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Convergence analysis for platooning over coloured additive noise channels

Fernando Sanhueza, Marco Gordon, Alejandro I. Maass, Andrés Peters, Francisco Vargas

***Abstract*— We consider a platooning control problem where the communication channels between vehicles are subject to coloured additive noises. Due to the stochastic nature of these channels, our analysis delves into examining the convergence of both the mean and variance of the vehicle tracking errors. We study the convergence as both time and number of vehicles grow unbounded. Our results include necessary and sufficient conditions for convergence and reveal that the colour of the noise does not impact the convergence characteristics of the error statistics, although it affects the values of the tracking error variances. Our findings offer insights into string stabilization. Numerical examples illustrate our results.**

***Index Terms*— Convergence, coloured additive noise channels, platooning, string stability**

I. Introduction

Platooning involves coordinating autonomous vehicles to optimize road use, cut fuel consumption, improve travel times, and enhance safety [1]. In this context, Cooperative Adaptive Cruise Control (CACC) is crucial for platooning analysis. CACC coordinates vehicles at predefined speeds, ensuring consistent inter-vehicle distances through wireless communication and spacing methods [2].

In one-dimensional platooning, the fundamental control challenge in CACC pertains to maintaining a desired distance from the preceding vehicle whenever possible. Under undisturbed conditions, the control algorithm of each vehicle should achieve this desired inter-vehicle gap in steady state, ensuring that the entire platoon travels at the cruising speed of the lead vehicle. However, the occurrence of disturbances, such as sudden changes in the leader's velocity, introduces transient tracking errors in each vehicle. These errors have the potential to propagate and increase along the chain of vehicles. If this occurs, the platoon is said to be string unstable [3]. Conversely, if these transient errors attenuate along the string of vehicles, the platoon is deemed string stable. Ensuring string stability is indispensable as it improves platoon performance, facilitates platoon scalability, diminishes collision probabilities, and augments the safety and fluidity of vehicular movement.

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Due to its significance, researchers have focused on obtaining conditions for string stability in different scenarios, but mostly deterministic. One relevant factor that affects the study of string stabilization relates to the vehicle models, which can be linear [4] or nonlinear [5], [6], described in the time domain [7] or frequency domain [8], and in continuous-time [9] or discrete-time setups [7]. The network communication topology also plays a crucial role, with the Predecessor-Following (PF) topology being one of the simplest and most studied ones [4]. However, there are several others, such as Bidirectional (BD) [10] and Predecessor-Leader-Following (PLF) [11], among others. The spacing policy is also an important parameter in the study of platoon systems [12]. A more extensive classification of string stability for platoon systems for deterministic setups can be found in [3] and [13].

An important issue in real applications is the case where inter-vehicle communication channels are subject to random communication phenomena. Indeed, an increasing interest in studying their effect on platooning applications has been observed during the last years, mainly focused on studying platoons with data loss [7], [14]–[16] and random delays [17]–[20]. We noted that platooning over additive channels is still incipient [21], although the study of additive noise channels in control problems has been extensively studied in other contexts [22]–[24]. In particular, in [21] the inter-vehicle communication is assumed to be affected by additive white noisy channels, and the string stability is numerically studied based on the convergence of the mean and variance of the tracking errors.

A disadvantage of studying platoons considering these random phenomena is the lack of a general framework to address string stability in stochastic environments. Many of the aforementioned works partially analyze string stability [15], [18], [25], assess it numerically using the mean and variance of the collected data [26], [27], or simply do not formally address string stability and instead focus on proposing techniques to mitigate the effects of communication channels [16], [20]. This lack of a comprehensive definition of string stability for stochastic environments has constrained theoretical analysis, although most of the available approaches coincide in that the convergence of the statistics of the signals of interest (mean and variance) is a necessary condition to have a string stable behavior.

In this work, our interest is to study platoons whose inter-vehicle communication is affected by coloured additive noise, which is a more general and realistic model compared to the white noise channel model [28]. We consider one-dimensional platoons with predecessor-following topology and a constant time headway policy, in a discrete-time setting. Our goal is to analyze the effect of the noise colour on the convergence properties of the tracking error second-order statistics (mean and variance). We consider convergence not only on time but also when the number of vehicles in the platoon increases unbounded, providing analytical results that would serve as a basis for string stability analysis. Our findings include necessary and sufficient conditions for platoon convergence and reveal that, while the stationary values of the tracking error variance are significantly influenced by the noise colors, it does not impact the convergence properties of the platoon. Our results are complemented with simulation results that allow a numerical evaluation of string stability and its connection with the convergence conditions.

II. PROBLEM SETUP

A. Notation

Let $M \in \mathbb{R}^{n\times n}$ be a square matrix, its spectral radius is denoted as $\rho(M) = \max\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\}$, where λ_i are the eigenvalues of M. Given a matrix $N \in \mathbb{R}^{n\times m}$, we use $vec(\cdot)$ to denote the vectorization, whose inverse operation $vec^{-1}(\cdot)$ is such that $N = vec^{-1}(vec(N))$. To denote the Kronecker product and transpose of a matrix we use $\otimes$, and $(\cdot)^\top$, respectively.

Let $x(k)$, with $k \in \mathbb{N}$, be a discrete-time stochastic process. The mean $\mu_x(k) \in \mathbb{R}^{n_x}$, and variance matrix $P_x(k) \in \mathbb{R}^{n_x\times n_x}$ are respectively defined as $\mu_x(k) \triangleq \mathcal{E}\{x(k)\}$, and $P_x(k) \triangleq \mathcal{E}\left\{(x(k)-\mu_x(k))(x(k)-\mu_x(k))^T\right\}$, where $\mathcal{E}\{\cdot\}$ denotes the expectation operator. The process x is said to be mean square stable if and only if the mean and variance satisfy $\lim_{k\to\infty}\mu_x(k) = \mu_x$ and $\lim_{k\to\infty}P_x(k) = P_x$, where μ_x and P_x are stationary well-defined constant values.

Consider a discrete-time LTI system whose transfer function is given by $W(z)$. The output of the system to a stochastic process input $u(k)$ is given by $y(k) = w(k) * u(k)$, where $*$ denotes the convolution, and $w(k)$ is the impulse response. To simplify the notation, we will adopt the notation $y = Wu$. Finally, we define $W(z)^\sim \triangleq W^\top(z^{-1})$.

B. Platooning framework

We consider a one-dimensional homogeneous platoon formed by N vehicles modelled as linear time-invariant discrete-time systems. The position of the vehicle i is denoted by $y_i(k)$, where k is the discrete-time index. We assume a predecessor following (PF) communication topology in which each vehicle i transmits its own position to its immediate follower $i+1$. However, in our setup, we assume that the transmitted signal $y_i(k)$ is corrupted by the communication channel and, thus, the received signal by the follower is $\tilde{y}_i(k) = y_i(k)+v_i(k)$, where v_i is coloured noise. We consider the following assumptions on v_i:

- The noise v_i is wide-sense stationary (*WSS*), with mean $\mu_{v_i} = 0$ and variance P_{v_i}.
- The power spectrum of v_i, S_v, is known and is such that $S_v(z) = \Omega_v(z)\Omega_v(z)^\sim$, where $\Omega_v(z)$ is stable, minimum phase and strictly proper spectral factor.

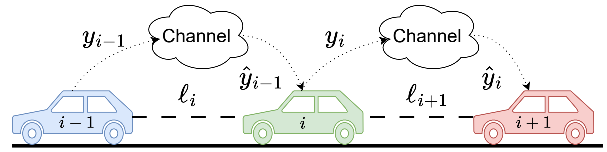

Fig. 1. Platoon with noisy communication.

- The noise $v_i(k)$ is uncorrelated with the initial conditions of the platoon vehicles.

The platoon tracking task consists of maintaining the inter-vehicle distance $\ell_i = y_{i-1}(k) - y_i(k)$ equal to a desired reference $r_i(k)$ if it is possible. Figure 1 depicts the described platoon's configuration with noisy communication. We assume that the leader ($i = 0$) is unaffected by noise since it does not follow any vehicle. We assume that the leader follows a virtual reference, which allows us to describe its trajectory (see e.g. [15]).

The adopted spacing policy is such that $r_i(k)$ should be wide enough for high speeds, and can be narrower for low speeds. We thus consider the time-headway spacing policy

$$r_i(k) = h\left[y_i(k) - y_i(k-1)\right], \tag{1}$$

where h is a positive time headway constant. It is known that string stability, that is, the lack of amplifications of disturbances as they propagate along the string, depends on the election of h when ideal communications are considered [3]. Applying the $\mathcal{Z}$ transform we can write $r_i(k)$ in the frequency domain as

$$R_i(z) = \frac{hz-h}{z}\,Y_i(z) = H(z)Y_i(z). \tag{2}$$

The tracking error in the vehicle i is denoted by $\zeta_i(k)$, and is given by

$$\begin{aligned}\zeta_i(k) &= \ell_i(k) - r_i(k)\\ &= y_{i-1}(k) - y_i(k) - h\left[y_i(k) - y_i(k-1)\right].\end{aligned} \tag{3}$$

However, $\zeta_i(k)$ is not available at the i-th vehicle due to the channel noise. Instead, each vehicle has access to a noisy version which is denoted as $e_i(k)$ and given by

$$e_i(k) = y_{i-1}(k) - y_i(k) + v_i(k) - r_i(k) \tag{4}$$

The movement of each vehicle i is determined by a local control loop which is limited to employing e_i, rather than ζ_i, to define the control action u_i. We recall that the leader follows a virtual reference and hence we can define a virtual error $\zeta_0(k) = r_0(k) - y_0(k) - h[y_0(k) - y_0(k-1)]$ which can be considered as a known external input.

Given the definition of $r_i(k)$, we can define the local control loop of each vehicle as the one in Figure 2, where the coloured noise v_i is modelled as the output of the spectral factor $\Omega_v(z)$ when its input is a white noise denoted by d_i (details of the noise modelling are in Section III). In Figure 2, $K(z)$, $G(z)$ and $H(z)$ are the transfer functions of the controller, the vehicle, and the system $H(z)$ in (2), which are the same for each vehicle since the platoon is

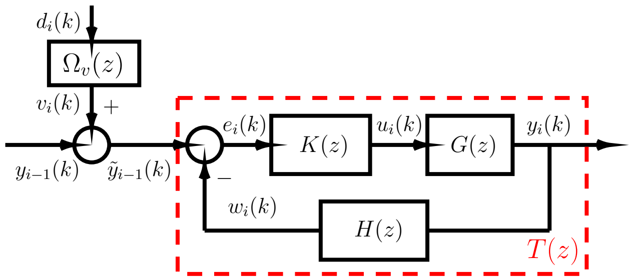

Fig. 2. Feedback control loop of the i-th vehicle

homogeneous. We assume that $G(z)K(z)$ has as least two integrators, which is required to achieve perfect tracking for ramp signals [29]. We also assume that $G(z)$ is strictly proper.

The closed-loop transfer function $T(z)$ is given by

$$T(z) = \frac{K(z)G(z)}{1 + K(z)G(z)H(z)}, \tag{5}$$

and the sensitivity $S(z)$ as $S(z) = 1 - H(z)T(z)$. From Figure 2, we note that the tracking errors ζ_i satisfy

$$\zeta_i = \begin{cases} T\zeta_0 - HT\Omega_v d_1 & \text{For } i = 1 \\ T\zeta_{i-1} + T\Omega_v d_{i-1} - HT\Omega_v d_i & \text{For } i > 1. \end{cases} \tag{6}$$

In this paper, our goal is to study the convergence of the tracking error when both the time k and the number of vehicles N grow unbounded. Since the noises are stochastic processes, we study the convergence properties of the mean and variance of the tracking errors.

III. State Space Description

A. Coloured noise modelling

Assume that $\Omega_v = (A_v, B_v, C_v, 0)$ is a minimal realization of Ω_v, which is strictly proper, stable and minimum phase. We model the noise v_i as the output of Ω_v, that is,

$$x_{v_i}(k+1) = A_v x_{v_i}(k) + B_v d_i(k) \tag{7a}$$
$$v_i(k) = C_{vi} x_{v_i}(k), \tag{7b}$$

where $d_i(k)$ is a white noise with mean $\mu_{d_i} = \Omega_v(1)^{-1}\mu_v = 0$ and unit variance. Note that the spectrum of v_i in (7) is S_v since $d_i(k)$ is white, with unit variance and $S_v(z) = \Omega_v(z)\Omega_v(z)^{\sim}$. In (7), x_{v_i} denotes the state of the system Ω_v that models the noise v_i. We consider that the initial condition $x_{v_i}(0)$ is a random variable satisfying

$$\mu_{x_{v_i}}(0) = (I - A_v)^{-1} B_v \mu_{d_i} = 0 \tag{8a}$$
$$P_{x_{v_i}}(0) = vec^{-1}\{(I - A_v \otimes A_v)^{-1} vec(B_v B_v^T)\}. \tag{8b}$$

Given the above choice of $\mu_{x_{v_i}}(0)$ and $P_{x_{v_i}}(0)$, it is easy to verify that the mean and variance of v_i in (7) are constant for all $k \geq 0$. Therefore, the signal v_i in (7) can be considered as a wide-sense stationary process with the same statistics as the original coloured noise if we focus our analysis for $k \geq 0$. In Figure 2, the noise modelling as described in this section is also depicted.

B. Representation based on an alternative state

For a given vehicle i, the corresponding coloured noise can be analyzed jointly with its closed-loop transfer function T. Thus, we consider a state space representation of the closed-loop transfer function T given by

$$x_i(k+1) = Ax_i(k) + B\tilde{y}_{i-1}(k) \tag{9a}$$
$$y_i(k) = Cx_i(k), \tag{9b}$$

where $x_i(k)$ is the state of the closed-loop system of vehicle i. Using (9) and (7) it is possible to obtain an extended model valid $\forall i \geq 1$, which is given by

$$\begin{bmatrix} x_i(k+1) \\ x_{v_i}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A & BC_v \\ 0 & A_v \end{bmatrix}}_{A_a} \underbrace{\begin{bmatrix} x_i(k) \\ x_{v_i}(k) \end{bmatrix}}_{x_{a_i}(k)} + \underbrace{\begin{bmatrix} 0 \\ B_v \end{bmatrix}}_{B_a} d_i(k) + \underbrace{\begin{bmatrix} BC & 0 \\ 0 & 0 \end{bmatrix}}_{B_p} \underbrace{\begin{bmatrix} x_{i-1}(k) \\ x_{v_{i-1}}(k) \end{bmatrix}}_{x_{a_{i-1}}(k)} \tag{10a}$$

$$y_i(k) = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_a} \begin{bmatrix} x_i(k) \\ x_{v_i}(k) \end{bmatrix}. \tag{10b}$$

Moving at a constant speed implies that the positions y_i can be modelled as ramp signals. Hence, some of the states in x_{a_i} may increase unbounded, which is expected. However, it is convenient to describe the platoon system in terms of a bounded state. Hence, we define the alternative state

$$\xi_i(k) = x_{a_{i-1}}(k) - x_{a_i}(k) - h[x_{a_i}(k) - x_{a_i}(k-1)]. \tag{11}$$

The main advantage of this state is that if the controller is properly designed the mean and variance of the signal ξ_i converges to fixed values, even if some terms in x_{a_i} become ramp signals. Given (11) and the state space description in (10), it is possible to characterize ξ_i and the tracking error ζ_i for any platoon member with $i > 0$ by

$$\xi_i(k+1) = A_a\xi_i(k) + B_p\xi_{i-1}(k) - (1-h)B_a d_i(k) + B_a d_{i-1}(k) + hB_a d_i(k-1) \tag{12a}$$
$$\zeta_i(k) = C_a\xi_i(k). \tag{12b}$$

C. State space representation for the concatenated system

For convenience, we can obtain a state space description for the whole platoon as a unique LTI system. Define

$$\zeta(k) = \begin{bmatrix} \zeta_i(k) \\ \vdots \\ \zeta_N(k) \end{bmatrix}, \quad \xi(k) = \begin{bmatrix} \xi_i(k) \\ \vdots \\ \xi_N(k) \end{bmatrix}, \quad d(k) = \begin{bmatrix} d_i(k) \\ \vdots \\ d_N(k) \end{bmatrix}.$$

It is not difficult to show, given (12a) and (12b), that

$$\xi(k+1) = A_g\xi(k) + B_0\zeta_0(k) + B_{ga}d(k) + B_{gb}d(k-1) \tag{13a}$$
$$\zeta(k) = C_g\xi(k), \tag{13b}$$

where

$$A_g = \begin{bmatrix} A_a & & & \\ B_p & A_a & & \\ & \ddots & \ddots & \\ & & B_p & A_a \end{bmatrix} C_g = \begin{bmatrix} C_a & & \\ & C_a & \\ & & \ddots & \\ & & & C_a \end{bmatrix}$$

$$B_{ga} = \begin{bmatrix} (h-1)B_a & & & \\ B_a & (h-1)B_a & & \\ & & \ddots & \ddots \\ & & B_a & (h-1)B_a \end{bmatrix}$$

$$B_{gb} = \text{diag}\{hB_a, \ldots, hB_a\}, \quad B_0 = [B^T, 0, \ldots, 0]^T.$$

This representation allows us to describe the dynamics of the whole platoon, having as external inputs the noises d and the virtual error ζ_0. The output is the vector of the tracking errors of the whole platoon. Naturally, the dimensions of the state space matrices increase as N increases.

IV. Convergence analysis

We consider the convergence of the tracking errors $\zeta_i(k)$, with $i = 1, \ldots, N$, when $k \to \infty$, and also when $N \to \infty$. In our stochastic setup, the convergence is studied in terms of the error statistics, that is, means and variances.

A. Convergence in time

From the system proposed on (13) it is possible to show that the state mean and the tracking error mean satisfy recursive equations.

Proposition 4.1: For any $N \in \mathbb{N}$, the mean of ξ and ζ in (13), denoted by μ_ξ and μ_ζ respectively, satisfy

$$\mu_\xi(k+1) = A_g \mu_\xi(k+1) + B_0 \zeta_0(k) \tag{14}$$

$$\mu_\zeta(k) = C_g \mu_\xi(k). \tag{15}$$

Additionally, the corresponding covariance matrices P_ξ and P_ζ satisfy

$$P_\xi(k+1) = A_g P_\xi(k) A_g^T + B_{ag} B_{ag}^T + B_{gb} B_{gb}^T + A_g B_{ag} B_{gb}^T + B_{gb}^T B_{gb}^T B_{ga}^T A_g^T \tag{16}$$

$$P_\zeta(k) = C_g P_\xi(k) C_g^T. \tag{17}$$

■

Since the system in (13) is linear and time-invariant, the recursions in Proposition 4.1 are standard results [30].

Now we focus on studying the statistics in Proposition 4.1 when $k \to \infty$.

Lemma 4.2: For any $N \in \mathbb{N}$, the mean and variance of ξ and ζ in (13) converge to constant values if and only if $\rho(A) < 1$.

Proof: It is easy to note that $\mu_\zeta(k)$ and $P_\zeta(k)$ converge to constant values if and only if $\mu_\xi(k)$ and $P_\xi(k)$ converge, respectively. Also, since the system in (13) is linear, it is easy to see that $\rho(A_g) < 1$ is a necessary and sufficient condition for $\mu_\xi(k)$ and $P_\xi(k)$ to converge to constant values [30]. Given the specific block bi-diagonal structure of the matrix A_g, it follows that $\rho(A_g) < 1$ if and only if $\rho(A_a) < 1$. Finally, noting that $eig(A_a) = eig(A) \cup eig(A_v)$, and given that $\rho(A_v) < 1$ since the spectral factor Ω_v is stable, then we conclude that $\rho(A) < 1$ is a necessary and sufficient condition for time-convergence. ■

Note that $\rho(A) < 1$ ensures that the system in (13) is internally stable but also mean square stable, which ensures the existence of the second order statistics when $k \to \infty$ for both the tracking errors and system states. This is regardless of the platoon length N. Satisfying $\rho(A) < 1$ is a basic task, which can be easily done by a proper controller design.

B. Convergence in the number of vehicles

Assuming $\rho(A) < 1$, we guarantee the existence of the stationary spectrum for ζ_i, and hence we can study the convergence when $N \to \infty$ in the frequency domain using the power spectral density of the tracking error. In order to do that, based on (6) we can propose the following result.

Proposition 4.3: Assume that $\rho(A) < 1$. Then,

$$\mu_{\zeta_i}(e^{j\theta}) = T(e^{j\theta}) \mu_{\zeta_{i-1}}(e^{j\theta}) \tag{18}$$

$$\begin{aligned} \phi_{\zeta_i}(e^{j\theta}) &= |T(e^{j\theta})|^2 \phi_{\zeta_{i-1}}(e^{j\theta}) \\ &\quad + |T(e^{j\theta})|^2 |S(e^{j\theta})|^2 |\Omega_v(e^{j\theta})|^2 + \\ &\quad + (1 - |T(e^{j\theta})|^2) |1 - S(e^{j\theta})|^2 |\Omega_v(e^{j\theta})|^2, \end{aligned} \tag{19}$$

where $\phi_{\zeta_1}(e^{j\theta}) = \left| H(e^{j\theta}) T(e^{j\theta}) \right|^2 \left| \Omega_v(e^{j\theta}) \right|^2$. ■

The terms in Proposition 4.3 are now recursive in i. This allows us to easily analyze the convergence when $N \to \infty$.

Lemma 4.4: Assume $\rho(A) < 1$. The mean and variance of the tracking errors of the platoon converge to constant values as $N \to \infty$ if and only if

$$|T(e^{j\theta})| < 1, \qquad \forall \theta > 0. \tag{20}$$

In this case, $\forall N \in \mathbb{N}$ and $i \in \{1, 2, ..., N\}$, the stationary mean μ_{ζ_i} is zero, and the stationary variance P_{ζ_i} satisfies:

1) $P_{\zeta_{i-1}} \leq P_{\zeta_i}, \ \forall i > 1$.
2) $\max_i P_{\zeta_i} = P_{\zeta_N} = \|F_{\zeta_N}\|_2^2$.
3) $\lim_{N \to \infty} P_{\zeta_N} = \left(||HT\Omega_v||_2^2 + \left|\left| \frac{ST\Omega_v}{M} \right|\right|_2^2 \right)$,

where $F_{\zeta_N} = \begin{bmatrix} ST^{N-1} & ST^{N-2} & \cdots & ST & -HT \end{bmatrix} \Omega_v$ and $M(z)$ is a stable and minimum phase spectral factor such that $1 - T(z)T(z)^\sim = M(z)M(z)^\sim$.

Proof: It is evident from Proposition 4.3 that both recursions converge as i grows if and only if $|T(e^{j\theta})| < 1$. Additionally, it is also easy to see from (18) that $\lim_{i \to \infty} \mu_{\zeta_i}(e^{j\theta}) = 0$ when $|T(e^{j\theta})| < 1$. On the other hand, by performing some manipulations, (19) can be alternatively written as

$$\phi_{\zeta_i}(e^{j\theta}) = \phi_{\zeta_{i-1}}(e^{j\theta}) + |S(e^{j\theta}) T^{i-1}(e^{j\theta}) \Omega_v(e^{j\theta})|^2. \tag{21}$$

Hence, the stationary value of P_{ζ_i} is given by

$$P_{\zeta_i} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{\zeta_i}(e^{j\theta}) \, d\theta = P_{\zeta_{i-1}} + \left|\left| ST^{i-1} \Omega_v \right|\right|_2^2. \tag{22}$$

From (22), it is clear that $P_{\zeta_i} > P_{\zeta_{i-1}}$. This means that the stationary variance increases with the number of vehicles. Therefore, for a platoon composed of N vehicles, the maximum stationary variance of the tracking error reaches its maximum value at P_{ζ_N}. Using (22), said value can be

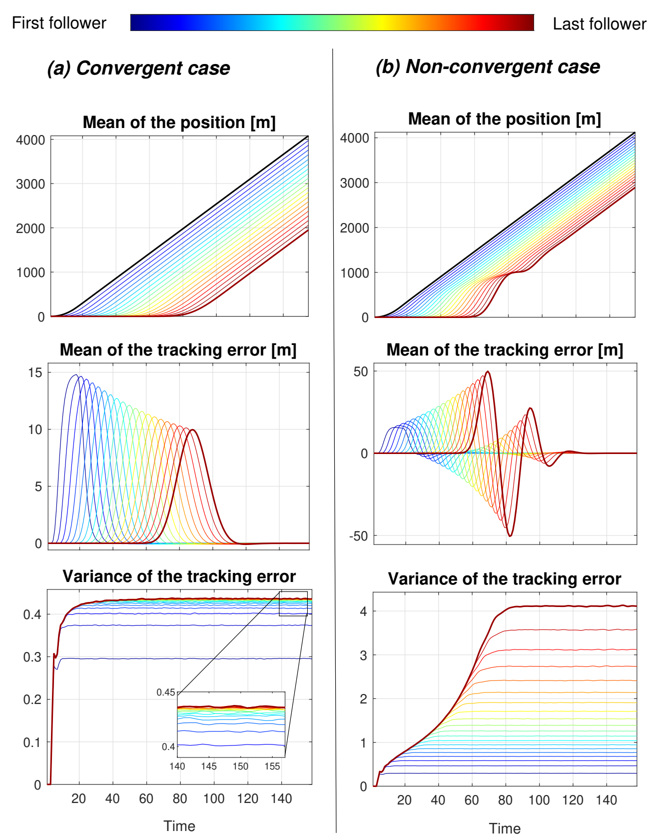

Fig. 3. Means and variances: (a) Convergent case (First column). (b) Non Convergent case (Second column).

obtained as $P_{\zeta_N} = \|F_{\zeta_N}\|_2^2$, with F_{ζ_N} as in Lemma 4.4. On the other hand, if the platoon satisfies the conditions in (4.2) and (4.4), it is also expected that P_{ζ_N} converges to a constant value when $N \to \infty$. Let $M(e^{j\theta})$ be a spectral factor such that $|M(e^{j\theta})|^2 = 1 - |T(e^{j\theta})|^2$, and $\phi_{\zeta_\infty}(e^{j\theta}) \triangleq \lim_{N\to\infty} \phi_{\zeta_N}(e^{j\theta})$. Then, from (19) it follows that

$$\begin{aligned}\phi_{\zeta_\infty}(e^{j\theta}) =& |T(e^{j\theta})|^2\phi_{\zeta_\infty}(e^{j\theta}) + |S(e^{j\theta})T(e^{j\theta})\Omega_v(e^{j\theta})|^2 \\ &+ |M(e^{j\theta})|^2|H(e^{j\theta})T(e^{j\theta})\Omega_v(e^{j\theta})|^2 \\ =& |H(e^{j\theta})T(e^{j\theta})\Omega_v(e^{j\theta})|^2 + \frac{|S(e^{j\theta})T(e^{j\theta})\Omega_v(e^{j\theta})|^2}{|M(e^{j\theta})|^2}.\end{aligned}$$

Finally, the maximum stationary variance is given by

$$\begin{aligned}\lim_{N\to\infty} P_{\zeta_N} =& \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\theta})T(e^{j\theta})\Omega_v(e^{j\theta})|^2 \, d\theta \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|S(e^{j\theta})T(e^{j\theta})\Omega_v(e^{j\theta})|^2}{|M(e^{j\theta})|^2} \, d\theta \\ =& ||HT\Omega_v||_2^2 + \left|\left|\frac{ST\Omega_v}{M}\right|\right|_2^2.\end{aligned}$$

■

Conditions in Lemmas 4.2 and 4.4 ensure temporal convergence and convergence in the number of vehicles in a platoon affected by coloured noise. Lemma 4.4 also establishes the existence of a maximum value to which all stationary variances converge. Such limit strongly depends on the colour of the noise. This limit, in turn, guarantees the boundedness of the platoon when both convergence conditions are satisfied.

C. Connection with the string stability property

It is important to highlight that the obtained conditions in Lemmas 4.2 and 4.4 coincide with those that ensure string stability for the ideal (and deterministic) case where channel noises are removed. However, in this work, we are not formally proving string stability. Instead, we have investigated convergence over time and the number of vehicles, and we have shown that, under certain conditions, we can guarantee that the mean of the errors converges to zero and that the error variances are bounded by an upper limit. This is closely related to the concept of string stability, so we believe that these conditions will be at least necessary conditions for string stability for stochastic scenarios.

V. NUMERICAL SIMULATION

Consider the platoon described in Section III. Each vehicle has a closed-loop transfer function $T(z)$ defined by $H(z) = (1+h) - h/z$,

$$G(z) = \frac{1}{z-1}, \quad K(z) = \frac{0.228z^2 - 0.1824z}{z^3 - 0.95z^2 - 0.73z + 0.68}.$$

To analyze the convergence we select two values of the constant h, namely, $h = 3.8$ and $h = 2.2$. Remember that as the value h increases, so does the desired inter-vehicle spacing. These values are selected to show a convergent and non-convergent system. It is important to mention that the system $T(z)$, for both time-headway scenarios, exhibits internal stability, i.e. $\rho(A) < 1$ (Lemma 4.2). This condition, although trivial, but necessary, can be observed in the convergence over time, as depicted in Fig. 3, where both cases show the mean and variance of the tracking error converging to stationary values when $k \to \infty$. In this section, we mainly focus on analyzing the behaviour of the mean and variance when the number of vehicles increases ($N \to \infty$). Through simulations, we validate and discuss the results obtained in Lemma 4.4.

The coloured noise is modelled using a white Gaussian noise $d_i(k)$ with mean $\mu_{d_i} = 0$ and variance $P_{d_i} = 1$, $\forall i \in 1, \ldots, N$; and the spectral factor $\Omega_v(z)$ is given by

$$\Omega_v(z) = \frac{0.021z^3 + 0.071z^2 + 0.689z + 0.28}{z^2 - 0.755z + 0.28}.$$

The output of this system, $v_i(k)$, is a zero-mean pink noise. Figure 3 shows the evolution of the mean of the positions, and the mean and variance of the tracking errors of a platoon of 20 vehicles which follow a ramp reference and start from rest with zero initial conditions. Each vehicle is denoted by a colour code, with the first follower depicted in dark blue and the last vehicle in dark red. The colour legend is presented at the top of the figure for reference. The results are obtained by using a Monte Carlo simulation with 10^6 realizations.

For the convergent case ($h = 3.8$) shown in Figure 3(a), we observe good reference tracking on the position mean. The mean of the tracking error converges to zero and also the peak values are reduced as the number of vehicles increases. Also, we observe that the tracking error variance converges to a value different from zero bounded by a maximum value. This means that the difference between the stationary variance of two consecutive vehicles $\Delta_i = P_{\zeta_i} - P_{\zeta_{i-1}}$ tends to zero as i increases.

On the other hand, on the non-convergent case ($h = 2.2$) presented in Figure 3(b), even if the individual variances converge to stationary values, we could note that as the number of vehicle increases so does the difference Δ_i. Additionally, in the non-convergent case, it is clear that collisions can exist given that the mean of the error presents oscillations with maximum peaks that grow as the number of vehicles increases. Note that given the double integration, the mean of the tracking error tends to zero, but the oscillations are significant on the transient.

VI. CONCLUSION

We explored a platooning problem wherein the communication channels among vehicles are affected by coloured additive noises. Due to the stochastic nature of the channels, we analyzed the convergence behaviour of the mean and variance of vehicle tracking errors over time and with an increasing number of vehicles. Our findings establish necessary and sufficient conditions for convergence, revealing that, in our setup, the noise colour did not impact the convergence characteristics of error statistics. However, it did influence the values of tracking error variances. Our results shed light on the impact of coloured noise on string stabilization.

REFERENCES

[1] M. Muratori, J. Holden, M. Lammert, A. Duran, S. Young, and J. Gonder. Potentials for platooning in US highway freight transport. Technical report, National Renewable Energy Lab.(NREL), Golden, CO (United States), 2017.

[2] K. C. Dey, L. Yan, X. Wang, Y. Wang, H. Shen, M. Chowdhury, L. Yu, C. Qiu, and V. Soundararaj. A review of communication, driver characteristics, and controls aspects of cooperative adaptive cruise control (CACC). *IEEE Transactions on Intelligent Transportation Systems*, 17(2):491–509, 2016.

[3] S. Feng, Y. Zhang, S. Eben Li, Z. Cao, H. Liu, and L. Li. String stability for vehicular platoon control: Definitions and analysis methods. *Annual Reviews in Control*, 2019.

[4] G. Naus, R. Vugts, J. Ploeg, R. van de Molengraft, and M. Steinbuch. String-stable CACC design and experimental validation: A frequency-domain approach. *IEEE Transactions on vehicular Technology Vol.59*, 2010.

[5] W. B. Dunbar and D. S. Caveney. Distributed receding horizon control of vehicle platoons: Stability and string stability. *IEEE Transactions on Automatic Control*, 57(3):620–633, 2012.

[6] F. Sanhueza, F. Vargas, H. Ramirez, and A. Peters. Port hamiltonian based model for platooning applications including air drag effects. volume 56, pages 3917–3922, 2023. 22nd IFAC World Congress.

[7] K. Halder, U. Montanaro, S. Dixit, M. Dianati, A. Mouzakitis, and S. Fallah. Distributed H_∞ controller design and robustness analysis for vehicle platooning under random packet drop. *IEEE Transactions on Intelligent Transportation Systems*, 23(5):4373–4386, 2022.

[8] F. Ma, J. Wang, Y. Yang, L. Wu, S. Zhu, S. Y. Gelbal, B. Aksun-Guvenc, and L. Guvenc. Stability design for the homogeneous platoon with communication time delay. *Automotive Innovation*, 3(2):101–110, Jun 2020.

[9] C. Viel, U. Vautier, J. Wan, and L. Jaulin. Platooning control for heterogeneous sailboats based on constant time headway. *IEEE Transactions on Intelligent Transportation Systems*, 21(5):2078–2089, 2020.

[10] S. Knorn, A. Donaire, J. C. Agüero, and R. H. Middleton. Passivity-based control for multi-vehicle systems subject to string constraints. *Automatica*, 50(12):3224–3230, 2014.

[11] S. Darbha and J. Hedrick. Constant spacing strategies for platooning in automated highway systems. *Journal of Dynamic Systems Measurement and Control-transactions of The Asme - J DYN SYST MEAS CONTR*, 121, 09 1999.

[12] B. Wang, J. Zheng, Q. Ren, Y. Wu, and C. Li. Analysis of the intra-platoon message delivery delay in a platoon of vehicles. *IEEE Transactions on Vehicular Technology*, 70(7):7012–7026, 2021.

[13] Y. Zhang, Z. Xu, Z. Wang, X. Yao, and Z. Xu. Impacts of communication delay on vehicle platoon string stability and its compensation strategy: A review. *Journal of Traffic and Transportation Engineering (English Edition)*, 10(4):508–529, 2023.

[14] F. Acciani, P. Frasca, G. Heijenk, and A. A. Stoorvogel. Stochastic string stability of vehicle platoons via cooperative adaptive cruise control with lossy communication. *IEEE Transactions on Intelligent Transportation Systems*, 23(8):10912–10922, 2022.

[15] A. Elahi, A. Alfi, and H. Modares. Distributed consensus control of vehicular platooning under delay, packet dropout and noise: Relative state and relative input-output control strategies. *IEEE Transactions on Intelligent Transportation Systems*, 23(11):20123–20133, 2022.

[16] C. Wu, Y. Lin, and A. Eskandarian. Cooperative adaptive cruise control with adaptive Kalman filter subject to temporary communication loss. *IEEE Access*, 7:93558–93568, 2019.

[17] J. Zheng, L. Xu, L. Xie, and K. You. Consensusability of discrete-time multiagent systems with communication delay and packet dropouts. *IEEE Trans. on Automatic Control*, 64(3):1185–1192, 2019.

[18] W. B. Qin, M. M. Gomez, and G. Orosz. Stability and frequency response under stochastic communication delays with applications to connected cruise control design. *IEEE Transactions on Intelligent Transportation Systems*, 18(2):388–403, 2017.

[19] C. Zhao, X. Duan, L. Cai, and P. Cheng. Vehicle platooning with non-ideal communication networks. *IEEE Transactions on Vehicular Technology*, 70(1):18–32, 2021.

[20] A. Elahi, A. Alfi, and H. Modares. H_∞ consensus of homogeneous vehicular platooning systems with packet dropout and communication delay. *IEEE Trans. on Systems, Man, and Cybernetics: Systems*, 2021.

[21] M. A. Gordon, F. J. Vargas, A. A. Peters, and A. I. Maass. Platoon stability conditions under inter-vehicle additive noisy communication channels. *IFAC-PapersOnLine*, 53(2):3150–3155, 2020. 21st IFAC World Congress.

[22] F J Vargas, E. I. Silva, and J. Chen. Stabilization of two-input two-output systems over SNR-constrained channels. *Automatica*, 49(10):3133–3140, 2013.

[23] R. A. González, F. J. Vargas, and J. Chen. Mean square stabilization over SNR-constrained channels with colored and spatially correlated additive noises. *IEEE Trans. on Automatic Control*, 64(11):4825–4832, 2019.

[24] Y. Feng and H. Sun. Robust optimal control for discrete-time LTI systems over multiple additive white gaussian noise channels. *IEEE Transactions on Automatic Control*, 2022.

[25] C. Zhao, L. Cai, and P. Cheng. Stability analysis of vehicle platooning with limited communication range and random packet losses. *IEEE Internet of Things Journal*, 8(1):262–277, 2020.

[26] Y. Tang, M. Yan, P. Yang, and L. Zuo. Consensus based control algorithm for vehicle platoon with packet losses. In *2018 37th Chinese Control Conference (CCC)*, pages 7684–7689. IEEE, 2018.

[27] M. A. Gordon, F. J. Vargas, and A. A. Peters. Mean square stability conditions for platoons with lossy inter-vehicle communication channels. *Automatica*, 147:110710, 2023.

[28] A. J. Rojas. Signal-to-noise ratio constrained feedback control: Robust stability analysis. *ISA Transactions*, 95:235–242, 2019.

[29] S. E. Li, Y. Zheng, K. Li, Le-Yi Wang, and H. Zhang. Platoon control of connected vehicles from a networked control perspective: Literature review, component modeling, and controller synthesis. *IEEE Transactions on Vehicular Technology*, pages 1–1, 2017.

[30] K. J. Åström. *Introduction to stochastic control theory*. Courier Corporation, 2012.