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Observer Design for Nonlinear Systems with Delayed Output Measurement

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***Abstract*— This paper deals with nonlinear observer design for systems with delayed nonlinear outputs. The main idea behind this work consists of using a dynamic extension technique to transform a system with delayed nonlinear outputs into a system with linear outputs and a delay-dependent integral term in the dynamic process. First, a general result for arbitrary nonlinear structures is proposed, and then further contributions are provided for a specific family of systems, namely systems in companion form for which we obtain novel high-gain observer synthesis conditions.**

***Index Terms*— Nonlinear systems; time-delay systems; observer design; delayed outputs.**

I. Introduction and Problem Formulation

A. Brief introduction

While observer design for nonlinear systems becomes more and more a popular topic due to its important and primordial role in control design schemes, diagnosis procedures, and health monitoring, the presence of delayed outputs makes it highly more interesting and useful for several modern applications. Indeed, delayed outputs are naturally encountered in remote estimation [1], cyberattacks detection [2], and multiagent systems in general [3]. On the other hand, from a mathematical standpoint, the problem of observer design with delayed outputs is highly more challenging than systems without delay. Without unnecessarily expanding this brief introduction, to avoid repetition, the following section I-B formulates the problem clearly and provides the positioning of this paper in relation to the state of the art in existing literature.

B. Problem Formulation

The motivation of the work consists of developing a simple method to deal with nonlinear systems with delayed nonlinear output measurements. The aim is to establish novel design conditions allowing high values of the maximum allowable delay in the output measurement while ensuring the exponential convergence of the observer. We stand out from the literature by proposing a simple but useful method.

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The class of systems we consider in this paper is described by the following equations:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h\big(x(t-\tau(t))\big) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ represents the output measurements vector. The delay $\tau(t) \geq 0$ is assumed to be known and bounded, i.e.: there exists a positive constant $\tau^\star$ such that $\tau(t) \leq \tau^\star, \forall t \geq 0$.

The main objective consists of estimating the system state, $x(t)$, in real-time from the delayed measurements $y(t)$. While the problem in the case of delay-free outputs, is relatively easy to handle, however, the presence of the delay makes the problem challenging. To cope with this issue, several methods have been developed in the literature [4]–[11]. To deal with arbitrarily long delay in the measurement, some methods are based on the use of a chain of observers [4], [7] while other methods exploit predictors-based observers [8]. For some families of systems, namely feed-forward systems, the problem may be solved by using the time-scaling technique as in [10]. On the other hand, different methods based on the high-gain observer methodology have been proposed for systems in triangular form with some recent improvements. However, these methods are valid for only systems with small values of the upper bound of the delay, $\tau^\star$. The aim of this paper is to overcome this limitation and propose a novel approach that will be both simple and enhance the maximum allowable value of $\tau^\star$.

C. Preliminary tools

Before stating the main results of the paper, we introduce the following simple and well-known mathematical tools.

Lemma 1 ([12], [13]): Consider a continuous, piecewise $\mathcal{C}^1$, and non-negative function ϑ defined in the interval $[-\tau^\star, +\infty)$ such that

$$\dot{\vartheta}(t) \leq -c_1\vartheta(t) + c_2 \sup_{s\in[t-\tau^\star,t]} \vartheta(s). \tag{2}$$

Assume that $c_1 > c_2 > 0$. Then, there exist two scalars $\alpha > 0$ and $\beta > 0$ such that

$$\vartheta(t) \leq \alpha \mathrm{e}^{-\beta t} \sup_{s\in[-\tau^\star,0]} \vartheta(s), \forall t \geq 0. \tag{3}$$

Lemma 2 (The Differential Mean Value Theorem [14]): Let $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^q$ be a differentiable function and let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be two vectors. Then, there exists

$$\boldsymbol{z} \triangleq \begin{bmatrix} \boldsymbol{z}^1 & \boldsymbol{z}^2 & \dots & \boldsymbol{z}^q \end{bmatrix}^\top \in \mathbb{R}^{nq},\ \boldsymbol{z}^i \in \mathbf{Co}(x, y) \tag{4}$$

where $\mathbf{Co}(x, y)$ stands for the convex hull of convex combinations of x and y, such that

$$\Psi(x) - \Psi(y) = \nabla_x^{\Psi}(\boldsymbol{z})(x - y) \tag{5}$$

where

$$\nabla_x^{\Psi}(\boldsymbol{z}) \triangleq \begin{bmatrix} \frac{\partial \Psi_1(\boldsymbol{z}^1)}{\partial x} & \frac{\partial \Psi_2(\boldsymbol{z}^2)}{\partial x} & \dots & \frac{\partial \Psi_q(\boldsymbol{z}^q)}{\partial x} \end{bmatrix}^{\top}. \tag{6}$$

Proof: The proof is omitted. ■

Finally, we need the following Lemma 3.

Lemma 3: Let $\phi : \mathbb{I} \to \mathbb{R}$ be a non-negative function, where $\mathbb{I}$ is an interval of $\mathbb{R}$. Then the following identity holds:

$$\left[\sup_{s \in \mathbb{I}} \left(\phi(s)\right)\right]^2 = \sup_{s \in \mathbb{I}} \left(\phi^2(s)\right). \tag{7}$$

II. Main Results

This section is devoted to the main contributions of this paper. We, first, present a preliminary result on which the main contributions are based. It will be used straightforwardly as a tool to conclude the main results.

A. Preliminary result

Consider the class of systems described by the following equations:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t), u(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g(\zeta(s), u(s))\mathrm{d}s \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{8}$$

where $\zeta(t) \in \mathbb{R}^{n_\zeta}$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, and $y_\zeta(t) \in \mathbb{R}^{p_\zeta}$ represents the output measurements vector. The delay $\tau(t) \geq 0$ is assumed to be known and bounded, i.e.: there exists a positive constant $\tau^\star$ such that $\tau(t) \leq \tau^\star, \forall t \geq 0$. Without loss of generality, we assume that the functions f_ζ and g are $\gamma_{f_\zeta}-$Lipschitz and γ_g-Lipschitz, respectively, with respect to ζ uniformly on $u(t)$. Assume also that $\zeta(t) = \zeta_0, \forall t \in [-\tau^\star, 0]$.

As a preliminary result, we develop a simple state observer design method for the system (8). To this end, we consider the following state observer :

$$\begin{aligned} \dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), u(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} g(\hat{\zeta}(s), u(s))\mathrm{d}s \\ + L\left(y_\zeta(t) - C\hat{\zeta}(t)\right), \end{aligned} \tag{9}$$

where $L \in \mathbb{R}^{n_\zeta \times p_\zeta}$ is the observer gain matrix to be determined such that the estimation error $\epsilon(t) \triangleq \zeta(t) - \hat{\zeta}(t)$ converges exponentially towards zero. Then, the estimation error dynamics is given as:

$$\begin{aligned} \dot{\epsilon}(t) = \Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) - LC\epsilon(t) \\ + B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s. \end{aligned} \tag{10}$$

For the sake of obtaining convenient checkable stability conditions, we have to transform the nonlinear term $\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right)$ by using Lemma 2. Then, there exists $\boldsymbol{z}_t \in \mathbf{Co}\left(\zeta(t), \hat{\zeta}(t)\right)$ as in (4) such that

$$\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) = \nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)\epsilon(t)$$

where $\nabla_\zeta^{f_\zeta}$ is defined as in (6). Notice that here $\boldsymbol{z}_t$ depends on $u(t)$ but for the sake of brevity, we use $\boldsymbol{z}_t$ instead of $\boldsymbol{z}_t(u(t))$. It follows that the error system (10) is under the form:

$$\begin{aligned} \dot{\epsilon}(t) = \left[\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right]\epsilon(t) \\ + B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s. \end{aligned} \tag{11}$$

Since f_ζ is $\gamma_{f_\zeta}-$Lipschitz, then there exist constant matrices $\mathcal{A}_j^\zeta \in \mathbb{R}^{n_\zeta \times n_\zeta}$ and functions $\lambda_j(\boldsymbol{z}_t)$, $j = 1, \dots \bar{n}_\zeta$ such that the generalized Jacobian $\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)$ belongs to the convex polytopic set defined as:

$$\mathcal{H}_{f_\zeta} \triangleq \left\{ \sum_{j=1}^{\bar{n}_\zeta} \lambda_j(\boldsymbol{z}_t)\mathcal{A}_j^\zeta, \sum_{j=1}^{\bar{n}_\zeta} \lambda_j(\boldsymbol{z}_t) = 1, \lambda_j(\boldsymbol{z}_t) \geq 0 \right\} \tag{12}$$

Notice that the matrices $\mathcal{A}_j^\zeta \in \mathbb{R}^{n_\zeta \times n_\zeta}$, represent the vertices of the polytope $\mathcal{H}_{f_\zeta}$. Also, the jacobian $\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)$ is affine on the variables $\lambda_j(\boldsymbol{z}_t)$, $j = 1, \dots \bar{n}_\zeta$.

Before stating the preliminary proposition, notice that since the function g is γ_g-Lipschitz with respect to ζ, then we have

$$\left\|\Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\right\| \leq \gamma_g \|\epsilon(s)\|. \tag{13}$$

Now we are ready to state the main theorem based on the use of the standard quadratic Lyapunov function, i.e.: $\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$, where $\mathcal{P} = \mathcal{P}^\top > 0$. We can use a more general Lyapunov function with a matrix $\mathcal{P}(\epsilon(t))$ depending on $\epsilon(t)$, however, we obtain non-constructive conditions difficult to deal with using numerical software algorithms. The second objective of using the standard quadratic Lyapunov function is to compare with available methods in the literature based on the same Lyapunov function.

Theorem 1: Assume that there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the following conditions hold:

$$\left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathcal{P} \leq 0, \quad j = 1, \dots, \bar{n}_\zeta \tag{14a}$$

$$\tau^\star < \frac{\mu\lambda_{\min}(\mathcal{P})}{2\gamma_g\|\mathcal{P}B_\zeta\|} \tag{14b}$$

Then the observer (9), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially.

Proof: By computing the derivative of the Lyapunov function $\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$ along the trajectories of (11), we obtain

$$\dot{\vartheta}(\epsilon(t)) = \epsilon^\top(t)\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t)$$
$$+ 2\epsilon^\top(t)\mathcal{P}B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s$$
$$\leq \epsilon^\top(t)\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t)$$
$$+ 2\gamma_g\|\mathcal{P}B_\zeta\|\|\epsilon(t)\|\int_{t-\tau^\star}^{t}\|\epsilon(s)\|\mathrm{d}s \quad (15)$$

Conditions (14a) and the convexity principle lead to

$$\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right) \leq -\mu\mathcal{P}. \quad (16)$$

On the other hand, from Lemma 3, we get

$$\begin{aligned}\|\epsilon(t)\|\int_{t-\tau^\star}^{t}\|\epsilon(s)\|\mathrm{d}s &\leq \tau^\star\|\epsilon(t)\|\sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\| \\ &\leq \tau^\star\left(\sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\|\right)^2 \\ &= \tau^\star\sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\|^2 \\ &\leq \frac{\tau^\star}{\lambda_{\min}(\mathcal{P})}\sup_{s\in[t-\tau^\star,t]}\vartheta(\epsilon(s)).\end{aligned} \quad (17)$$

Hence, from (16) and (17), we deduce that

$$\dot{\vartheta}(\epsilon(t)) \leq -\mu\vartheta(\epsilon(t)) + \frac{2\gamma_g\|\mathcal{P}B_\zeta\|}{\lambda_{\min}(\mathcal{P})}\tau^\star\sup_{s\in[t-\tau^\star,t]}\vartheta(\epsilon(s)). \quad (18)$$

Consequently, from (14b) and Lemma 1, there exist two positive scalars α abd β such that

$$\vartheta(t) \leq \alpha\mathrm{e}^{-\beta t}\sup_{s\in[-\tau^\star,0]}\vartheta(s), \forall t \geq 0, \quad (19)$$

which means that the estimation error $\epsilon(t)$ is exponentially stable. This completes the proof. ■

B. Main result

In this section, we propose a simple observer design method for the class of systems (1) with nonlinear delayed-output measurement, which is the main motivation of this paper. As stated in Section I-B, to handle the delay in the output measurements, several techniques have been proposed in the literature. In this paper, we propose a novel and different observer design technique. To this end, we, first, introduce the new state variable, $z(t) \in \mathbb{R}^{n_z \times n_z}$, defined by:

$$\begin{cases} \dot{z}(t) = f_z(z(t), u(t)) + Y_z y(t) \\ z(0) = z_0, \end{cases} \quad (20)$$

where f_z is a known globally Lipschitz function, the matrix $Y_z \in \mathbb{R}^{n_z \times p}$ is known and constant, and $z_0 \in \mathbb{R}^{n_z}$ is a known constant vector. The idea consists in using a state augmentation approach to get a new system for which the created $z(t)$ is the output measurement. Indeed, since $f_z, u(t), Y_z$, and z_0 are all known, then the state $z(t)$ is known in real-time from the measured output $y(t)$ of the original system (1). Before introducing the main transformation, notice that from the *Newton-Leibniz* formula, $y(t)$ can be written under the form:

$$y(t) = h(x(t)) - \int_{t-\tau(t)}^{t}\frac{\partial h}{\partial x}\big(x(s)\big)f\big(x(s), u(s)\big)\mathrm{d}s. \quad (21)$$

By exploiting (20) and (21), the system (1) can be transformed into the form (8) with

$$\zeta(t) \triangleq \begin{bmatrix} z(t) \\ x(t) \end{bmatrix}, y_\zeta \triangleq z(t),\ C \triangleq \begin{bmatrix}\mathbb{I}_{n_z} & 0\end{bmatrix}, \quad (22)$$

$$f_\zeta\big(\zeta(t), u(t)\big) \triangleq \begin{bmatrix} f_z(z(t), u(t)) + Y_z h(x(t)) \\ f(x(t), u(t)) \end{bmatrix}, \quad (23)$$

$$g\big(\zeta(t), u(t)\big) \triangleq \frac{\partial h}{\partial x}\big(x(t)\big)f\big(x(t), u(t)\big),\ B_\zeta \triangleq \begin{bmatrix} -Y_z \\ 0 \end{bmatrix} \quad (24)$$

Then, we propose the following generalized state observer:

$$\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), u(t)\right) + B_\zeta\int_{t-\tau(t)}^{t} g(\hat{\zeta}(s), u(s))\mathrm{d}s + L\left(y_\zeta(t) - C\hat{\zeta}(t)\right) \quad (25a)$$

$$\hat{x}(t) = \begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix}\hat{\zeta}(t). \quad (25b)$$

Before summarizing the result in a corollary, we need the following assumption.

Assumption 1: The function g defined in (24) is γ_g-Lipschitz with respect to $x(t)$, uniformly on $u(t)$.

Now all the conditions to apply Theorem 1 are satisfied, we can summarize the result.

Corollary 1: Consider the system (8) with the parameters and functions given in (22)–(24). Assume that Assumption 1 is satisfied and there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the conditions (14a)–(14b) hold. Let $L = \mathcal{P}^{-1}\mathcal{R}^\top$ be the gain matrix of (25a). Then, the estimated state $\hat{x}(t)$ given by (40b) converges exponentially to the state $x(t)$ of the original system (1).

Remark 1: Without loss of generality, we consider in (20) a linear function f_z depending on $z(t)$ only, i.e: $f_z(z(t), u(t)) = A_z z(t)$. Even, for simplification, we can take $f_z(z(t), u(t)) \equiv 0$. In addition, these considerations allow reducing the dimension of the corresponding polytopic set $\mathcal{H}_{f_\zeta}$, which reduces then the number of LMIs (14a) to solve.

III. Further Results: A Particular Family of Systems

This section considers the high-gain observer and its robustness with respect to the delay in the output measurement. Although several techniques have been proposed in the literature for this class of systems, we show that our method is simple and applies straightforwardly to this class of systems under the companion form.

A. System description and assumptions

Without loss of generality, we consider the family of systems described by (1) with

$$\begin{aligned} f(x(t),u(t)) &= Ax(t) + Bf_x\big(x(t)\big) \\ h\big(x(t-\tau(t))\big) &= h_x\big(C_x x(t-\tau(t))\big) \\ (A)_{ij} &= \begin{cases} 1 & \text{if} \quad j = i+1 \\ 0 & \text{if} \quad j \neq i+1 \end{cases} \\ C_x &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^\top . \end{aligned} \tag{26}$$

where f_x and h_x are $\gamma_{f_x}-$Lipschitz and $\gamma_{h_x}-$Lipschitz, respectively, with respect to their arguments.

For the sake of observability, the following assumption is necessary.

Assumption 2: There exists $\delta_h > 0, \delta_h \leq \gamma_{h_x}$ such that

$$\delta_h \leq \frac{\partial h_x}{\partial v}(v) \leq \gamma_{h_x}, \ \forall v \in \mathbb{R}. \tag{27}$$

Without loss of generality, for the sake of simplification, we assume that

$$\gamma_{h_x} \triangleq \sup_{v\in\mathbb{R}} \left(\frac{\partial h_x}{\partial v}(v) \right) = 1. \tag{28}$$

Otherwise, we use for the observer the output

$$y_{\text{new}}(t) \triangleq \frac{y(t)}{\sup\limits_{v\in\mathbb{R}} \left(\frac{\partial h_x}{\partial v}(v) \right)}. \tag{29}$$

As in the previous section, we create the following new variable, $z(t) \in \mathbb{R}$, as in (20):

$$\begin{cases} \dot{z}(t) = \gamma y(t) \\ z(t) = z(0) = z_0, \forall t \in [-t^\star, 0]. \end{cases} \tag{30}$$

where $\gamma > 0$ is a constant scalar, which is considered a tuning parameter. Also, in this case, from the *Newton-Leibniz* formula, (21) is reduced to

$$y(t) = h(x(t)) - \int_{t-\tau(t)}^{t} \frac{\partial h_x}{\partial x_1}\big(x_1(s)\big)x_2(s)\mathrm{d}s. \tag{31}$$

It is quite clear that the function

$$\phi(x_1, x_2) \triangleq \frac{\partial h_x}{\partial x_1}\big(x_1\big)x_2 \tag{32}$$

is globally Lipschitz with respect to x_2 because h_x is $\gamma_{h_x}-$Lipschitz, and then $\frac{\partial h_x}{\partial x_1}(x_1)$ is bounded. However, it is not globally Lipschitz with respect to x_1. Then, we need an additional assumption on x_2.

As in the previous section, we consider the transformed system (8) with the parameters (22)–(24) with $Y_z = \gamma$, and according to (26) as follows:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g(\zeta(s))\mathrm{d}s \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{33}$$

where $g(\zeta) = \phi(\zeta_2, \zeta_3)$.

B. System transformation: High-gain observer

By construction of the corresponding augmented system (33), without the integral term, the triangular companion form is preserved. Then, we can apply the high-gain observer methodology. To this end, we perform a second transformation, which is usual in this context, although it is often applied to the error system. Let us introduce the following linear transformation:

$$\xi = \mathbb{T}_\theta \zeta, \ \text{where } \mathbb{T}_\theta \triangleq \operatorname{diag}\left(\frac{1}{\theta}, \dots, \frac{1}{\theta^{n+1}}\right) \tag{34}$$

which transforms (33) into

$$\begin{cases} \dot{\xi}(t) = \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(s)\right)\mathrm{d}s \\ y_\zeta(t) = C\mathbb{T}_{\frac{1}{\theta}}\xi(t) \end{cases} \tag{35}$$

To make the developments easy to follow and for any convenience, we express the system (35) in the following detailed form:

$$\begin{cases} \dot{\xi}(t) = \begin{bmatrix} \frac{\gamma}{\theta}h_x(\theta^2\xi_2(t)) \\ \theta[0 \quad A]\xi(t) \end{bmatrix} + \dfrac{1}{\theta^{n+1}}\begin{bmatrix} 0 \\ B \end{bmatrix} f_x\left([0 \quad \mathbb{I}_n]\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) \\ \qquad + \dfrac{1}{\theta}B_\zeta \displaystyle\int_{t-\tau(t)}^{t} \phi\big(\theta^2\xi_2(s), \theta^3\xi_3(s)\big)\mathrm{d}s \\ y_\zeta(t) = \theta C\xi(t) = \theta\xi_1(t) \end{cases} \tag{36}$$

Before summarizing the result, let us define the function $\breve{g}$ as the Lipschitz extension of ϕ introduced in (32):

$$\breve{g}(z_1, z_2) = \phi\big(z_1, \pi_{\mathcal{I}}(z_2)\big) \tag{37}$$

where $\pi_{\mathcal{I}}(z_2)$ stands for the *Hilbert* projection of z_2 on $\mathcal{I}$ for any closed interval $\mathcal{I} \subset \mathbb{R}$. Define the matrix $\mathcal{A}_\gamma(.)$ as

$$\mathcal{A}_\gamma(\boldsymbol{v}) \triangleq \begin{bmatrix} 0 & \begin{bmatrix} \gamma\boldsymbol{v} & 0_{1\times n-1} \end{bmatrix} \\ 0_{n\times 1} & A \end{bmatrix}, \ \forall \boldsymbol{v} \in \mathbb{R}. \tag{38}$$

Theorem 2: Assume that the component $x_2(t)$ of the system (1), with (26), belongs to a compact interval $\mathcal{I} \subset \mathbb{R}$ and that the function ϕ defined in (32) is Lipschitz in $\mathbb{R} \times \mathcal{I}$. Assume there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p \times n+1}$, and positive scalars μ and γ such that the following conditions hold:

$$\left(\mathcal{A}_\gamma(\ell)\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_\gamma(\ell) - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathbb{I}_{n+1} \leq 0, \quad \ell \in \{\delta_h, 1\} \tag{39a}$$

$$\theta > \max\left(1, \frac{2\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right) \tag{39b}$$

$$\tau^\star < \frac{\lambda_{\min}(\mathcal{P})\left[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\right]}{2\theta^2\gamma\kappa_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}B_\zeta\|} \tag{39c}$$

Then the output $\hat{x}(t)$ of the following observer

$$\dot{\hat{\xi}}(t) = \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\mathrm{d}s + L\left(y_\zeta(t) - C\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) \tag{40a}$$

$$\hat{x}(t) = \begin{bmatrix} 0_{n\times 1} & \mathbb{I}_n \end{bmatrix} \mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t) \tag{40b}$$

with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially to the state $x(t)$ of the original system (1) with the particular parameters in (26).

Proof: It is sufficient to show that $\tilde{\xi}(t) \triangleq \xi(t) - \hat{\xi}(t)$ is exponentially stable. Then, the dynamics of the estimation error is given by:

$$\begin{aligned}\dot{\tilde{\xi}}(t) = \theta\left[\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right]\tilde{\xi}(t) \\ + \Big(\psi(\xi(t)) - \psi(\hat{\xi}(t))\Big) \\ + B_\zeta\int_{t-\tau(t)}^{t}\left[g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) - \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\right]\mathrm{d}s \quad (41)\end{aligned}$$

where

$$\psi(\xi(t)) \triangleq \frac{1}{\theta^{n+1}}\begin{bmatrix}0\\B\end{bmatrix} f_x\left([0 \quad \mathbb{I}_n]\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) \quad (42)$$

and $h_x(\theta^2\xi_2(t)) - h_x(\theta^2\hat{\xi}_2(t)) = \theta^2\frac{\partial h_x}{\partial v}(w(t))\tilde{\xi}_2(t)$, $w(t) \in$ **Co** $\Big(\theta^2\xi_2(t), \theta^2\hat{\xi}_2(t)\Big)$, from the differential mean value theorem in Lemma 2 applied to the scalar function h_x.
In addition, since f_x is $\gamma_{f_x}-$Lipschitz and from the structure of ψ in (42), there exists a constant $\kappa_f \geq \gamma_{f_x}$, independent from θ, such that

$$\Big\|\psi(\xi(t)) - \psi(\hat{\xi}(t))\Big\| \leq \kappa_f\|\tilde{\xi}(t)\|. \quad (43)$$

Since $\theta^3\xi_3(t) = x_2(t) \in \mathcal{I}$ and the *Hilbert* projection preserves the Lipchitz constant in $\mathbb{R}^2$ and from the structure of g in (36), there exists $\kappa_g \geq \gamma_g$ such that

$$\Big\|g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) - \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\Big\| \leq \kappa_g\theta^2\|\tilde{\xi}(t)\|. \quad (44)$$

Now, after computing the derivative of $\vartheta(\tilde{\xi}(t)) \triangleq \tilde{\xi}^\top(t)\mathcal{P}\tilde{\xi}(t)$ along the trajectories of (41), and by considering the bounds (43)-(44), we get

$$\begin{aligned}\dot{\vartheta}(\tilde{\xi}(t)) \leq \tilde{\xi}^\top(t)\Bigg[\left(\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right)^\top\mathcal{P} \\ + \mathcal{P}\left(\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right)\Bigg]\tilde{\xi}(t) \\ + 2\theta^2\gamma\kappa_g\|\mathcal{P}B_\zeta\|\|\tilde{\xi}(t)\|\int_{t-\tau^\star}^{t}\|\tilde{\xi}(s)\|\mathrm{d}s \\ + 2\kappa_f\lambda_{\max}(\mathcal{P})\|\tilde{\xi}(t)\|^2. \quad (45)\end{aligned}$$

It follows from (39a) and the convexity principle that

$$\begin{aligned}\dot{\vartheta}(\tilde{\xi}(t)) \leq -\frac{\Big(\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\Big)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde{\xi}(t)) \\ + \tau^\star\frac{2\theta^2\gamma\kappa_g\|\mathcal{P}B_\zeta\|}{\lambda_{\min}(\mathcal{P})}\sup_{s\in[t-\tau^\star,t]}\vartheta(\tilde{\xi}(s)) \quad (46)\end{aligned}$$

Then, according to Lemma 1 and $\theta \geq 1$, the exponential convergence of $\tilde{\xi}(t)$ is inferred if

$$\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P}) > 0$$

and

$$\frac{\Big(\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\Big)}{\lambda_{\max}(\mathcal{P})} > \tau^\star\frac{2\theta^2\gamma\kappa_g\|\mathcal{P}B_\zeta\|}{\lambda_{\min}(\mathcal{P})}$$

which are equivalent to (39b) and (39c), respectively. ■

IV. Conclusion

In this paper, we proposed several observer design techniques for nonlinear systems in the presence of delayed and nonlinear outputs. Through a state augmentation technique and output transformation, the problem of the presence of delay and nonlinearities in the output measurement is easily solved by transferring the delay and the nonlinearities to the dynamic process. Such a transfer is achieved by creating a new output measurement and extending the dynamics of the system. For the specific class of systems considered in this paper, namely systems in companion form, novel synthesis conditions are proposed, which are less conservative than those existing in the literature.

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