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Stochastic ISS of Impulsive Evolution Equations with Randomly Distributed Jump Instants

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Abstract— **This paper studies stochastic input-to-state stability of impulsive evolution equations with randomly distributed jump instants. We model these random jump instants via a Poisson process. Our approach derives the stability conditions by employing candidate Lyapunov functions parameterized by nonlinear rates. We apply our results to the cooling mechanism of a metal string modeled by a partial differential equation with jumps.**

Index Terms— **Impulsive evolution equation, stochastic stability, ISS, Poisson process.**

I. Introduction

Impulsive evolution equations offer the plausibility to dynamically model various real-world phenomena due to their combined continuous and discrete behavior, referred to as *flow* and *jump*, respectively. The flow is thereby typically described by a partial differential equation (PDE) interrupted by state jumps at certain time instants. Impulsive evolution can model quenching [1], population models [2], and neural networks [3]. An essential property of these dynamical models is their sensitivity to external perturbations. The notion of input-to-state stability (ISS), introduced in [4], is a useful tool for characterizing the system's tolerance to such perturbations. ISS was initially developed for systems described by ordinary differential equations (ODEs). ISS of ODEs has now evolved into a mature research area with many significant applications.

The efficacy of the ISS theory for ODEs and the requirement of robust stability analysis tools for PDEs motivated the study of ISS for impulsive evolution equations in [5]. However, it only covers impulsive systems with fixed jump instants, and it does not apply to systems with state jumps occurring at random time instants while, in practical scenarios, environmental factors may affect the instants of the impulses [6]. Particularly, in wireless networks, because of synchronisation routines, acknowledgement packets, waiting times, etc., transmissions instants are naturally random [7]. This motivated us to study impulsive evolution equations with random jump instants. For such systems, the difficulty arises from the random nature of jump instants. Due to the influence of random jump instants, any solution of the impulsive evolution equation with random jump instants is a stochastic process. ISS framework for impulsive evolution equations with fixed jump instants remains inconclusive for impulsive evolution equations with random jump instants as it does not support a direct extension of the notion of convergence to a probabilistic setting. Therefore, it seems inevitable to us to introduce a stochastic framework of ISS (as in [8] and [9]) and use tools from stochastic theory to conclude corresponding statements for impulsive evolution equations with random jump instants. To the best of the authors' knowledge, stochastic input-to-state stability (SISS) of impulsive evolution equations with random jump instants has not been studied before.

In this paper, we focus on characterizing SISS of impulsive evolution equations with random jump instants. We model these random jump instants via a Poisson process, i.e., the time between any two consecutive jump instants is exponentially distributed. The motivation for studying impulsive evolution equations with Poissonian jumps comes from their flexibility in modeling biological systems such as complex chemical reaction networks of a cell. In such networks, Poisson process-induced genetic toggle happen when a certain gene is expressed. This gene expression then results in the transcription of a relevant mRNA that serves as a template for producing certain proteins. The latter being the functional units of a cell, consequently, triggers different chemical pathways to become active. The jump dynamics of such complex reaction networks of a cell can be modeled using a Poisson process. Moreover, it is common to model the transmissions instants in wireless networks by a Poisson process [7]. To establish SISS, we employ candidate SISS-Lyapunov functions parameterized by nonlinear rates, which facilitates their construction. We investigate two cases: (i) the discrete dynamics is ISS, but the flow may not be ISS, and (ii) the continuous dynamics is ISS, but the jump sequence may not be ISS.

The rest of the paper unfolds as follows. In Section II, we provide the system description and the notion of SISS. In Section III, we discuss our main results on SISS of impulsive evolution equations with Poisson-distributed jump instants. In Section IV, we apply our results to the cooling mechanism of a metal spring modeled by a PDE with jumps.

II. System Description and SISS

We denote the set of natural numbers by $\mathbb{N}$, the set of non-negative integers by $\mathbb{N}_0$, the set of real numbers by

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$\mathbb{R}$, and the set of non-negative real numbers by $\mathbb{R}_0^+$. Let $(Y, \|\cdot\|)$ be a Banach space representing the state space. Let the Banach space $(U, \|\cdot\|)$ represent the input space. Let $t_0 \in \mathbb{R}$ be the initial time and $I := [t_0, \infty)$. By U_c, we denote the space of bounded functions from I to U with norm $\|u\|_\infty := \sup_{t\in I}\{\|u(t)\|\}$. By $f^-(t)$, we denote the left limit of a function f at t. We denote the space of continuous functions from X to Y by $\mathcal{C}(X, Y)$. By $\mathcal{PC}(I, Y)$, we denote the space of piecewise continuous functions from I to Y, which are right-continuous and the left limit exists for all times $t \in I$. Let $S = (t_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of random variables defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with Ω being a sample space, $\mathcal{F}$ being the σ-algebra, and P being a probability measure. The random variables $t_n : \Omega \to (t_0, \infty)$, for all $n \in \mathbb{N}$, represent the random arrival times of the impulses. Let $(N_t)_{t\geq t_0}$ be a homogeneous Poisson process with rate parameter λ, representing the counting process associated with the sequence S, whose probability distribution is given as

$$P\{N_t - N_{t_0} = n\} = P\{N_{t-t_0} = n\} = e^{-\lambda(t-t_0)}\frac{\lambda^n(t-t_0)^n}{n!}$$

for all $t > t_0$ and $n \in \mathbb{N}$. The process N_{t-t_0} and the sequence S are related as $t_n = \inf\{t > t_0 : N_{t-t_0} = n\}$ for all $n \in \mathbb{N}$.

Consider an impulsive evolution equation described by interacting continuous and discontinuous evolution maps:

$$\begin{aligned} \partial_t y(t) &= Ay(t) + f(t, y(t), u(t)), & t \notin S, \\ y(t_i) &= g_i\big(y^-(t_i), u(t_i)\big), & i \in \mathbb{N}, \end{aligned} \tag{1}$$

where $t_i \in S$, $u \in U_c$, and $y(t) \in Y$. Here, A is a closed linear operator and the infinitesimal generator of a C_0-semigroup on Y, $f : I\times Y\times U \to Y$, and $g_i : Y\times U \to Y$ for all $i \in \mathbb{N}$. The solution of equation (1) is a stochastic process $y := \phi_\omega(t; t_0, y_0, u)$, where the ω-parametrization indicates the dependency of the solution on the sequence S of the random impulse times $t_n, n \in \mathbb{N}$. Thereby, we assume that every realization $y : \Omega \to \mathcal{PC}(I, Y)$ is uniquely defined, i.e. a unique forward global mild solution exists for every $\omega \in \Omega$, initial condition $y(t_0) = y_0,\ y_0 \in Y$ and $u \in U_c$. Sufficient conditions for the existence and uniqueness of solutions of equation (1) in Banach space can be obtained by applying the technique of [10] in the pathwise sense.

Definition 1: We define the following function classes also referred to as *comparison functions*:

1) *Class* $\mathcal{P}$ is the set of all continuous functions $\gamma : [0, \infty) \to [0, \infty)$, which satisfy $\gamma(0) = 0$ and $\gamma(r) > 0$ for all $r > 0$.
2) *Class* $\mathcal{K}$ is the set of all continuous functions $\gamma : [0, \infty) \to [0, \infty)$, which are strictly increasing and $\gamma(0) = 0$. *Class* $\mathcal{K}_\infty$ is the subset of class $\mathcal{K}$ for which additionally $\gamma(s) \to \infty$ as $s \to \infty$.
3) *Class* $\mathcal{KL}$ is the set of all continuous functions $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ for which $\beta(s, r)$ is class $\mathcal{K}$ for every fixed $r \geq 0$, and for each fixed $s > 0$, the mapping $\beta(s, r)$ is strictly decreasing with respect to r and $\beta(s, r) \to 0$ as $r \to \infty$.

We next introduce the notion of SISS (as in [8]) and candidate SISS-Lyapunov function.

Definition 2: Given an impulsive time sequence S, the impulsive evolution equation (1) is *SISS*, if for any $\varepsilon \in (0, 1)$, there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for every initial values $(t_0, y_0) \in I \times Y$, every bounded input function u,

$$\begin{aligned} &P\{\|\phi_\omega(t; t_0, y_0, u)\| \leq \beta(\|y_0\|, t - t_0) + \gamma(\|u\|_\infty)\} \\ &\geq 1 - \varepsilon \end{aligned} \tag{2}$$

for all $t \in [t_0, \infty)$.

Definition 3: We call a function $V \in \mathcal{C}\big(Y, \mathbb{R}_0^+\big)$, a *candidate SISS-Lyapunov function* for system (1) if, for all $\omega \in \Omega$, it fulfills the following conditions:

1) There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that
$$\alpha_1(\|y\|) \leq V(y) \leq \alpha_2(\|y\|)\,. \tag{3}$$
2) There exist functions $\chi \in \mathcal{K}_\infty$, $\psi \in \mathcal{P}$ and $\varphi \in \mathcal{C}([0, \infty), \mathbb{R})$ with $\varphi(0) = 0$ such that for every $u \in U_c$, whenever $V(y) \geq \chi(\|u\|_\infty)$, the differential and jump inequalities
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}V(y) \leq \varphi(V(y))\,,\ t \notin S, \\ &V(g_i(y, u(t_i))) \leq \psi(V(y))\,,\ t_i \in S,\ i \in \mathbb{N} \end{aligned}$$
hold true. Here, $\frac{\mathrm{d}}{\mathrm{d}t}V(y)$ defines the pathwise Dini-derivative:
$$\frac{\mathrm{d}}{\mathrm{d}t}V(y) = \limsup_{s\searrow 0}\frac{1}{s}(V(\phi_c(s; 0, y, u)) - V(y))\,,$$
where ϕ_c is a transition map that corresponds to the continuous part of system (1), i.e., $\phi_c(t; t_0, y, u)$ is the state of system (1) at time t if the state at time $t_0 := 0$ was y and no impulses occur.
3) There exists a function $\alpha_3 \in \mathcal{K}$ such that for every $u \in U_c$ whenever $V(y) < \chi(\|u\|_\infty)$, the following jump inequality holds true:
$$V(g_i(y, u(t_i))) \leq \alpha_3(\|u(t_i)\|)\,,\ t_i \in S,\ i \in \mathbb{N}. \tag{4}$$

III. Main Results

In this section, we provide sufficient conditions for SISS of system (1).

A. Input-to-state stable discrete dynamics

We derive SISS conditions for system (1) when the discrete dynamics is ISS, but the continuous dynamics may not be ISS.

Theorem 1: Consider system (1) with a rate parameter $\lambda > 0$. Assume that there exists a candidate SISS-Lyapunov function $V \in \mathcal{C}\big(Y, \mathbb{R}_0^+\big)$ for system (1) with rates $\varphi, \psi \in \mathcal{P}$ as in Definition 3. Let there exist a constant $\delta > 0$ such that

$$\int_{\psi(a)}^{a}\frac{1}{\varphi(s)}\,\mathrm{d}s \geq \frac{1}{\lambda} + \delta \tag{5}$$

holds for all $a > 0$. Then, system (1) is SISS.

Proof: See Appendix A for the proof. ■

Note that the right hand side of inequality (5) is always greater than zero as λ and δ are positive constants. Moreover, the integrand on the left hand side of the inequality (5) is always positive by the definition of φ. Therefore, the integral in the inequality (5) can only be positive if $\psi(a) < a$ for all $a \in \mathbb{R}^+$, which implies that the discrete dynamics is ISS. The flow, however, may not be ISS.

B. Input-to-state stable continuous dynamics

We now discuss SISS of system (1) when the flow is ISS, but the jumps possibly not.

Theorem 2: Consider system (1) with a rate parameter $\lambda > 0$. Assume that there exists a candidate SISS-Lyapunov function for system (1) with rates $-\varphi, \psi \in \mathcal{P}$ as in Definition 3. Let there exist constants $\delta > 0$ such that the inequality

$$\int_{\psi(a)}^{a} \frac{1}{\varphi(s)}\,\mathrm{d}s \leq \frac{1}{\lambda} - \delta \tag{6}$$

holds for all $a > 0$. Then, system (1) is SISS.

Proof: See Appendix B for the proof. ■

The restriction that the flow must be ISS in Theorem 2 is given explicitly by our choice $-\varphi \in \mathcal{P}$. This implies that the Lyapunov value of the continuous dynamics has to be decreasing along the trajectories as long as it is outside the perturbation radius $\chi(\|u\|_\infty)$, i.e., it must be ISS. However, in this case, the jumps may not be ISS.

IV. Application: Cooling mechanism of a metal string

Consider a metal string spanned vertically with thermal diffusivity $\frac{1}{\pi^2}$ and external heat source $\frac{\tanh(|u(t,x)|)}{2y(t,x)} + \frac{1}{\pi^2}y(t,x)$. The water drops, running down the string and simultaneously cooling it, are assumed to induce jumps in temperature distribution $y(t,x)$, which are modelled using a Poisson process with parameter λ. This system can be described by an *inhomogeneous heat equation* with jumps $y : I \to \mathcal{L}^2([0,1])$,

$$\begin{aligned}\frac{\partial}{\partial t}y(t,x) &= \frac{1}{\pi^2}\Delta y(t,x) + \frac{\tanh(|u(t,x)|)}{2y(t,x)}\\ &\quad + \frac{1}{\pi^2}y(t,x), \quad t \notin S,\\ y(t_i,x) &= \begin{cases} \frac{1}{2}\frac{y^-(t_i,x)}{\sqrt{\|y^-(t_i,x)\|}}, & \|y^-(t_i,x)\| > 1,\\ \frac{1}{2}y^-(t_i,x)\cdot\|y^-(t_i,x)\|, & \text{else,}\end{cases}\end{aligned} \tag{7}$$

for $t_i \in S$, $i \in \mathbb{N}$, with boundary conditions $y(t,0) = y(t,1) = 0$, where Δ is the Laplace Operator $\Delta y(t,x) = \frac{\partial^2}{\partial x^2}y(t,x)$. Let us consider the input $u : I \to \mathcal{L}^2([0,1])$ and choose rate functions as

$$\varphi(s) = \tanh(s), \qquad \psi(s) = \begin{cases}\frac{1}{2}\sqrt{s}, & s > 1,\\ \frac{1}{2}s^2, & \text{else,}\end{cases}$$

and the candidate Lyapunov function as

$$V(y) = \|y\|^2$$

with perturbation radius $\chi(s) = s$. Note that the discrete dynamics is ISS in this example, which is evident from our choice of the function ψ.

We next show that V is indeed a candidate Lyapunov function as given in Definition 3 with according rates:

1) Condition (1) in Definition 3 can be verified trivially.

2) We choose $y \in Y$ and $u \in U_c$ such that

$$V(y) \geq \|u\|_\infty, \tag{8}$$

and calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (y(t,x))^2\,\mathrm{d}x\\ &= \int_0^1 2y(t,x)\left(\frac{1}{\pi^2}\Delta y(t,x)\right.\\ &\qquad \left.+\frac{\tanh(|u(t,x)|)}{2y(t,x)} + \frac{1}{\pi^2}y(t,x)\right)\mathrm{d}x\\ &\leq \int_0^1 \tanh(|u(t,x)|)\,\mathrm{d}x\\ &\leq \tanh\left(\int_0^1 |u(t,x)|\,\mathrm{d}x\right)\\ &\leq \tanh(\|u(t,\cdot)\|) \leq \tanh(V(y)),\end{aligned}$$

where we used

$$\begin{aligned}\int_0^1 y(t,x)\Delta y(t,x)\,\mathrm{d}x &= \int_0^1 -\nabla y(t,x)\nabla y(t,x)\,\mathrm{d}x\\ &\leq -\int_0^1 y^2(t,x)\,\mathrm{d}x\end{aligned}$$

in the third step, which follows from Green's first identity and Friedrichs's inequality. In the fourth step, we applied Jensen's inequality and in the fifth step Hölder's inequality. The rest follows from (8).
Similarly, for $\|y\| > 1$, it follows that

$$V(g_i(y,u(t_i))) = \tfrac{1}{4}\|y\| = \tfrac{1}{4}\sqrt{V(y)},$$

and for $\|y\| \leq 1$,

$$V(g_i(y,u(t_i))) = \tfrac{1}{4}\|y\|^4 = \tfrac{1}{4}V^2(y) \tag{9}$$

holds.

3) Condition (9) holds independently of u and together with $V(y) \geq \|u\|_\infty$, it can be seen that

$$V(g_i(y,u)) \leq \begin{cases}\frac{1}{4}\sqrt{V(y)} = \frac{1}{4}\sqrt{\|u\|_\infty}, & \|y\| > 1,\\ \frac{1}{4}V^2(y) = \frac{1}{4}\|u\|_\infty^2, & \text{else,}\end{cases}$$

gives us the necessary bounds.

It remains to show that (5) holds true. Indeed, we have

$$\begin{aligned}\int_{\psi(a)}^{a}\frac{1}{\varphi(s)}\,\mathrm{d}s &= \ln\left(\frac{\sinh(a)}{\sinh\left(\frac{1}{4}\sqrt{a}\right)}\right)\\ &= \ln\left(1+\sqrt{e}+e+e^{\frac{3}{2}}\right) - \tfrac{3}{4},\end{aligned}$$

for $a > 1$ and

$$\begin{aligned}\int_{\psi(a)}^{a}\frac{1}{\varphi(s)}\,\mathrm{d}s &= \ln\left(\frac{\sinh(a)}{\sinh\left(\frac{1}{4}a^2\right)}\right)\\ &= \ln\left(1+\sqrt{e}+e+e^{\frac{3}{2}}\right) - \tfrac{3}{4},\end{aligned}$$

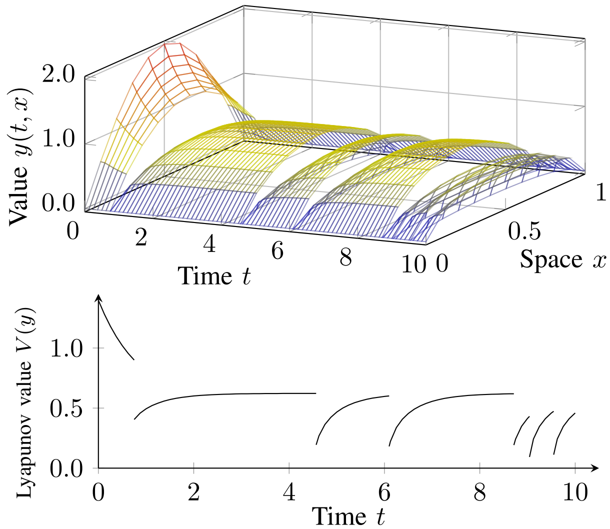

Fig. 1. Simulation of the heat equation (7) for a single realization of the Poisson-distributed jumps with rate parameter $\lambda = 0.64$. In the upper graph, a three dimensional plot of the system behavior with time is depicted. Below, the corresponding candidate Lyapunov function is illustrated.

for $a \leq 1$, respectively. Therefore, for all λ and δ which satisfy $\ln\left(1+\sqrt{e}+e+e^{\frac{3}{2}}\right) - \frac{3}{4} \geq \frac{1}{\lambda} + \delta$, i.e., $\lambda < 0.650$, system (7) is SISS.

In Fig. 1, simulation of system (7) with initial condition $y(0,x) = 2\sin(\pi x)$ and input $u(t,x) \equiv 1$ is shown for a single realization of the Poisson-distributed jumps with rate parameter $\lambda = 0.64$. In the upper graph, the system state is plotted in a three dimensional graph. In the lower graph, the development of the candidate Lyapunov function $V(y(t)) = \|y(t)\|$ with time is plotted. It can be seen that the system state varies in such a manner that it stays within the perturbation radius $\chi(\|u\|) = 1$.

V. Conclusions

We provided sufficient conditions for SISS via candidate SISS-Lyapunov functions for impulsive evolution equations with Poisson-distributed jumps. To this end, we employed Lyapunov functions with nonlinear rates φ and ψ, which facilitated their construction. We illustrated the efficacy of our result by applying it to cooling mechanism of a metal string. A possible future research direction is to consider generic Lévy process to model the random jump instants of the impulsive evolution equations as in [11].

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Appendix

A. Proof of Theorem 1

Without loss of generality, we can assume that $\delta < \frac{1}{2\lambda}\sqrt{3} - \frac{1}{2\lambda}$. The proof can be divided in two steps.

Step 1: We define the set $A_1 := \{y \in Y : V(y) < \chi(\|u\|_\infty)\}$. We first assume that $y \notin A_1$ i.e. $V(y) \geq \chi(\|u\|_\infty)$. Then, by Definition 3, the candidate Lyapunov function V satisfies the inequalities:

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(y)) \leq \varphi(V(y)),\ t \notin S, \tag{10}$$

$$V(g_i(y, u(t_i))) \leq \psi(V(y)),\ t_i \in S,\ i \in \mathbb{N}. \tag{11}$$

We define a function $F : (0,\infty) \to \mathbb{R}$ as $F(q) := \int_1^q \frac{1}{\varphi(s)}\,\mathrm{d}s$. Note that F is strictly increasing for all $q \in (0,\infty)$ because φ is positive. The image of F is an open interval of the form $(-\infty, M)$ for some constant $M \in \mathbb{R} \cup \{\infty\}$ since

$$\begin{aligned}\lim_{q\to 0}\int_1^q \frac{1}{\varphi(s)}\,\mathrm{d}s &= \int_1^{\psi(1)} \frac{1}{\varphi(s)}\,\mathrm{d}s + \int_{\psi(1)}^{\psi^2(1)} \frac{1}{\varphi(s)}\,\mathrm{d}s + \dots \\ &= \sum_{k=0}^{\infty}\int_{\psi^k(1)}^{\psi^{k+1}(1)} \frac{1}{\varphi(s)}\,\mathrm{d}s \leq -\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}+\delta\right) \to -\infty.\end{aligned}$$

Above, we used the fact that $a > \psi(a) > 0$ for all $a > 0$ as ψ is a $\mathcal{P}$-function in the first equation. Here, by ψ^k, we mean the k-times composition $\psi \circ \dots \circ \psi$, where $\psi^0(a) = a$. $\mathrm{im}(F)$ is right-open as F is a strictly increasing function. Therefore, F is invertible and $F^{-1} : (-\infty, M) \to (0,\infty)$ is also strictly increasing.

Until further notice, the analysis focuses on the evolution of V with respect to time along a single realization, i.e. with $\omega \in \Omega$ kept fixed. Thus, for the sake of brevity, we define $v(t) := V(\phi_\omega(t; t_0, y_0, u))$. Based on the arrival sequence S of the impulses, $F(v(t)) - F(v(t_0))$, for all $t \in [t_0, \infty)$, can be written as

$$F(v(t))-F(v(t_0))=\int_{v(t_0)}^{v(t)}\frac{1}{\varphi(s)}\,\mathrm{d}s$$
$$=\sum_{k=1}^{N_{t-t_0}}\left(\int\limits_{v(t_{k-1})}^{v^-(t_k)}\frac{1}{\varphi(s)}\,\mathrm{d}s+\int\limits_{v^-(t_k)}^{\psi(v^-(t_k))}\frac{1}{\varphi(s)}\,\mathrm{d}s\right)+\int\limits_{v(t_{N_{t-t_0}})}^{v(t)}\frac{1}{\varphi(s)}\,\mathrm{d}s. \tag{12}$$

The first integral appearing in the sum in the above equation can be upper bounded as follows: since (10) holds for every $t\notin S$, in particular for $t\in(t_{k-1},t_k)$, for arbitrary $k\in N$, we have

$$\int_{t_{k-1}}^{t_k}\frac{\frac{\mathrm{d}}{\mathrm{d}t}v(t)}{\varphi(v(t))}\mathrm{d}t\le\int_{t_{k-1}}^{t_k}\frac{\varphi(v(t))}{\varphi(v(t))}\mathrm{d}t=t_k-t_{k-1}. \tag{13}$$

Note that $v(t)$ can be equal to zero for some t, however, this is not an issue, since $\lim_{s\to0}\frac{\varphi(s)}{\varphi(s)}=1$ by l'Hôpital's rule. Substituting $s:=v(t)$, we get

$$\int_{v(t_{k-1})}^{v^-(t_k)}\frac{1}{\varphi(s)}\,\mathrm{d}s\le t_k-t_{k-1}. \tag{14}$$

Plugging (14) and (5) in (12), we obtain for every $t\in I$,

$$\begin{aligned}&F(v(t))-F(v(t_0))\\&\le\sum_{k=1}^{N_{t-t_0}}\left(t_k-t_{k-1}-\left(\frac{1}{\lambda}+\delta\right)\right)+(t-t_{N_{t-t_0}})\\&=t-t_0-N_{t-t_0}\left(\tfrac{1}{\lambda}+\delta\right).\end{aligned} \tag{15}$$

Multiplying both sides of the above equation by $\lambda^2\delta$ and taking the exponential, we obtain

$$e^{\lambda^2\delta F(v(t))}\le e^{\lambda^2\delta F(v(t_0))}e^{\lambda^2\delta(t-t_0)-\lambda^2\delta N_{t-t_0}\left(\frac{1}{\lambda}+\delta\right)}. \tag{16}$$

Taking expectation on both sides of (16) for all possible realizations of jump time sequence S yields

$$\mathbb{E}\Big[e^{\lambda^2\delta F(v(t))}\Big]\le e^{\lambda^2\delta F(v(t_0))}\mathbb{E}\Big[e^{\lambda^2\delta(t-t_0)-N_{t-t_0}\left(\frac{1}{\lambda}+\delta\right)}\Big],$$

where we can upper-bound the expectation on the right hand side of the above inequality as

$$\begin{aligned}&\mathbb{E}\Big[e^{\lambda^2\delta(t-t_0)-\lambda^2\delta N_{t-t_0}\left(\frac{1}{\lambda}+\delta\right)}\Big]\\&=\sum_{n=0}^{\infty}e^{\lambda^2\delta(t-t_0)-n\lambda\delta(1+\lambda\delta)}e^{-\lambda(t-t_0)}\frac{\lambda^n(t-t_0)^n}{n!}\\&=\sum_{n=0}^{\infty}e^{(\lambda^2\delta-\lambda)(t-t_0)}\left(e^{-\lambda\delta(1+\lambda\delta)}\right)^n\frac{\lambda^n(t-t_0)^n}{n!}\\&\le\sum_{n=0}^{\infty}e^{(\lambda^2\delta-\lambda)(t-t_0)}\\&\quad\times\left(1-\lambda\delta(1+\lambda\delta)+\tfrac{1}{2}\lambda^2\delta^2(1+\lambda\delta)^2\right)^n\frac{\lambda^n(t-t_0)^n}{n!}\\&=e^{(\lambda^2\delta-\lambda)(t-t_0)}e^{\left(1-\lambda\delta(1+\lambda\delta)+\frac{1}{2}\lambda^2\delta^2(1+\lambda\delta)^2\right)\lambda(t-t_0)}\\&=e^{\lambda^3\delta^2\left(-\frac{1}{2}+\lambda\delta+\lambda^2\delta^2\right)(t-t_0)}\\&=e^{-c(t-t_0)},\end{aligned}$$

where we set $c:=\lambda^3\delta^2\left(\frac{1}{2}-\lambda\delta-\lambda^2\delta^2\right)$. By the initial assumption $\delta<\frac{1}{2\lambda}\sqrt{3}-\frac{1}{2\lambda}$, the bound $c>0$ holds true. For the inequality appearing in the third step of above equation, we used that $e^{-z}\le1-z+\frac{1}{2}z^2$ for $z\ge0$. Furthermore, we employed the Taylor series of the exponential function $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ to obtain the third last equality. Now we can define the $\mathcal{KL}$-function

$$\tilde\beta(v(t_0),t-t_0):=e^{\lambda^2\delta F(v(t_0))-c(t-t_0)}$$

such that

$$\mathbb{E}\Big[e^{\lambda^2\delta F(v(t))}\Big]\le\tilde\beta(v(t_0),t-t_0)$$

is satisfied as long as $V(y(t,\cdot))\ge\chi(\|u\|_\infty)$. We define the $\mathcal{KL}$-function

$$\beta(r,s):=\alpha_1^{-1}\left(F^{-1}\left(\frac{1}{\lambda^2\delta}\ln\left(\frac{1}{\varepsilon}\cdot\tilde\beta(\alpha_2(r),s)\right)\right)\right)$$

for an arbitrarily small parameter $\varepsilon\in(0,1)$. Then, it follows that

$$\begin{aligned}&P\{\|y(t)\|\le\beta(\|y_0\|,t-t_0)\}\\&\ge P\left\{v(t)\le F^{-1}\left(\frac{1}{\lambda^2\delta}\ln\left(\frac{1}{\varepsilon}\cdot\tilde\beta(v(t_0),t-t_0)\right)\right)\right\}\\&=1-P\left\{e^{\lambda^2\delta F(v(t))}>\frac{1}{\varepsilon}\cdot\tilde\beta(v(t_0),t-t_0)\right\}\\&\ge1-\frac{\mathbb{E}\left[e^{\lambda^2\delta F(v(t))}\right]}{\frac{1}{\varepsilon}\cdot\tilde\beta\left(v(t_0),t-t_0\right)}\ge1-\varepsilon,\end{aligned} \tag{17}$$

where we used (3) to obtain the first inequality and for the bound in the second last inequality, we applied Markov's inequality (cf. [12], Chapter II, Lemma 18.1).

Step 2: We now investigate the case that $V(y)<\chi(\|u\|_\infty)$ or $y(t)\in A_1$. Therefore, we investigate the behavior when an arbitrarily chosen realization of the trajectory leaves A_1. We fix an initial condition (t_0,y_0). By construction, (17) holds for all t, which satisfy $y(t)\notin A_1$. So, we define $t^*:=\inf\{t\in[t_0,\infty]\,|\,y(t)\in A_1\}$. Then,

$$P\{\|y(t;t_0,y_0,u)\|\le\beta\left(\|y_0\|,t-t_0\right)\}\ge1-\varepsilon \tag{18}$$

for $t\in[t_0,t^*]$. In the case $t^*=\infty$, inequality (18) holds for $t\in I$. It follows $y(t)\to0$ almost surely as $t\to\infty$.

We further define

$$\begin{aligned}A_2&:=\{y\in Y\,|\,V(y)\le\gamma_1(\|u\|_\infty)\},\\A_3&:=\{y\in Y\,|\,V(y)\le\gamma_2(\|u\|_\infty)\},\end{aligned}$$

where $\gamma_1,\gamma_2\in\mathcal{K}_\infty$, and

$$\begin{aligned}\gamma_1(s)&:=\max\{\alpha_3(s),\chi(s)\},\\\gamma_2(s)&:=\max\{\alpha_3(s),\chi(s),\alpha_2\big(\beta\big(\alpha_1^{-1}(\gamma_1(s),0)\big)\big)\}.\end{aligned}$$

We show that every trajectory starting in A_1 remains in A_3 with probability $1-\varepsilon$. By definition $A_1\subset A_2\subset A_3$ holds. All trajectories that leave A_1 by a jump remain in A_2 due to (4). On the other hand, the trajectories that leave A_1 by flow, have to cross the boundary ∂A_1.

In both cases, there must be a $t'\in I$ such that $y(t',\cdot)\subset A_2\backslash A_1$. We can apply (17) with $t=t_0=t'$. As $\alpha_2^{-1}(v(t))\le$

$\|y(t)\|$ and $\|y(t')\| \leq \alpha_1^{-1}(v(t')) \leq \alpha_1^{-1}(\gamma_1(\|u\|_\infty))$ by (3), all the trajectories that leave A_1 will stay in A_3 with probability greater than $1-\varepsilon$. So for all $y(t') \in A_1$, it holds

$$P\{v(t) \leq \gamma_2(\|u\|_\infty)\} \geq 1-\varepsilon.$$

Finally, by defining $\gamma \in \mathcal{K}_\infty$, $\gamma := \alpha_1^{-1} \circ \gamma_2$, we obtain

$$P\{\|y(t;t_0,y_0,u)\| \leq \gamma(\|u\|_\infty)\} \geq 1-\varepsilon \tag{19}$$

for all $t > t^*$. Combining (18) and (19) then yields the desired result

$$P\{\|y(t)\| \leq \beta(\|y_0\|, t-t_0) + \gamma(\|u\|_\infty)\} \geq 1-\varepsilon$$

for all $t \geq t_0$ as in (2). This completes the proof.

B. Proof of Theorem 2

Analogously to the proof of Theorem 1, we partition the proof into two steps.

Step 1: Let us fix a realization y and define a function $F : (0,\infty) \to \mathbb{R}$ as $F(q) := \int_1^q -\frac{1}{\varphi(s)}\,\mathrm{d}s$, which is strictly increasing for all $q \in (0,\infty)$, because φ is strictly negative for all $s > 0$. The image of F is an open interval of the form (m, M) for some constants $m \in \{-\infty\} \cup \mathbb{R}$ and $M \in \mathbb{R} \cup \{\infty\}$. F is strictly increasing which means that $\mathrm{im}(F)$ is indeed open. Furthermore, F can be inverted and $F^{-1} : (m,M) \to (0,\infty)$ is also a strictly increasing function.

By analogous reasoning as in the proof of Theorem 1, we obtain the bound for all $t \in I$,

$$\begin{aligned}
&F(v(t)) - F(v(t_0)) = -\int_{v(t_0)}^{v(t)} \frac{1}{\varphi(s)}\,\mathrm{d}s \\
&= -\sum_{k=1}^{N_{t-t_0}} \left(\int_{v(t_{k-1})}^{v^-(t_k)} \frac{1}{\varphi(s)}\,\mathrm{d}s + \int_{v^-(t_k)}^{\psi(v^-(t_k))} \frac{1}{\varphi(s)}\,\mathrm{d}s \right) - \int_{v(t_{N_{t-t_0}})}^{v(t)} \frac{1}{\varphi(s)}\,\mathrm{d}s \\
&\leq -\sum_{k=1}^{N_{t-t_0}} \left(t_k - t_{k-1} - \left(\frac{1}{\lambda} - \delta\right)\right) - (t - t_{N_{t-t_0}}) \\
&= -(t-t_0) + N_{t-t_0}\left(\tfrac{1}{\lambda} - \delta\right).
\end{aligned} \tag{20}$$

Note that the inequality (20) loses validity when the right hand side equals to $m - F(v(t_0))$ because (13) does not hold anymore. Therefore, we set

$$\begin{aligned}
&F(v(t)) - F(v(t_0)) \\
&\leq \max\left\{-(t-t_0) + N_{t-t_0}\left(\tfrac{1}{\lambda} - \delta\right), m - F(v(t_0))\right\}.
\end{aligned}$$

Applying the transformation $e^{\lambda^2\delta F(v(t_0))} e^{\lambda^2\delta(\,\cdot\,)}$ to the above inequality yields

$$\begin{aligned}
&e^{\lambda^2\delta F(v(t))} \leq e^{\lambda^2\delta \max\left\{F(v(t_0)) - (t-t_0) + N_{t-t_0}\left(\frac{1}{\lambda}-\delta\right), m\right\}} \\
&\leq \max\left\{e^{\lambda^2\delta\left(F(v(t_0)) - (t-t_0) + N_{t-t_0}\left(\frac{1}{\lambda}-\delta\right)\right)}, e^{\lambda^2\delta m}\right\} \\
&\leq e^{\lambda^2\delta\left(F(v(t_0)) - (t-t_0) + N_{t-t_0}\left(\frac{1}{\lambda}-\delta\right)\right)} + e^{\lambda^2\delta m},
\end{aligned}$$

where we defined $e^{\lambda^2\delta m} := 0$ in case $m = -\infty$. From this, we obtain the upper-bound

$$\begin{aligned}
&e^{\lambda^2\delta F(v(t))} - e^{\lambda^2\delta m} \\
&\leq e^{\lambda^2\delta F(v(t_0))} e^{-\lambda^2\delta(t-t_0) + \lambda^2\delta N_{t-t_0}\left(\frac{1}{\lambda}-\delta\right)}.
\end{aligned} \tag{21}$$

Taking expectation on both sides of (21) for all possible realizations of jump time sequences yields

$$\begin{aligned}
&\mathbb{E}\left[e^{\lambda^2\delta F(v(t))} - e^{\lambda^2\delta m}\right] \\
&\leq e^{\lambda^2\delta F(v(t_0))} \mathbb{E}\left[e^{-\lambda^2\delta(t-t_0) + \lambda^2\delta N_{t-t_0}\left(\frac{1}{\lambda}-\delta\right)}\right],
\end{aligned}$$

where we can upper bound the expectation on the right hand side of the above inequality as

$$\begin{aligned}
&\mathbb{E}\left[e^{-\lambda^2\delta(t-t_0) + \lambda^2\delta N_{t-t_0}\left(\frac{1}{\lambda}-\delta\right)}\right] \\
&= \sum_{n=0}^{\infty} e^{-\lambda^2\delta(t-t_0) + n\lambda\delta(1-\lambda\delta)} e^{-\lambda(t-t_0)} \frac{\lambda^n (t-t_0)^n}{n!} \\
&= \sum_{n=0}^{\infty} e^{(-\lambda^2\delta - \lambda)(t-t_0)} \left(e^{\lambda\delta(1-\lambda\delta)}\right)^n \frac{\lambda^n (t-t_0)^n}{n!} \\
&\leq e^{(-\lambda^2\delta - \lambda)(t-t_0)} \\
&\times \left(\sum_{n=0}^{\infty} \left(1 + \lambda\delta(1-\lambda\delta) + \tfrac{3}{4}\lambda^2\delta^2(1-\lambda\delta)^2\right)^n \frac{\lambda^n (t-t_0)^n}{n!} \right) \\
&= e^{(-\lambda^2\delta - \lambda)(t-t_0)} e^{(\lambda + \lambda^2\delta(1-\lambda\delta) + \frac{3}{4}\lambda^3\delta^2(1-\lambda\delta)^2)(t-t_0)} \\
&= e^{\frac{1}{4}\lambda^3\delta^2(-1-6\lambda\delta+3\lambda^2\delta^2)(t-t_0)} \\
&= e^{-c(t-t_0)}.
\end{aligned}$$

Here, we defined $c := \frac{1}{4}\lambda^3\delta^2\left(1 + 6\lambda\delta - 3\lambda^2\delta^2\right)$. Without loss of generality, we assume that δ is small enough to fulfill $\delta < \frac{1}{\lambda} + \frac{2}{3\lambda}\sqrt{3}$ such that we can bound $c > 0$. For the inequality appearing in the third step of the above equation, we used $e^z \leq 1 + z + \frac{3}{4}z^2$ for $z \in [0,1]$. In the third last step, we applied Taylor series of the exponential function.

Now we have

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda^2\delta F(v(t))} - e^{\lambda^2\delta m}\right] &\leq e^{\lambda^2\delta F(v(t_0)) - c(t-t_0)} \\
&=: \tilde{\beta}(v(t_0), t-t_0)
\end{aligned}$$

for all y, which satisfy $V(y(t)) \geq \chi(\|u\|_\infty)$. Here, we defined the $\mathcal{KL}$-function $\tilde{\beta}$.

For an arbitrary but fixed parameter $\varepsilon \in (0,1)$, we define the $\mathcal{KL}$-function

$$\beta(r,s) := \alpha_1^{-1}\left(F^{-1}\left(\tfrac{1}{\lambda^2\delta}\ln\left(\tfrac{1}{\varepsilon}\tilde{\beta}(\alpha_2(r), s) + e^{\alpha m}\right)\right)\right)$$

such that

$$\begin{aligned}
&P\{\|y(t)\| \leq \beta(\|y_0\|, t-t_0)\} \\
&\geq P\left\{v(t) \leq F^{-1}\left(\tfrac{1}{\lambda^2\delta}\ln\left(\tfrac{1}{\varepsilon}\tilde{\beta}(v(t_0), t-t_0) + e^{\lambda^2\delta m}\right)\right)\right\} \\
&= 1 - P\left\{e^{\lambda^2\delta F(v(t))} - e^{\lambda^2\delta m} > \tfrac{1}{\varepsilon}\tilde{\beta}(v(t_0), t-t_0)\right\} \\
&\geq 1 - \frac{\mathbb{E}\left[e^{\lambda^2\delta F(v(t))} - e^{\lambda^2\delta m}\right]}{\frac{1}{\varepsilon}\tilde{\beta}(v(t_0), t-t_0)} \geq 1-\varepsilon
\end{aligned}$$

holds true. Here, we applied (3) in the first inequality and Markov's inequality in the second last inequality.

Step 2: The second step proceeds completely analogous to the Step 2 in the proof of Theorem 1. Therefore, we omit it here.