2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Design of Robust PD State Feedback Controllers for Descriptor Systems

Kawthar Zaidan, Israa Azzam, Dany Abou Jaoude, and Elie Shammas

***Abstract*— This paper deals with the normalization and asymptotic and exponential stabilization of linear time-invariant (LTI) and uncertain polytopic descriptor systems using proportional-derivative (PD) state feedback controllers. The synthesis problems are formulated as semidefinite programs (SDPs). The formulation allows for the optimization of the PD gains. A numerical example illustrates the proposed theory.**

I. INTRODUCTION

Descriptor systems are constrained dynamical systems described by algebraic-differential equations that admit implicit state-space representations. Descriptor systems are effective in modeling several types of systems such as robotic, mechanical, chemical, and electrical systems [1].

The literature is rich in works that study the properties of descriptor systems. For instance, [2] derives conditions for solvability, controllability, and observability of continuous-time descriptor systems, and [3], [4] deal with the design of proportional-derivative (PD) state feedback controllers for linear time-invariant (LTI) descriptor systems. Most related to our present work are the works that deal with uncertain descriptor systems and their robust stabilization using linear matrix inequality (LMI) techniques. For instance, [5], [6] derive analysis conditions for robust stability and admissibility of polytopic descriptor systems, respectively; and in [7], [8], proportional (P) state feedback controllers are designed to ensure the admissibility of the closed-loop system. [7] deals with uncertain continuous-time descriptor systems in which only the derivative matrix E is subjected to norm-bounded uncertainties and is of constant rank. [8] deals with discrete-time and continuous-time uncertain descriptor systems having polytopic uncertainties and satisfying the same rank condition on E. The synthesis technique in [8] places the closed-loop eigenvalues in a desired region. When a derivative (D) term is used in the control law, the problem of control of descriptor systems becomes that of stabilization and regularization or normalization. The reader is referred to [9] and the references therein for a list of applications motivating the use of D control. The works of [9]–[12] address the stabilization and normalization problem of descriptor systems using LMI techniques. Synthesizing D state controllers, [9] deals with the asymptotic and exponential stabilization of LTI and uncertain descriptor systems. It further explores bounding the output peak and computing optimized D gains. In [10], PD state feedback controllers are designed to stabilize LTI descriptor systems and robustly stabilize uncertain descriptor systems in which the system matrices lie in compact sets. Therein, the stabilization problem is converted into two related quadratic stabilization problems. [11] designs a PD controller that robustly stabilizes a descriptor system with norm-bounded disturbances and achieves guaranteed H_∞-performance. Finally, [12] derives a PD state feedback controller that stabilizes and normalizes uncertain descriptor systems having norm-bounded perturbations. [12] also formulates bilinear matrix inequalities (BMIs) for the synthesis of a PD output feedback controller.

The contribution of this paper is a novel robust PD synthesis technique for LTI and uncertain polytopic descriptor systems based on the Schur complement lemma and its reverse. The derived gains ensure normalization alongside asymptotic and exponential stabilization. The synthesis problems are formulated as semi-definite programs (SDPs) to be solved using LMI solvers. The proposed approach allows for gain minimization. The paper is based on the thesis [13], to which the reader is referred for more details.

The paper is structured as follows. Section II gives the notation and preliminary results. Section III formulates the problems. Section IV gives the synthesis results and Section V shows how to optimize the PD gains. An example is given in Section VI. The paper concludes with Section VII.

II. NOTATION AND PRELIMINARIES

The sets $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{S}^n$, and $\mathbb{S}^n_{++}$ denote the sets of real vectors of dimension n, real matrices of dimensions $n \times m$, symmetric matrices of dimension n, and positive definite matrices of dimension n, respectively. The strict inequality $A \succ B$ means that $A - B \in \mathbb{S}^n_{++}$. The identity matrix is denoted by I. X^T and X^{-1} denote the transpose and inverse of matrix X, respectively. $rank[X]$ denotes the rank of matrix X. $\lambda_{max}(X)$ represents the maximum eigenvalue of the symmetric matrix X. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as follows: $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$.

Consider the autonomous LTI explicit system

$$\dot{x}(t) = Ax(t), \tag{1}$$

where t denotes continuous time, x is the state, and $A \in \mathbb{R}^{n\times n}$ is the state matrix. If the system is subjected to parametric uncertainties, the uncertain system equation becomes

$$\dot{x}(t) = A(\varphi)x(t), \tag{2}$$

where φ is the uncertain parameter vector. Denote by Δ_φ the compact set of allowable values of φ.

The system in (1) is asymptotically stable if all the eigenvalues of A lie in the open left-half of the complex plane,

Department of Mechanical Engineering, Maroun Semaan Faculty of Engineering and Architecture, American University of Beirut, Lebanon.
Corresponding author: D. Abou Jaoude da107@aub.edu.lb
This work is supported by the University Research Board (URB) at the American University of Beirut (AUB).

i.e., the system trajectories $x(t)$ satisfy $\lim_{t\to\infty} \|x(t)\| = 0$ for all initial conditions $x(0) = x_0 \in \mathbb{R}^n$. The system is exponentially stable with a decay rate α if the system trajectories further satisfy $\lim_{t\to\infty} e^{\alpha t}\|x(t)\| = 0$ for all $x_0 \in \mathbb{R}^n$. Similar robust stability definitions can be made for the uncertain system (2) that must hold for all $\varphi \in \Delta_\varphi$.

This section concludes with the following lemmas.

Lemma 1 ([14]): System (1) is asymptotically stable if and only if there exists $P \in \mathbb{S}^n_{++}$ such that $A^T P + PA \prec 0$.

Lemma 2 ([14]): System (1) is exponentially stable with a decay rate $\alpha \succ 0$ if and only if there exists $P \in \mathbb{S}^n_{++}$ such that $A^T P + PA \prec -2\alpha P$.

Lemma 3 ([15]): If $G \in \mathbb{R}^{n\times n}$ satisfies $G + G^T \prec 0$, then G is nonsingular.

Lemma 4 ([14]): The uncertain system (2) is robustly asymptotically stable if there exists $P \in \mathbb{S}^n_{++}$ such that $A(\varphi)^T P + PA(\varphi) \prec 0$ for all $\varphi \in \Delta_\varphi$.

Lemma 5 ([14]): System (2) is robustly exponentially stable with a decay rate $\alpha \succ 0$ if there exists $P \in \mathbb{S}^n_{++}$ such that $A(\varphi)^T P + PA(\varphi) \prec -2\alpha P$ for all $\varphi \in \Delta_\varphi$.

III. Problem Setup

A. LTI Descriptor System

Consider the continuous-time LTI descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

where x is the state, u is the input, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $E \in \mathbb{R}^{n\times n}$ is the derivative matrix and has rank $r \le n$, and $B \in \mathbb{R}^{n\times m}$ is the input matrix and is assumed to have full column rank, i.e., $rank[B] = m \le n$. It is assumed that $rank \begin{bmatrix} E & B \end{bmatrix} = n$. This assumption is required for the existence of a matrix F_1 such that $E + BF_1$ is nonsingular [3], [4], [10]; however, may be restrictive in practice as stated in [9]. Consider the PD state feedback control law

$$u(t) = -Kx(t) - F\dot{x}(t), \tag{4}$$

where K is the proportional (P) gain and F is the derivative (D) gain. Substituting (4) in (3) gives

$$(E + BF)\dot{x}(t) = (A - BK)x(t). \tag{5}$$

Assuming that the gain F is designed such that $(E + BF)$ is invertible, equation (5) becomes

$$\dot{x}(t) = (E + BF)^{-1}(A - BK)x(t). \tag{6}$$

The first problem addressed in this paper is therefore to use Lemmas 1 and 2 to design a PD state feedback controller for system (3) such that $(E + BF)$ is nonsingular and the resulting closed-loop system (6) is asymptotically stable or exponentially stable.

B. Polytopic Uncertain Descriptor System

Consider the following uncertain descriptor system:

$$E(\rho)\dot{x}(t) = A(\delta)x(t) + B(\beta)u(t). \tag{7}$$

The derivative matrix $E(\rho)$, state matrix $A(\delta)$, and input matrix $B(\beta)$ are subjected to parametric uncertainties ρ, δ, and β, and are assumed to lie in the following convex polytopes, respectively:

$$\mathcal{E} = \left\{\sum\nolimits_{k=1}^{v_e} \rho_k E_k \;\middle|\; \rho_k \ge 0, \sum\nolimits_{k=1}^{v_e} \rho_k = 1\right\}, \tag{8}$$

$$\mathcal{A} = \left\{\sum\nolimits_{i=1}^{v_a} \delta_i A_k \;\middle|\; \delta_i \ge 0, \sum\nolimits_{i=1}^{v_a} \delta_i = 1\right\}, \tag{9}$$

$$\mathcal{B} = \left\{\sum\nolimits_{j=1}^{v_b} \beta_j B_j \;\middle|\; \beta_j \ge 0, \sum\nolimits_{j=1}^{v_b} \beta_j = 1\right\}, \tag{10}$$

where E_k, A_i, and B_j are the matrices at the k^{th}, i^{th}, and j^{th} vertices of $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$, respectively, and v_e, v_a, and v_b are the total number of vertices in each polytope.

Applying the constant-gains PD state feedback control law (4) to the polytopic descriptor system (7) and assuming that the derivative gain F is designed such that $(E(\rho) + B(\beta)F)$ is nonsingular for all parameter values, we get

$$\dot{x}(t) = (E(\rho) + B(\beta)F)^{-1}(A(\delta) - B(\beta)K)x(t). \tag{11}$$

The second problem addressed in this paper is thus to leverage Lemmas 4 and 5 to design a robust PD state feedback controller for the uncertain implicit system (7) such that $(E(\rho) + B(\beta)F)$ is nonsingular for all parameter values and the resulting explicit uncertain closed-loop system (11) is robustly asymptotically stable or exponentially stable.

IV. Control Design

A. PD State Control Design for LTI Descriptor System

Theorem 1: Consider the LTI descriptor system (3). If there exist $Q \in \mathbb{S}^n_{++}$, $X_1 \in \mathbb{S}^n$, and Y_1 and $Y_2 \in \mathbb{R}^{m\times n}$ that satisfy the following LMIs:

$$\begin{pmatrix} Q & 0 & (BY_1)^T \\ 0 & Q & (BY_2)^T \\ BY_1 & BY_2 & X_1 \end{pmatrix} \succ 0, \tag{12}$$

$$\begin{pmatrix} B(Y_1 + Y_2) + (B(Y_1 + Y_2))^T - cvx - X_1 & Q \\ Q & Q \end{pmatrix} \succ 0, \tag{13}$$

where $cvx = \Lambda + \Lambda^T$ with $\Lambda = EQA^T - EY_1^T B^T + BY_2 A^T$, then the choice of $K = Y_1 Q^{-1}$ and $F = Y_2 Q^{-1}$ in (4) renders the closed-loop system in (6) asymptotically stable.

Proof: Inequality (12) is equivalent to the following inequality by the Schur complement formula [16]:

$$X_1 \succ \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}^T \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}, \tag{14}$$

since $Q \succ 0$. Using one of the properties in [17, Section 2.3.3], inequality (13) is equivalent to

$$cvx - B(Y_1 + Y_2)Q^{-1}(B(Y_1 + Y_2))^T + X_1 \prec 0.$$

Then, using (14), it follows that

$$cvx - B(Y_1 + Y_2)Q^{-1}(B(Y_1 + Y_2))^T + \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}^T \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix} \prec 0. \tag{15}$$

Substituting $Y_1 = KQ$, $Y_2 = FQ$, and $cvx = \Lambda + \Lambda^T$ in (15) yields the following inequality:

$$(E+BF)Q(A-BK)^T + (A-BK)Q(E+BF)^T \prec 0. \tag{16}$$

By Lemma 3, $(E+BF)Q(A-BK)^T$ is invertible, and so is $(E+BF)$. By multiplying (16) by $Q^{-1}(E+BF)^{-1}$ from the left and its transpose from the right, and substituting $Q^{-1} = P \succ 0$, we see that the condition in Lemma 1 is satisfied for the closed-loop system in (6), namely,

$$(A-BK)^T(E+BF)^{-T}P+P(E+BF)^{-1}(A+BK) \prec 0,$$

where the superscript $-T$ denotes the inverse transpose. ∎

Theorem 1 designs a PD state feedback controller that normalizes and asymptotically stabilizes the descriptor system (3). Exponential stabilization is achieved by Theorem 2.

Theorem 2: Consider the LTI descriptor system (3). If there exist $Q \in \mathbb{S}^n_{++}$, X_1 and $X_2 \in \mathbb{S}^n$, and Y_1 and $Y_2 \in \mathbb{R}^{m\times n}$ that satisfy (12) and the following inequalities:

$$\begin{pmatrix} -(cvx+X_1-X_2) & (EQ+BY_2) \\ (EQ+BY_2)^T & Q/2\alpha \end{pmatrix} \succ 0, \quad (17)$$

$$\begin{pmatrix} B(Y_1+Y_2)+(B(Y_1+Y_2))^T-X_2 & Q \\ Q & Q \end{pmatrix} \succ 0, \quad (18)$$

where $cvx = \Lambda+\Lambda^T$ with $\Lambda = EQA^T - EY_1^TB^T + BY_2A^T$, then the choice of $K = Y_1Q^{-1}$ and $F = Y_2Q^{-1}$ in (4) renders the closed-loop system in (6) exponentially stable with a decay rate α.

Proof: By the Schur complement formula, inequality (17) is equivalent to

$$cvx+X_1-X_2 \prec -2\alpha(EQ+BY_2)Q^{-1}(EQ+BY_2)^T. \quad (19)$$

By one of the properties in [17, Section 2.3.3], (18) is equivalent to

$$X_2 \prec B(Y_1+Y_2)Q^{-1}(B(Y_1+Y_2))^T.$$

From the above inequality, (14), and (19), it follows that

$$cvx - B(Y_1+Y_2)Q^{-1}(B(Y_1+Y_2))^T + \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix}^T \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} (BY_1)^T \\ (BY_2)^T \end{pmatrix} \prec -2\alpha(EQ+BY_2)Q^{-1}(EQ+BY_2)^T. \quad (20)$$

Substituting $Y_1 = KQ$, $Y_2 = FQ$, $cvx = \Lambda + \Lambda^T$, and $Q^{-1} = P \succ 0$ in (20) gives

$$(E+BF)P^{-1}(A-BK)^T + (A-BK)P^{-1}(E+BF)^T \prec -2\alpha(E+BF)P^{-1}(E+BF)^T,$$

which can be rearranged into

$$(E+BF)P^{-1}((A-BK)+\alpha(E+BF))^T + ((A-BK)+\alpha(E+BF))P^{-1}(E+BF)^T \prec 0. \quad (21)$$

By Lemma 3, it follows that $(E+BF)P^{-1}((A-BK)+\alpha(E+BF))^T$ and $(E+BF)$ are nonsingular matrices. The proof is concluded by pre- and post-multiplying (21) by $P(E+BF)^{-1}$ and its transpose, respectively, to obtain

$$(A-BK)^T(E+BF)^{-T}P+P(E+BF)^{-1}(A-BK) \prec -2\alpha P.$$

∎

When using Theorem 2, the bisection method can be used to attain the maximum feasible decay rate α.

B. PD State Control Design for Polytopic Uncertain System

Expanding on the results of Theorems 1 and 2, Theorems 3 and 4 follow a parameter-independent Lyapunov function approach to derive a robust PD state feedback controller for descriptor systems with polytopic uncertainties.

Theorem 3: Consider the polytopic uncertain descriptor system (7). If there exists $Q \in \mathbb{S}^n_{++}$, $X_1 \in \mathbb{S}^n$, and Y_1 and $Y_2 \in \mathbb{R}^{m\times n}$ that satisfy the following LMIs:

$$\begin{pmatrix} Q & 0 & (B_jY_1)^T \\ 0 & Q & (B_jY_2)^T \\ B_jY_1 & B_jY_2 & X_1 \end{pmatrix} \succ 0, \quad (22)$$

$$\begin{pmatrix} B_j(Y_1+Y_2)+(B_j(Y_1+Y_2))^T - cvx_{ijk} - X_1 & Q \\ Q & Q \end{pmatrix} \succ 0, \quad (23)$$

where $cvx_{ijk} = \Lambda_{ijk} + \Lambda_{ijk}^T$ with $\Lambda_{ijk} = E_kQA_i^T - E_kY_1^TB_j^T + B_jY_2A_i^T$, for $i = 1,\dots,v_a$, $j = 1,\dots,v_b$, and $k = 1,\dots,v_e$, then the choice of $K_R = Y_1Q^{-1}$ and $F_R = Y_2Q^{-1}$ in (4) renders the closed-loop system in (11) robustly asymptotically stable for all permissible parameters ρ, δ, and β.

Proof: For each $j = 1,\dots,v_b$, multiply inequalities (22) and (23) by $\beta_j \geq 0$ such that $\sum_{j=1}^{v_b}\beta_j = 1$. Summing the resulting inequalities, we get

$$\begin{pmatrix} Q & 0 & (B(\beta)Y_1)^T \\ 0 & Q & (B(\beta)Y_2)^T \\ B(\beta)Y_1 & B(\beta)Y_2 & X_1 \end{pmatrix} \succ 0,$$

$$\begin{pmatrix} B(\beta)(Y_1+Y_2)+(B(\beta)(Y_1+Y_2))^T - cvx_{ik} - X_1 & Q \\ Q & Q \end{pmatrix} \succ 0, \quad (24)$$

where $cvx_{ik} = \Lambda_{ik} + \Lambda_{ik}^T$ with $\Lambda_{ik} = E_kQA_i^T - E_kY_1^TB(\beta)^T + B(\beta)Y_2A_i^T$ and $B(\beta) = \sum_{j=1}^{v_b}\beta_jB_j$ is in the set $\mathcal{B}$ defined in (10).
For each $i = 1,\dots,v_a$, multiply inequality (24) by $\delta_i \geq 0$ such that $\sum_{i=1}^{v_a}\delta_i = 1$ and sum the resulting inequalities to get

$$\begin{pmatrix} B(\beta)(Y_1+Y_2)+(B(\beta)(Y_1+Y_2))^T - cvx_k - X_1 & Q \\ Q & Q \end{pmatrix} \succ 0, \quad (25)$$

where $cvx_k = \Lambda_k + \Lambda_k^T$, $\Lambda_k = E_kQA(\delta)^T - E_kY_1^TB(\beta)^T + B(\beta)Y_2A(\delta)^T$, and $A(\delta) = \sum_{i=1}^{v_a}\delta_iA_i$ lies in the set $\mathcal{A}$ defined in (9). Finally, for each $k = 1,\dots,v_e$, multiply inequality (25) by $\rho_k \geq 0$ such that $\sum_{k=1}^{v_e}\rho_k = 1$, and sum the resulting inequalities to get

$$\begin{pmatrix} B(\beta)(Y_1+Y_2)+(B(\beta)(Y_1+Y_2))^T - cvx - X_1 & Q \\ Q & Q \end{pmatrix} \succ 0,$$

where $cvx = \Lambda(\rho,\delta,\beta) + \Lambda(\rho,\delta,\beta)^T$, $\Lambda(\rho,\delta,\beta) = E(\rho)QA(\delta)^T - E(\rho)Y_1^TB(\beta)^T + B(\beta)Y_2A(\delta)^T$, and $E(\rho) = \sum_{k=1}^{v_e}\rho_kE_k$ is in the set $\mathcal{E}$ defined in (8). From here on, the proof proceeds in a similar manner to the proof of Theorem 1, ultimately calling on Lemma 4. ∎

Theorem 4: Consider the polytopic uncertain descriptor system (7). If there exist $Q \in \mathbb{S}^n_{++}$, X_1 and $X_2 \in \mathbb{S}^n$, and Y_1 and $Y_2 \in \mathbb{R}^{m\times n}$ that satisfy (22) and the following inequalities:

$$\begin{pmatrix} -(cvx_{ijk}+X_1-X_2) & (E_kQ+B_jY_2) \\ (E_kQ+B_jY_2)^T & Q/2\alpha \end{pmatrix} \succ 0,$$

$$\begin{pmatrix} B_j(Y_1+Y_2)+(B_j(Y_1+Y_2))^T-X_2 & Q \\ Q & Q \end{pmatrix} \succ 0,$$

where $cvx_{ijk} = \Lambda_{ijk} + \Lambda_{ijk}^T$ with $\Lambda_{ijk} = E_kQA_i^T - E_kY_1^TB_j^T + B_jY_2A_i^T$, for $i = 1,\dots,v_a$, $j = 1,\dots,v_b$, and $k = 1,\dots,v_e$, then the choice of $K_R = Y_1Q^{-1}$ and $F_R = Y_2Q^{-1}$ in (4) renders the closed-loop system in (11) robustly exponentially stable with a decay rate α for all permissible parameters ρ, δ, and β.

Proof: The proof follows the same procedures as those of Theorems 2 and 3, ultimately calling on Lemma 5. ■

V. Gain Minimization

In this section, we show how to compute "small" control gains K and F that stabilize the closed-loop system and at the same time minimize an appropriately-chosen objective function. This discussion builds on the heuristic for gain minimization found in [9], [15].

To minimize the PD gains, we perform the following optimization when applying Theorems 1-4:

$$\begin{aligned} &\text{minimize } \mu_1\varrho_1+\mu_2\varrho_2, \\ &\text{subject to } Q\succ\xi I, \begin{pmatrix} \varrho_1 I & Y_1 \\ Y_1^T & I \end{pmatrix}\succ 0, \begin{pmatrix} \varrho_2 I & Y_2 \\ Y_2^T & I \end{pmatrix}\succ 0, \end{aligned}$$

where $\xi > 0$ and $\mu = (\mu_1,\mu_2)$ are defined by the user such that $\mu_1,\mu_2 \geq 0$ and $\mu_1+\mu_2 = 1$. By finding optimal values for ϱ_1 and ϱ_2, the PD gains are minimized since $KK^T \prec (\varrho_1/\xi^2)I$ and $FF^T \prec (\varrho_2/\xi^2)I$. To see this, use the Schur complement formula to rewrite $\begin{pmatrix} \varrho_1 I & Y_1 \\ Y_1^T & I \end{pmatrix}\succ 0$ equivalently as $Y_1Y_1^T \prec \varrho_1 I$. Also, $Q\succ \xi I$ is equivalent to $QQ \succ \xi Q$. Pre- and post-multiplying the previous inequality by K and its transpose, respectively, leads to $KQQK^T \succeq \xi KQK^T$, where $\succeq$ denotes the generalized inequality on the positive semidefinite cone. Substituting $Y_1 = KQ$ gives $KQK^T \preceq (1/\xi)Y_1Y_1^T \prec (\varrho_1/\xi)I$. Furthermore, $Q\succ\xi I$ implies that $KQK^T \succeq \xi KK^T$. Hence, $KK^T \prec (\varrho_1/\xi^2)I$. Following similar steps yields $FF^T \prec (\varrho_2/\xi^2)I$.

VI. Illustrative Example

Consider the RLC network system studied in [1] and shown in Figure 1, which consists of two capacitors, a resistor, and an inductor. The values of the capacitances are $C_{1,2} = 2F\pm20\%$, resistance $R = 2\Omega\pm30\%$, and inductance $L = 2H \pm 20\%$. Let the voltages across the capacitors be denoted by u_{c1} and u_{c2}, respectively, and the currents through the capacitors be denoted by I_1 and I_2, respectively. The control input is the source voltage, i.e., $u = u_e$. The state vector is chosen as $x = (u_{c1}, u_{c2}, I_2, I_1)$. Based on Kirchoff's second law, we establish the following uncertain implicit state-space representation of the system:

$$E(\rho)\dot{x} = A(\delta)x + Bu,$$

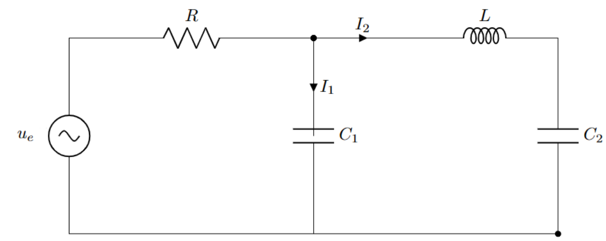


Fig. 1. RLC network circuit.

$$E=\begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -R & -R \end{pmatrix}, B=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\rho = (C_1, C_2, L)$ and $\delta = R$. The uncertain parameters lie in intervals, e.g., $C_1 \in [C_{1,min}, C_{1,max}]$, and have nominal values at the midpoint of these intervals, e.g., $\bar{C}_1 = 0.5(C_{1,min} + C_{1,max})$. By adopting the descriptor system representation, the dependence of the state-space matrices on the parameters is made linear and the resulting uncertain system is polytopic. The condition $rank\begin{bmatrix} E & B \end{bmatrix} = n = 4$ is satisfied for all parameter values.

PD state feedback controllers are designed using the methods derived in the paper. Firstly, we apply the methods of Section IV-A to the nominal LTI descriptor system obtained using the nominal values of the parameters. Accounting for the uncertainties in the parameter values, we then apply the robust control methods of Section IV-B to the formulated polytopic uncertain system. This system has a total of 2^4 vertices. The gain minimization technique of Section V is also applied to the designed controllers. In exponential stabilization and exponential stabilization with gain minimization, the bisection method is used to obtain the maximum feasible decay rate α in the range $[0.01, 100]$ (ensuring strict numerical feasibility of the constraints). The following parameters are used for gain minimization: $\xi = 0.01$ and $\mu_1 = 0.5$. Eight controllers are designed in total corresponding to combinations of the following: nominal and robust gains, asymptotic and exponential stabilization, and optimized and non-optimized gains. For each proposed controller, we compute $\lambda_{max,K} = \lambda_{max}(KK^T)$ and $\lambda_{max,F} = \lambda_{max}(FF^T)$. The results are summarized in Table I, where it can be seen that the gain minimization heuristic results in a reduction of $\lambda_{max,K}$ and $\lambda_{max,F}$. The SDPs in this paper are solved using the parser Yalmip and solver SDPT3 on an Intel Core i7-1065G7, 1.30GHz processors, and 8GB of RAM running Windows 11. Example solver and CPU times for one feasible SDP are 0.5 sec and 0.26 sec for robust asymptotic stabilization, 0.6 sec and 0.12 sec for robust asymptotic stabilization with gain minimization, 0.3 sec and 0.06 sec for robust exponential stabilization, and 0.7 sec and 0.06 sec for robust exponential stabilization with gain minimization, respectively.

To validate the derived gains, we examine the response of the state variable u_{C1} to the initial condition $x(0) = x_0 =$

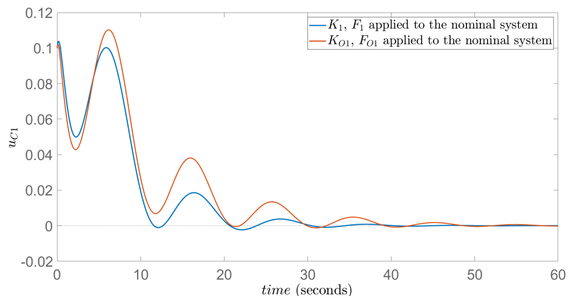


Fig. 2. Closed-loop nominal system response to x_0 using the nominal gains for asymptotic stabilization, without (blue) and with (red) gain minimization.

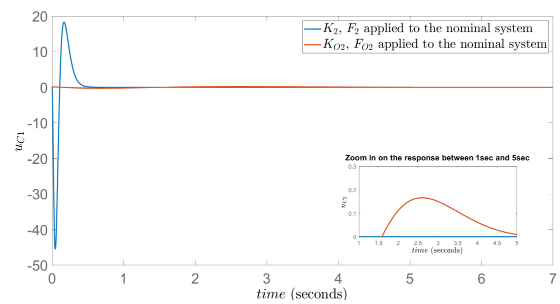


Fig. 3. Closed-loop nominal system response to x_0 using the nominal gains for exponential stabilization, without and with gain minimization.

$(0.1, 0.1, 0.1, 0.1)$ by simulating the closed-loop systems obtained by applying the computed nominal and robust PD gains to the nominal system. The results are shown in Figures 2-4. The nominal system is successfully stabilized using all controllers. Furthermore, the system exhibits the fastest response using the gains K_2 and F_2. However, this fast response, captured by the highest decay rate α, necessitates high gain values, as can be seen in Table I.

On the other hand, employing the gains designed for the nominal system on the uncertain system results in instability. To see this, the eigenvalues of the closed-loop system matrix $(E + BF)^{-1}(A - BK)$ associated with each controller are computed for vertex 1 of the polytopic system, i.e., $C_1 = C_{1,min}$, $C_2 = C_{2,min}$, $L = L_{min}$, and $R = R_{min}$. The results are presented in Table II. For this vertex, the nominal PD gains (K_1, F_1), (K_{O1}, F_{O1}), and (K_{O2}, F_{O2}) result in closed-loop stability. However, this cannot be guaranteed, as the nominal PD gains (K_2, F_2) render the closed-loop system unstable. However, as expected, the application of the robust PD gains achieves closed-loop system stabilization for all the vertices of the polytopic system.

Finally, we applied the PD synthesis methods of [10] to both the nominal and polytopic systems. These methods require the specification of some matrix T, here taken as $T = I$. The resulting gains are given in Table III.

For comparison, the nominal closed-loop system response to the initial condition x_0 is simulated using our nominal gains K_1 and F_1 and the nominal gains K_3 and F_3 of [10]. Figure 5(a) plots the system response and Figure 5(b) shows

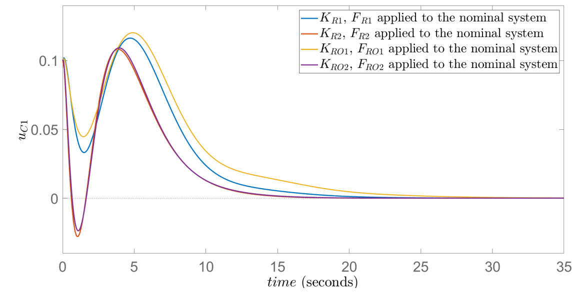


Fig. 4. Closed-loop nominal system response to x_0 using the robust gains for asymptotic stabilization (blue), exponential stabilization (red), asymptotic stabilization with gain minimization (orange), and exponential stabilization with gain minimization (violet).

TABLE I
NOMINAL AND ROBUST PD STATE CONTROL GAINS.

	PD Gains and Decay rate α	λ_{max}
Asymptotic Stabilization	**Robust PD by applying Theorem 3**	
	$K_{R1}= \begin{pmatrix} 4.11 & -2.06 & 2.81 & 0.76 \end{pmatrix}$ $F_{R1}= \begin{pmatrix} 1.1 & 0.24 & 3 & 0.79 \end{pmatrix}$	$\lambda_{max,K}$=29.68 $\lambda_{max,F}$=10.9
	Robust PD with optimized gains	
	$K_{RO1}= \begin{pmatrix} 3.02 & -1.75 & 2.09 & 0.53 \end{pmatrix}$ $F_{RO1}= \begin{pmatrix} 1.5 & 0.11 & 3.16 & 0.54 \end{pmatrix}$	$\lambda_{max,K}$=16.88 $\lambda_{max,F}$=12.53
	Nominal PD by applying Theorem 1	
	$K_1= \begin{pmatrix} 1.27 & -0.61 & 0.22 & 0.52 \end{pmatrix}$ $F_1= \begin{pmatrix} 1.23 & -0.35 & 1.22 & 0.87 \end{pmatrix}$	$\lambda_{max,K}$=2.3 $\lambda_{max,F}$=3.9
	Nominal PD with optimized gains	
	$K_{O1}= \begin{pmatrix} 0.14 & -0.22 & -0.13 & 0.05 \end{pmatrix}$ $F_{O1}= \begin{pmatrix} 0.54 & -0.22 & 0.74 & 0.36 \end{pmatrix}$	$\lambda_{max,K}$=0.088 $\lambda_{max,F}$=1.02
Exponential Stabilization	**Robust PD by applying Theorem 4**	
	$K_{R2}= \begin{pmatrix} 6.8 & -1.52 & 7.83 & 0.97 \end{pmatrix}$ $F_{R2}= \begin{pmatrix} -0.33 & 2.02 & 3.28 & 0.4 \end{pmatrix}$ $\alpha = 0.11$	$\lambda_{max,K}$=110.8 $\lambda_{max,F}$=15.1
	Robust PD with optimized gains	
	$K_{RO2}= \begin{pmatrix} 6.08 & -1.39 & 6.89 & 0.83 \end{pmatrix}$ $F_{RO2}= \begin{pmatrix} -0.15 & 1.78 & 3.13 & 0.41 \end{pmatrix}$ $\alpha = 0.11$	$\lambda_{max,K}$=86.6 $\lambda_{max,F}$=13.1
	Nominal PD by applying Theorem 2	
	$K_2= \begin{pmatrix} 9 & 83.78 & 29.85 & 1 \end{pmatrix}$ $F_2= \begin{pmatrix} -6 & -50.38 & -19.65 & 0 \end{pmatrix}$ $\alpha = 16.41$	$\lambda_{max,K}$=7992 $\lambda_{max,F}$=2960
	Nominal PD with optimized gains	
	$K_{O2}= \begin{pmatrix} 7.3 & 8.58 & 13.97 & 0.9 \end{pmatrix}$ $F_{O2}= \begin{pmatrix} -3.43 & -0.6 & -4.24 & 0.06 \end{pmatrix}$ $\alpha = 0.79$	$\lambda_{max,K}$=322.9 $\lambda_{max,F}$=30.12

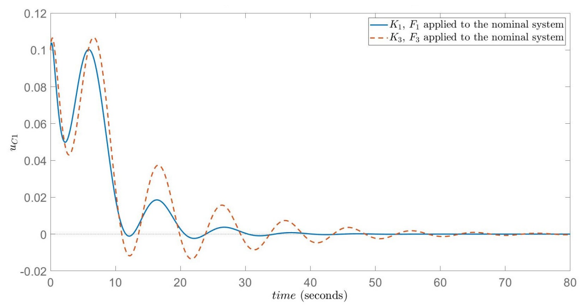
TABLE II
EIGENVALUES OF THE CLOSED-LOOP SYSTEM OBTAINED USING EACH CONTROLLER APPLIED TO VERTEX 1 OF THE POLYTOPIC SYSTEM.

Used gains	Eigenvalues
K_1, F_1	$-2.28 \quad -0.22 + 0.71i \quad 0.22 - 0.71i \quad -0.37$
K_{O1}, F_{O1}	$-4.37 \quad -0.16 - 0.76i \quad -0.16 - 0.76i \quad -0.23$
K_2, F_2	$78499 \quad -2 + 2i \quad -2 - 2i \quad 2$
K_{O2}, F_{O2}	$-0.43 + 7.21i \quad -0.4 - 7.2i \quad -0.86 + 0.68i \quad -0.86 - 0.68i$
K_{R1}, F_{R1}	$-1.39 + 1.18i \quad -1.39 - 1.18i \quad -0.42 + 0.33i \quad -0.42 - 0.33i$
K_{RO1}, F_{RO1}	$-3.58 \quad -0.63 + 0.52i \quad -0.63 - 0.52i \quad -0.42$
K_{R2}, F_{R2}	$-2.23 + 2.48i \quad -2.23 - 2.48 \quad -0.48 + 0.33i \quad -0.48 - 0.33i$
K_{RO2}, F_{RO2}	$-2.12 + 2.28i \quad -2.12 - 2.28i \quad -0.49 + 0.33i \quad -0.49 - 0.33i$

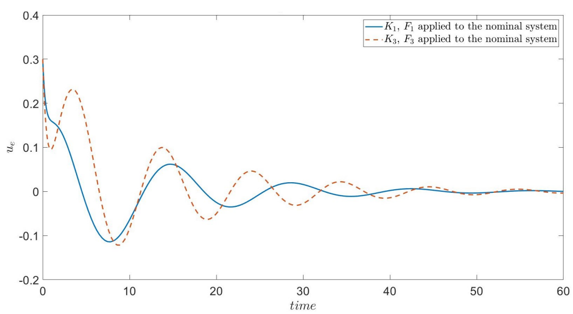
TABLE III

STABILIZING PD GAINS COMPUTED USING THE METHODS OF [10].

PD Gains	λ_{max}
Robust PD gains	
$K_{R3} = \begin{pmatrix} 8.17 & -1.66 & 9.05 & 10.69 \end{pmatrix}$	$\lambda_{max,K} = 265.74$
$F_{R3} = \begin{pmatrix} 5.51 & -1.12 & 6.11 & 7.21 \end{pmatrix}$	$\lambda_{max,F} = 120.99$
Nominal PD gains	
$K_3 = \begin{pmatrix} -6.8 & 1.52 & -7.83 & -0.97 \end{pmatrix}$	$\lambda_{max,K} = 157.26$
$F_3 = \begin{pmatrix} 3.8 & -0.4 & 4.53 & 5.96 \end{pmatrix}$	$\lambda_{max,F} = 70.6$



(a) Response to initial condition x_0



(b) Input Effort

Fig. 5. Closed-loop nominal system response to x_0 and control input effort using the nominal gains K_1 and F_1 (blue) and K_3 and F_3 (red).

the required input effort. The results are comparable, with slight superior performance when using K_1 and F_1. For example, the L_2-norm of the tracking error is 0.8624 and 1.0134 for (K_1, F_1) and (K_3, F_3), respectively. The same applies for the input effort and when simulating the polytopic system at all the vertices using the robust gains (K_{R1}, F_{R1}) and (K_{R3}, F_{R3}).

VII. CONCLUSION

This paper derives PD state feedback controllers that normalize and asymptotically and exponentially stabilize LTI descriptor systems using LMI techniques. It also computes robust PD gains for uncertain polytopic descriptor systems. The paper additionally deals with the optimization of the computed gains. Finally, the derived results are demonstrated on a numerical RLC circuit example. Future work will analyze the conservativeness of the current results and will investigate the benefit of starting the synthesis of robust gains from analysis results for uncertain explicit systems based on a parameter-dependent Lyapunov function approach, as opposed to the parameter-independent Lyapunov function approach used herein. Future work will also consider more general problem setups such as PD output feedback.

REFERENCES

[1] L. Dai, *Singular Control Systems*. Berlin, Heidelberg: Springer, 1989.

[2] E. Yip and R. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Transactions on Automatic Control*, vol. 26, no. 3, pp. 702–707, 1981.

[3] A. Bunse-Gerstner, V. Mehrmann, and N. K. Nichols, "Regularization of descriptor systems by derivative and proportional state feedback," *SIAM Journal on Matrix Analysis and Applications*, vol. 13, no. 1, pp. 46–67, 1992.

[4] R. Mukundan and W. Dayawansa, "Feedback control of singular systems—Proportional and derivative feedback of the state," *International Journal of Systems Science*, vol. 14, no. 6, pp. 615–632, 1983.

[5] R. Sakuwa and Y. Fujisaki, "Robust stability analysis of single-parameter dependent descriptor systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005, pp. 2933–2938.

[6] L. H. Yen and V. N. Phat, "Stability analysis of linear polytopic descriptor systems using a novel copositive matrix approach," *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4684–4690, 2019.

[7] C. Lin, J. L. Wang, G.-H. Yang, and J. Lam, "Robust stabilization via state feedback for descriptor systems with uncertainties in the derivative matrix," *International Journal of Control*, vol. 73, no. 5, pp. 407–415, 2000.

[8] J. Bai and H. Su, "Robust D stabilization of singular systems with polytopic uncertainties," in *Proceedings of the American Control Conference*, 2009, pp. 3706–3709.

[9] F. A. Faria, E. Assunçao, M. Teixeira, and R. Cardim, "Robust state-derivative feedback LMI-based designs for linear descriptor systems," *Mathematical Problems in Engineering*, vol. 2010, pp. 1–15, 2010.

[10] G. R. Duan, G. W. Irwin, and G. P. Liu, "Robust stabilization of descriptor linear systems via proportional-plus-derivative state feedback," in *Proceedings of the American Control Conference*, vol. 2, 1999, pp. 1304–1308.

[11] J. Ren and Q. Zhang, "Robust H_∞ control for uncertain descriptor systems by proportional–derivative state feedback," *International Journal of Control*, vol. 83, no. 1, pp. 89–96, 2010.

[12] C. Lin, Q.-G. Wang, and T. H. Lee, "Robust normalization and stabilization of uncertain descriptor systems with norm-bounded perturbations," *IEEE Transactions on Automatic Control*, vol. 50, no. 4, pp. 515–520, 2005.

[13] I. Azzam, "Semi-definite programming (SDP) approach for robust state PD control design for descriptor systems," Master's thesis, American University of Beirut, 2021.

[14] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics, 1994.

[15] E. Assunção, M. C. Teixeira, F. A. Faria, N. D. Silva, and R. Cardim, "Robust state-derivative feedback LMI-based designs for multivariable linear systems," *International Journal of Control*, vol. 80, no. 8, pp. 1260–1270, 2007.

[16] F. Zhang, *The Schur Complement and its Applications*. New York: Springer Science & Business Media, 2006, vol. 4.

[17] R. J. Caverly and J. R. Forbes, "LMI properties and applications in systems, stability, and control theory," *arXiv preprint arXiv:1903.08599*, 2019.