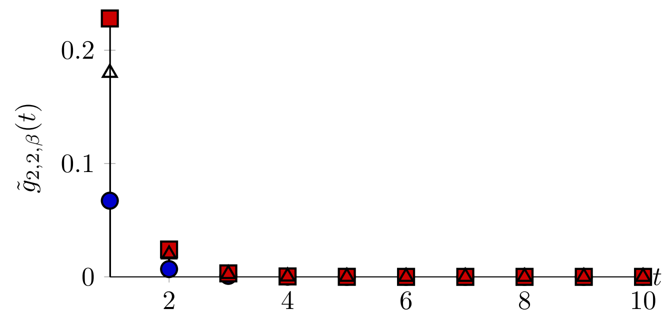

Fig. 2: Impulse responses $\tilde{g}_{2,2,\beta}(t)$ of $(\tilde{A}_2, b_{2,2,\beta}, \tilde{c}_2)$ in Theorem 2: ● $\tilde{g}_{2,2,\{1,2\}}(t)$, ■ $\tilde{g}_{2,2,\{1,3\}}(t)$ and △ $\tilde{g}_{2,2,\{2,3\}}(t)$ are strictly positive.

A. Example 1: OVD_1 $\mathcal{O}$

We begin by considering

$$A = \begin{pmatrix} -1.20 & -1.50 & -1.88 \\ 1.51 & 1.75 & 1.88 \\ -0.16 & -0.01 & 0.40 \end{pmatrix}, \quad c^\mathsf{T} = \begin{pmatrix} 1.16 \\ 1.8 \\ 3 \end{pmatrix}.$$

Since $A \not\geq 0$, A is not 2-positive and Proposition 5 is not applicable. However, by checking the impulse responses corresponding to consecutive minors of $\mathcal{O}$ (see Fig. 1 and 2), it follows from Corollary 3 that $\mathcal{O}$ is 2-positive. Thus, for all $b \in \mathbb{R}^3$ with $S^-(b) \leq 1$, it follows that $S^-(g) \leq S^+(g) \leq 1$.

B. Example 2: SVB_1 $\mathcal{O}$

Next, we consider

$$A = \begin{pmatrix} 0.7 & 0.6 & -2 \\ 0.15 & 0.15 & -0.25 \\ 0 & 0.03 & 0.1 \end{pmatrix}, \quad c^\mathsf{T} = \begin{pmatrix} 1.1 \\ 0.1 \\ -5.5 \end{pmatrix}$$

with

$$\mathcal{O}^3 = \begin{pmatrix} 1.10 & 0.10 & -5.50 \\ 0.79 & 0.51 & -2.78 \\ 0.63 & 0.46 & -1.98 \end{pmatrix}$$

Since $\mathcal{O}^3$ contains elements of mixed signs, $\mathcal{O}$ is not 0-variation diminishing. Hence, independent of the choice of b, no upper bound on the variation of the impulse response can be provided with variation diminishing arguments. Fortunately, as seen in Fig. 3 and 4, Theorem 2 implies that $\mathcal{O}$

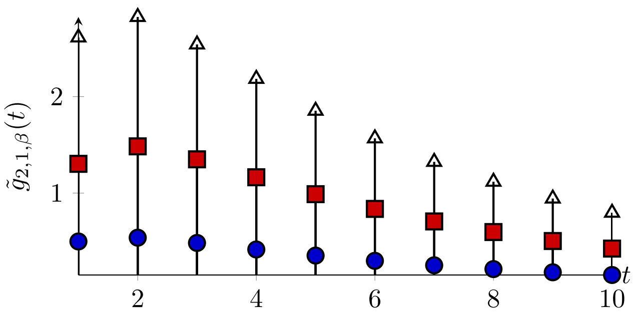

Fig. 3: Impulse responses $\tilde{g}_{2,1,\beta}(t)$ of $(\tilde{A}_1, b_{2,1,\beta}, \tilde{c}_1)$ in Theorem 2: $\tilde{g}_{2,1,\{1,2\}}(t)$, $\tilde{g}_{2,1,\{1,3\}}(t)$ and $\tilde{g}_{2,1,\{2,3\}}(t)$ are strictly positive.

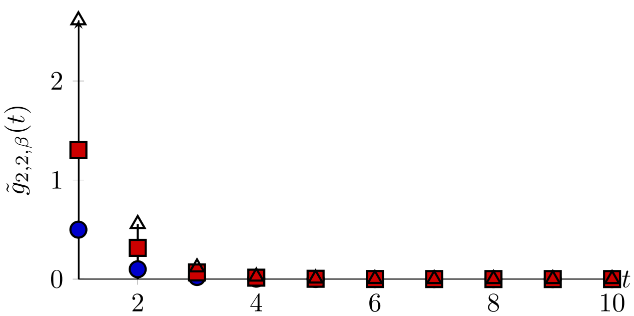

Fig. 4: Impulse responses $\tilde{g}_{2,2,\beta}(t)$ of $(\tilde{A}_2, b_{2,2,\beta}, \tilde{c}_2)$ in Theorem 2: $\tilde{g}_{2,2,\{1,2\}}(t)$, $\tilde{g}_{2,2,\{1,3\}}(t)$ and $\tilde{g}_{2,2,\{2,3\}}(t)$ are strictly positive.

is SSC_2. Then, for any $b \in \mathbb{R}^3$ with $S^-(b) \leq 1$ we have that $S^-(g) \leq S^+(g) \leq 1$.

V. Conclusion

In this work, we have derived a tractable approach to certify that the observability/controllability operator of a discrete-time LTI system is strictly k-variation bounding. The approach generalizes recent certificates for the more restrictive notion of k-variation diminishment [9]. We apply our results to the problem of upper bounding the number of sign changes in the impulse response of an LTI system. Interestingly, as a consequence of our characterization, it became evident that variation bounding and variation diminishing properties in system operators share the same pole requirements, i.e., the $k+1$ largest eigenvalues of A have to be real and positive. In particular, it turned out that strictly $n-1$-variation bounding implies order-preserving $n-1$-variation diminishment up to a simple state-space transformation.

In future work, we would like to extend our findings to the Hankel and Toeplitz operator without operator splitting and find out whether there are SSC_k observability operators that do not permit similarity transformation such that Proposition 5 can be used.

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