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Observer Design by Using a New Output-Based Dynamic Extension Technique

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Abstract— **This paper deals with the observer design for continuous-time nonlinear systems with external disturbances. It suggests a new observer structure that relies on an output-based dynamic extension strategy enabling the state observer to be less affected by measurement noise. The proposed observer is based on new Linear Matrix Inequality (LMI) condition guaranteeing the Input-to-State Stability (ISS) property of the estimation error.**

Index Terms— **Observer design, LMIs, Lyapunov methods, Continuous-time nonlinear systems.**

I. Introduction

State estimation is essential to ensuring effective control of systems in many real-world applications, ensuring their steady and optimal operation. It provides accurate estimates when direct measurement of system states is infeasible due to sensor limitations or cost constraints. To this end, several estimation schemes in the literature aim to build an accurate estimate of the system states under the observability conditions. For linear systems, Luenberger observers have been widely used in the literature [1].

However, nonlinear systems are the most common in real-world applications. Hence, estimation strategies methods able to handle these complexities are required. The first nonlinear observer design approaches have involved transforming a nonlinear system into an observable canonical form to design state observers with linear error dynamics such as in [2]. Beyond the well-established Extended Kalman Filter (EKF) method [3], another wide range of nonlinear state observer techniques have emerged applied directly on the nonlinear system. For instance, the sliding-mode observer, as explored by [4], tackles the design of the estimator through the algebraic Riccati equation, aiming to minimize an upper bound on the estimation error. Additionally, Moving Horizon Estimation (MHE), an optimization-based approach, is gaining traction for its ability to estimate the state of nonlinear systems over a finite time horizon, as demonstrated in [5], [6]. Nevertheless, the pursuit of more robust estimation schemes have lead to the emergence of another class of observers based on Linear Matrix Inequalities (LMIs) in the literature. This integration offer guaranteed performance in terms of convergence rate, disturbance rejection, and noise attenuation. The LMI technique, combined with Lyapunov methods, directly addresses the nonlinearities in the system dynamics, enabling the design of observers to accurately estimate the state variables even in the presence of nonlinearities and despite noisy sensor readings and system uncertainties as shown by researchers in [7], [8], [9]. Researchers have effectively applied this technique to various real-world nonlinear system applications, showcasing its versatility such as fault diagnosis of a formation of satellites and tumor growth prediction in [10], [11].

In this work, we suggest a more generic nonlinear observer design strategy with an extended dynamic output which expands the range of systems on which the method can be applicable. The main contribution of the paper is proposing a filtered-like output based observer, which allows reducing the effect of the measurement noise in the state observer. Furthermore, to guarantee optimal performance, a system must be Input-to-State (ISS) stable since it ensures global stability, accounting for an error term based on the input's essential supremum norm [12]. To this end, we propose a novel Linear Matrix Inequality (LMI) condition, ensuring the Input-to-State Stability (ISS) property of the estimation error.

The paper is organised as follows. Section II, presents the problem formulation where the new output variable is introduced. The observer design is detailed in Section III. Finally, conclusions and future work are outlined in Section IV.

II. Problem Formulation

A. System description and motivation

We begin by presenting basic concepts and preliminaries that are essential in the proposed methodologies. We consider nonlinear systems in the form

$$\begin{cases} \dot{x} = \psi(x, w) \\ y = \phi(x, v) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $y \in \mathbb{R}^p$ is the vector of output measurements of the system. $w \in \mathbb{R}^s$ and $v \in \mathbb{R}^q$ are unknown external disturbances. The functions $\psi(\cdot, \cdot)$ and $\phi(\cdot, \cdot)$ are assumed to be Lipschitz-continuous, with respect to x, on $\Omega \subseteq \mathbb{R}^n$, where Ω is a positively invariant compact and convex set of the system (1). Then, the functions $\psi(\pi_\Omega(\cdot), w)$ and $\phi(\pi_\Omega(\cdot), v)$ are globally Lipschitz in $\mathbb{R}^n$, where $\pi_\Omega(x)$ is the *Hilbert* projection of x on the convex set Ω.

Usually in the literature, we propose the following state observer:

$$\dot{\hat{x}} = \psi(\hat{x}, 0) + \mathbb{L}\Big(y - \phi(\hat{x}, 0)\Big), \tag{2}$$

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where $\hat{x} \in \mathbb{R}^n$ is the estimate of the system state x and $\mathbb{L} \in \mathbb{R}^{n \times p}$ is the constant observer gain to be determined such that the estimation error $e = x - \hat{x}$ converges exponentially towards zero. However, with this kind of classical and natural observer, the observer gain is multiplied by the output nonlinearity and then by the disturbance in the measurement. This coupling between the observer gain and the output measurement can amplify the effect of the measurement noise. In this paper, the main objective consists in proposing a different state observer for the system (1). We will propose a filtered-like output based observer, which allows both avoiding the output nonlinearity and disturbances to be explicitly multiplied by the observer gain.

B. Output-based dynamic extension

The initial system is given by the equations (1). We perform a change of output variable by introducing the following system:

$$\begin{cases} \dot{\eta}(t) = \alpha y(t) \\ \eta(0) = \eta_0 \text{ (known)} \end{cases} \tag{3}$$

Since η_0, α, and $y(t)$ are known, then the state $\eta(t)$ may be considered as the new output measurement generated from the measurement $y(t)$. Therefore, we consider $\boldsymbol{y}(t) = \eta(t)$ as the new output, expressed in terms of the extended state vector ξ as follows:

$$\boldsymbol{y}(t) = \eta(t) = \overbrace{\begin{bmatrix}\mathbb{I}_p & 0\end{bmatrix}}^{C} \xi, \text{ with } \xi(t) \triangleq \begin{bmatrix}\eta(t) \\ x(t)\end{bmatrix} \tag{4}$$

where $\mathbb{I}_p$ is the identity matrix of dimension p. Let us define also the vector $\boldsymbol{\omega}$ which combines the w and v as follows:

$$\boldsymbol{\omega}(t) \triangleq \begin{bmatrix}w(t) \\ v(t)\end{bmatrix}.$$

This allows us to transform the equations of the original system into a new system with the state variable ξ:

$$\begin{cases} \dot{\xi} = f(\xi, \boldsymbol{\omega}) \triangleq \begin{bmatrix}\alpha\phi(x, v) \\ \psi(x, w)\end{bmatrix} \\ \boldsymbol{y} = \eta = C\xi \end{cases} \tag{5}$$

This transformed system incorporates the relationships between the output vector $\boldsymbol{y}$ and the system states represented by ξ. Note that the dynamics of the states are now given by $f(\xi, \boldsymbol{\omega})$, which is a function of the generalized state ξ and extended disturbance vector $\boldsymbol{\omega} \in \mathbb{R}^{n_w}$, where $n_w = s + q$. The state observer we will consider in this paper corresponds to the system (5) with linear outputs. Then, the observer synthesis and the Lyapunov analysis become simpler.

C. Preliminary tools

Before tackling the observer design problem, we need the following lemma.

***Lemma** 1 ([13]):* Consider a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$. Assume that φ is $\gamma_\varphi-$Lipschitz on a nonempty subset $\Omega \subseteq \mathbb{R}^n$, with γ_{φ_i} the Lipschitz constant of each component φ_i of φ. Then, there exist

- a Lipschitz extension $\breve{\varphi} \triangleq \varphi \circ \pi_\Omega : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ of φ, with $\breve{\varphi}(x) = \varphi(x), \forall x \in \Omega$;
- functions $\varphi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$
- constants $\underline{\gamma}_{\varphi_{ij}}$ and $\bar{\gamma}_{\varphi_{ij}}$

such that for all $X, Y \in \mathbb{R}^n, X \neq Y$, we have

$$\breve{\varphi}(X) - \breve{\varphi}(Y) = \sum_{i=1}^{p}\sum_{j=1}^{n} \varphi_{ij}(t)\mathcal{H}_{ij}^{p,n}\Big(X - Y\Big) \tag{6}$$

and

$$-\gamma_{\varphi_{ij}} \leq \underline{\gamma}_{\varphi_{ij}} \leq \varphi_{ij}(t) \leq \bar{\gamma}_{\varphi_{ij}} \leq \gamma_{\varphi_{ij}} \tag{7}$$

where

$$\varphi_{ij}(t) \triangleq \varphi_{ij}\Big(X^{Y_{j-1}}, X^{Y_j}\Big),\ \mathcal{H}_{ij}^{p,n} \triangleq \mathrm{e}_p(i)\mathrm{e}_n^\top(j).$$

Proof: The detail of the proof is given in [13]. It is based on the use of the properties of the *Hilbert* projection function, π_Ω, on the convex set Ω. ∎

III. Observer Design by using Output-Based Dynamic Extension

This section is devoted to the observer design method proposed in this paper. In addition to the new observer structure and the output-based extended dynamics technique, we propose novel LMI-based synthesis procedure guaranteeing the Input-to-State (ISS) property of the estimation error.

A. Formulation of the observer design problem

Instead of the standard state observer form (2), we propose a new observer structure by exploiting the extended dynamics (5). The observer is driven by the created output measurement $\boldsymbol{y}(t)$. This new output $\boldsymbol{y}(t)$ plays the role of a filter which makes it possible to filter the disturbances or the measurement noise. The estimation strategy is depicted in Figure 1.

The state observer form is described by the following equations:

$$\begin{cases} \dot{\hat{\xi}} = \breve{f}(\hat{\xi}, 0) + \mathcal{L}\Big(\boldsymbol{y} - C\hat{\xi}\Big) \\ \hat{x} = \begin{bmatrix}0 & \mathbb{I}_n\end{bmatrix}\hat{\xi} \end{cases} \tag{8}$$

where

$$\breve{f}(\hat{\xi}, \boldsymbol{\omega}) = f\left(\pi_{\Omega_\xi}(\hat{\xi}), \boldsymbol{\omega}\right),\ \forall \boldsymbol{\omega} \in \Omega_{\boldsymbol{\omega}} \ni 0,\ \Omega_\xi \triangleq \mathbb{R}^p \times \Omega,$$

and $\mathcal{L} \in \mathbb{R}^{(n+p)\times p}$ is the constant observer gain to be computed such that the estimation error $\epsilon(t) \triangleq x(t) - \hat{x}(t)$ satisfies an exponential ISS bound to be determined later. To develop a synthesis method based on Lyapunov analysis, let us define the extended error vector $\epsilon_\xi(t) \triangleq \xi(t) - \hat{\xi}(t)$, which satisfies the following dynamics:

$$\dot{\epsilon}_\xi = \Big[f(\xi, \boldsymbol{\omega}) - \breve{f}(\hat{\xi}, 0)\Big] - \mathcal{L}C\epsilon_\xi. \tag{9}$$

We decompose the nonlinear term as follows:

$$f(\xi, \boldsymbol{\omega}) - \breve{f}(\hat{\xi}, 0) = \Big(f(\xi, \boldsymbol{\omega}) - \breve{f}(\hat{\xi}, \boldsymbol{\omega})\Big) + \Big(\breve{f}(\hat{\xi}, \boldsymbol{\omega}) - \breve{f}(\hat{\xi}, 0)\Big). \tag{10}$$

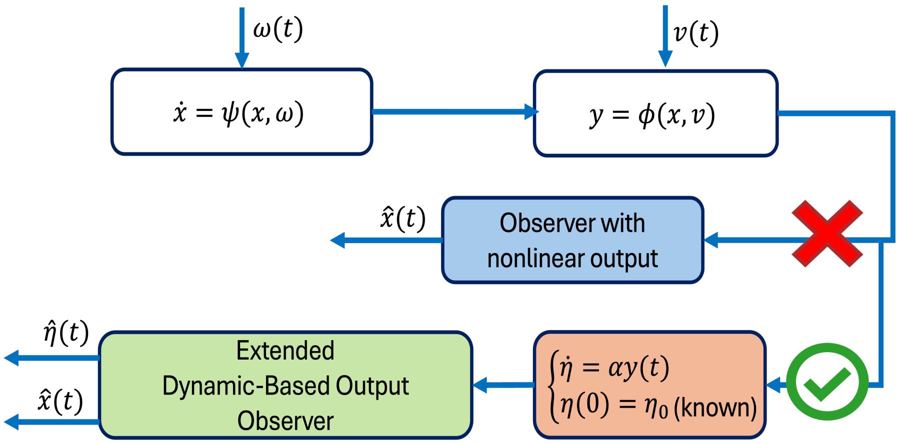

Fig. 1: Output-Based Dynamic Extension Technique.

Since $\hat{\xi} \in \Omega_\xi$, then we have $f(\xi, \boldsymbol{\omega}) = \check{f}(\xi, \boldsymbol{\omega})$. It follows that from Lemma 1, there exist function $f^\xi_{ij} : \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \longrightarrow \mathbb{R}$ and constants $\underline{\gamma}_{f^\xi_{ij}}$ and $\bar{\gamma}_{f^\xi_{ij}}$ such that

$$f(\xi, \boldsymbol{\omega}) - \check{f}(\hat{\xi}, \boldsymbol{\omega}) = \sum_{i=1}^{n_p}\sum_{j=1}^{n_p} f^\xi_{ij}(t)\mathcal{H}^{n_p}_{ij}\left(\xi - \hat{\xi}\right) \tag{11}$$

and

$$-\gamma_{f^\xi_{ij}} \leq \underline{\gamma}_{f^\xi_{ij}} \leq f^\xi_{ij}(t) \leq \bar{\gamma}_{f^\xi_{ij}} \leq \gamma_{f^\xi_{ij}}. \tag{12}$$

Similarly, there exist functions $f^{\boldsymbol{\omega}}_{ij} : \mathbb{R}^{n_w} \times \mathbb{R}^{n_w} \longrightarrow \mathbb{R}$ and constants $\underline{\gamma}_{f^{\boldsymbol{\omega}}_{ij}}$ and $\bar{\gamma}_{f^{\boldsymbol{\omega}}_{ij}}$ such that

$$\check{f}(\hat{\xi}, \boldsymbol{\omega}) - \check{f}(\hat{\xi}, 0) = \sum_{i=1}^{n_p}\sum_{j=1}^{n_w} f^{\boldsymbol{\omega}}_{ij}(t)\mathcal{H}^{n_p,n_w}_{ij}\boldsymbol{\omega}(t) \tag{13}$$

and

$$-\gamma_{f^{\boldsymbol{\omega}}_{ij}} \leq \underline{\gamma}_{f^{\boldsymbol{\omega}}_{ij}} \leq f^{\boldsymbol{\omega}}_{ij}(t) \leq \bar{\gamma}_{f^{\boldsymbol{\omega}}_{ij}} \leq \gamma_{f^{\boldsymbol{\omega}}_{ij}}. \tag{14}$$

Before tackling the Lyapunov analysis on the estimation error dynamics (9), we introduce some transformations and notations to write the dynamics equations in a simpler and convenient form. Then, we introduce the following notations:

$$\sum_{i=1}^{n_p}\sum_{j=1}^{n_p} f^\xi_{ij}(t)\mathcal{H}^{n_p}_{ij}\left(\xi - \hat{\xi}\right) = \mathcal{A}_{n_p} \times \Pi_\xi(\epsilon_\xi) \tag{15}$$

$$\sum_{i=1}^{n_p}\sum_{j=1}^{n_w} f^{\boldsymbol{\omega}}_{ij}(t)\mathcal{H}^{n_p,n_w}_{ij}\left(\xi - \hat{\xi}\right) = \mathcal{A}_{n_p,n_w} \times \Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \tag{16}$$

where

$$\mathcal{A}_{\ell_1,\ell_2} \triangleq \text{blkdiag}\Big(\overbrace{\mathbf{e}_{\ell_1,\ell_2}, \dots, \mathbf{e}_{\ell_1,\ell_2}}^{\ell_2 \text{ times}}\Big),$$

and

$$\mathbf{e}_{\ell_1,\ell_2} \triangleq \overbrace{\left[\mathrm{e}^\top_{\ell_1}(1) \quad \dots \quad \mathrm{e}^\top_{\ell_1}(\ell_2)\right]}^{\ell_2 \text{ times}}$$

by setting $\mathcal{A}_{\ell_1,\ell_1} = \mathcal{A}_{\ell_1}$. In (15) and (16), we have $\ell_1 = n_p$ and $\ell_2 = n_w$. The terms $\Pi_\xi(\epsilon_\xi)$ and $\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega})$ are defined in (17) and (18), respectively.

Hence, the estimation error equation (9) is rewritten as follows:

$$\dot{\epsilon}_\xi = \mathcal{A}_{n_p}\Pi_\xi(\epsilon_\xi) - \mathcal{L}C\epsilon_\xi + \mathcal{A}_{n_p,n_w}\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}). \tag{19}$$

B. New LMI-based ISS bound

This section is devoted to the main theorem, which provides new sufficient LMI conditions ensuring the ISS property of the estimation error $\epsilon(t)$. Before stating the main theorem, we introduce the following matrices defined by the bounds of the functions f^ξ_{ij} and $f^{\boldsymbol{\omega}}_{ij}$:

$$\underline{\Lambda}^\xi_i \triangleq \text{blkdiag}\Big(\underline{\gamma}_{f^\xi_{ij}}\mathbb{I}_{n_p}, j = 1, \dots, n_p\Big)$$

$$\bar{\Lambda}^\xi_i \triangleq \text{blkdiag}\Big(\bar{\gamma}_{f^\xi_{ij}}\mathbb{I}_{n_p}, j = 1, \dots, n_p\Big)$$

$$\underline{\Lambda}_\xi \triangleq \text{blkdiag}\Big(\underline{\Lambda}^{f^\xi}_i, i = 1, \dots, n_p\Big)$$

$$\bar{\Lambda}_\xi \triangleq \text{blkdiag}\Big(\bar{\Lambda}^{f^\xi}_i, i = 1, \dots, n\Big)$$

$$\underline{\Lambda}^{\boldsymbol{\omega}}_i \triangleq \text{blkdiag}\Big(\underline{\gamma}_{f^{\boldsymbol{\omega}}_{ij}}\mathbb{I}_{n_w}, j = 1, \dots, n_w\Big)$$

$$\bar{\Lambda}^{\boldsymbol{\omega}}_i \triangleq \text{blkdiag}\Big(\bar{\gamma}_{f^{\boldsymbol{\omega}}_{ij}}\mathbb{I}_{n_w}, j = 1, \dots, n_w\Big)$$

$$\underline{\Lambda}_{\boldsymbol{\omega}} \triangleq \text{blkdiag}\Big(\underline{\Lambda}^{\boldsymbol{\omega}}_i, i = 1, \dots, n_p\Big)$$

$$\bar{\Lambda}_{\boldsymbol{\omega}} \triangleq \text{blkdiag}\Big(\bar{\Lambda}^{\boldsymbol{\omega}}_i, i = 1, \dots, n_p\Big).$$

We also introduce the following symmetric and positive definite matrices S_ξ and $S_{\boldsymbol{\omega}}$ as follows:

$$S_\xi \triangleq \text{blkdiag}\Big(S^\xi_1, \dots, S^\xi_{n_p}\Big), \qquad S^\xi_i \triangleq \text{blkdiag}\Big(S^\xi_{i1}, \dots, S^\xi_{in_p}\Big) \tag{20}$$

$$\Pi_\xi(\epsilon_\xi) \triangleq \text{blkdiag}\Big(f^\xi_{ij}(t)\mathbb{I}_{n_p}, j=1,\dots,n_p, i=1,\dots,n_p\Big)\mathbb{H}\epsilon_\xi$$
$$= \begin{bmatrix} \begin{pmatrix} f^\xi_{11}\mathbb{I}_{n_p} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & f^\xi_{1n_p}\mathbb{I}_{n_p}\end{pmatrix} & 0 & \dots \\ \vdots & \ddots & \vdots \\ 0 & \dots & \begin{pmatrix} f^\xi_{n_p}\mathbb{I}_{n_p} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & f^\xi_{n_p}\mathbb{I}_{n_p}\end{pmatrix} \end{bmatrix} \overbrace{\begin{bmatrix} \left.\begin{pmatrix}\mathbb{I}_{n_p} \\ \vdots \\ \mathbb{I}_{n_p}\end{pmatrix}\right\} n_p \text{ times} \\ \vdots \\ \left.\begin{pmatrix}\mathbb{I}_{n_p} \\ \vdots \\ \mathbb{I}_{n_p}\end{pmatrix}\right\} n_p \text{ times}\end{bmatrix}}^{\mathbb{H}} \epsilon_\xi \tag{17}$$

$$\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \triangleq \text{blkdiag}\Big(f^{\boldsymbol{\omega}}_{ij}(t)\mathbb{I}_{n_w}, j=1,\dots,n_w, i=1,\dots,n_p\Big)\mathbb{J}\boldsymbol{\omega}$$
$$= \begin{bmatrix} \begin{pmatrix} f^{\boldsymbol{\omega}}_{11}\mathbb{I}_{n_w} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & f^{\boldsymbol{\omega}}_{1n_w}\mathbb{I}_{n_w}\end{pmatrix} & 0 & \dots \\ \vdots & \ddots & \vdots \\ 0 & \dots & \begin{pmatrix} f^{\boldsymbol{\omega}}_{n_p1}\mathbb{I}_{n_w} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & f^{\boldsymbol{\omega}}_{n_pn_w}\mathbb{I}_{n_w}\end{pmatrix} \end{bmatrix} \overbrace{\begin{bmatrix} \left.\begin{pmatrix}\mathbb{I}_{n_w} \\ \vdots \\ \mathbb{I}_{n_w}\end{pmatrix}\right\} n_w \text{ times} \\ \vdots \\ \left.\begin{pmatrix}\mathbb{I}_{n_w} \\ \vdots \\ \mathbb{I}_{n_w}\end{pmatrix}\right\} n_w \text{ times}\end{bmatrix}}^{\mathbb{J}} \boldsymbol{\omega} \tag{18}$$

$$S_{\boldsymbol{\omega}} \triangleq \text{blkdiag}\Big(S^{\boldsymbol{\omega}}_1,\dots,S^{\boldsymbol{\omega}}_{n_p}\Big),$$
$$S^{\boldsymbol{\omega}}_i \triangleq \text{blkdiag}\Big(S^{\boldsymbol{\omega}}_{i1},\dots,S^{\boldsymbol{\omega}}_{in_w}\Big). \tag{21}$$

where S^ξ_{ij} and $S^{\boldsymbol{\omega}}_{ij}$ are symmetric and positive definite matrices of appropriate dimensions.

The main result of this paper is formulated in the following theorem.

***Theorem** 1:* Assume that there exist a symmetric positive definite matrix $\mathcal{P}$, two symmetric positive definite matrices S_ξ and $S_{\boldsymbol{\omega}}$ under the form (20)-(21), and a matrix $\mathcal{X}$ of adequate dimensions such that the LMI (23) is satisfied. Then the estimation error $\epsilon(t)$ with the observer gain $\mathcal{L} = \mathcal{P}^{-1}\mathcal{X}$ satisfies the following exponential ISS bound:

$$\|\epsilon(t)\| \le \max\left(\sqrt{\frac{4\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})}}\|\epsilon_0\|\mathrm{e}^{-\frac{\beta}{2}t}, \sqrt{\frac{4\lambda}{\beta\lambda_{\min}(\mathcal{P})}}\sup_{s\in[0,\ t]}\|\boldsymbol{\omega}(s)\|\right). \tag{22}$$

Proof: Let $\vartheta(\epsilon_\xi) = \epsilon_\xi^\top\mathcal{P}\epsilon_\xi$ be the Lyapunov function candidate, where $\mathcal{P} = \mathcal{P}^\top > 0$. We will show that under the condition (23), we have

$$\theta(\epsilon_\xi,\boldsymbol{\omega}) \triangleq \dot{\vartheta}(\epsilon_\xi) + \beta\vartheta(\epsilon_\xi) - \lambda\boldsymbol{\omega}^\top\boldsymbol{\omega} < 0. \tag{24}$$

By developing the derivative of $\vartheta(\epsilon_\xi)$ along the trajectories of (19), and by changing the variable $\mathcal{X} = \mathcal{P}\mathcal{L}$, we obtain

$$\theta(\epsilon_\xi,\boldsymbol{\omega}) = \begin{bmatrix}\epsilon_\xi \\ \Pi_\xi(\epsilon_\xi) \\ \Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \\ \boldsymbol{\omega}\end{bmatrix}^\top \Omega \begin{bmatrix}\epsilon_\xi \\ \Pi_\xi(\epsilon_\xi) \\ \Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \\ \boldsymbol{\omega}\end{bmatrix} \tag{25}$$

where

$$\Omega \triangleq \begin{bmatrix} -\mathcal{X}-\mathcal{X}^\top+\beta\mathcal{P} & \mathcal{P}\mathcal{A}_{n_p} & \mathcal{P}\mathcal{A}_{n_p,n_w} & 0 \\ \mathcal{A}^\top_{n_p}\mathcal{P} & 0 & 0 & 0 \\ \mathcal{A}^\top_{n_p,n_w}\mathcal{P} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda\mathbb{I}_{n_w}\end{bmatrix}. \tag{26}$$

On other hand, from (12) and (14), we have

$$\vartheta_\xi \triangleq \Big(\Pi_\xi(\epsilon_\xi) - \bar{\Lambda}_\xi\mathbb{H}\epsilon_\xi\Big)^\top S_\xi\Big(\Pi_\xi(\epsilon_\xi) - \underline{\Lambda}_\xi\mathbb{H}\epsilon_\xi\Big) + \Big(\Pi_\xi(\epsilon_\xi) - \underline{\Lambda}_\xi\mathbb{H}\epsilon_\xi\Big)^\top S_\xi\Big(\Pi_\xi(\epsilon_\xi) - \bar{\Lambda}_\xi\mathbb{H}\epsilon_\xi\Big) \le 0 \tag{27}$$

$$\vartheta_\omega \triangleq \Big(\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) - \bar{\Lambda}_\omega\mathbb{J}\boldsymbol{\omega}\Big)^\top S_\omega\Big(\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) - \underline{\Lambda}_\omega\mathbb{J}\boldsymbol{\omega}\Big) + \Big(\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) - \underline{\Lambda}_\omega\mathbb{J}\boldsymbol{\omega}\Big)^\top S_\omega\Big(\Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) - \bar{\Lambda}_\omega\mathbb{J}\boldsymbol{\omega}\Big) \le 0 \tag{28}$$

It is obvious that $\theta(\epsilon_\xi,\boldsymbol{\omega}) \le \theta(\epsilon_\xi,\boldsymbol{\omega}) - (\vartheta_\xi + \vartheta_{\boldsymbol{\omega}})$. After

$$\Theta \triangleq \begin{bmatrix} -\mathbb{H}^\top \mathbb{A}_\xi(S_\xi)\mathbb{H} + \mathcal{C}(\mathcal{X}) + \beta\mathcal{P} & \mathcal{P}\mathcal{A}_{n_p} + \mathbb{H}^\top \left(\bar{\Lambda}_\xi + \underline{\Lambda}_\xi\right)^\top S_\xi & \mathcal{P}\mathcal{A}_{n_p,n_w} & 0 \\ (\star) & -2S_\xi & 0 & 0 \\ (\star) & (\star) & -2S_{\boldsymbol{\omega}} & S_{\boldsymbol{\omega}}\left(\bar{\Lambda}_{\boldsymbol{\omega}} + \underline{\Lambda}_{\boldsymbol{\omega}}\right)\mathbb{J} \\ (\star) & (\star) & (\star) & -\lambda\mathbb{I}_{n_w} - \mathbb{J}^\top \mathbb{A}_{\boldsymbol{\omega}}(S_{\boldsymbol{\omega}})\mathbb{J} \end{bmatrix} < 0 \tag{23}$$

$$\mathcal{C}(\mathcal{X}) \triangleq -\mathcal{X}C - C^\top \mathcal{X}^\top, \ \mathbb{A}_\xi(S_\xi) \triangleq \underline{\Lambda}_\xi^\top S_\xi \bar{\Lambda}_\xi + \bar{\Lambda}_\xi^\top S_\xi \underline{\Lambda}_\xi, \ \mathbb{A}_{\boldsymbol{\omega}}(S_{\boldsymbol{\omega}}) \triangleq \underline{\Lambda}_{\boldsymbol{\omega}}^\top S_{\boldsymbol{\omega}} \bar{\Lambda}_{\boldsymbol{\omega}} + \bar{\Lambda}_{\boldsymbol{\omega}}^\top S_{\boldsymbol{\omega}} \underline{\Lambda}_{\boldsymbol{\omega}}$$

expanding the terms ϑ_ξ and $\vartheta_{\boldsymbol{\omega}}$, we obtain

$$\theta(\epsilon_\xi, \boldsymbol{\omega}) - (\vartheta_\xi + \vartheta_{\boldsymbol{\omega}}) = \begin{bmatrix} \epsilon_\xi \\ \Pi_\xi(\epsilon_\xi) \\ \Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \\ \boldsymbol{\omega} \end{bmatrix}^\top \Theta \begin{bmatrix} \epsilon_\xi \\ \Pi_\xi(\epsilon_\xi) \\ \Pi_{\boldsymbol{\omega}}(\boldsymbol{\omega}) \\ \boldsymbol{\omega} \end{bmatrix} \tag{29}$$

where Θ is defined in (23). This means that under the condition (23), we have $\theta(\epsilon_\xi, \boldsymbol{\omega}) < 0$, which implies that

$$\dot{\vartheta}(\epsilon_\xi) < -\beta\vartheta(\epsilon_\xi) + \lambda\|\boldsymbol{\omega}(t)\|^2. \tag{30}$$

Hence, from the well-known comparison theorem, we deduce the following inequality:

$$\vartheta(\epsilon_\xi(t)) \leq \vartheta(\epsilon_\xi(0))\mathrm{e}^{-\beta t} + \lambda\mathrm{e}^{-\beta t}\int_0^t \mathrm{e}^{\beta s}\|\boldsymbol{\omega}(s)\|^2 \mathrm{d}s.$$

Since

$$\lambda_{\min}(\mathcal{P})\|\epsilon_\xi\|^2 \leq \vartheta(\epsilon_\xi(t)) \leq \lambda_{\max}(\mathcal{P})\|\epsilon_\xi\|^2$$

then by developing the integral term, we get

$$\|\epsilon_\xi(t)\|^2 \leq \frac{\lambda_{\max}(\mathcal{P})}{\lambda_{\min}(\mathcal{P})}\|\epsilon_\xi(0)\|^2\mathrm{e}^{-\beta t} + \frac{\lambda}{\beta\lambda_{\min}(\mathcal{P})}\sup_{s\in[0,\ t]}\|\boldsymbol{\omega}(s)\|^2. \tag{31}$$

By construction of $\eta(t)$ in (3), η_0 is known, then we can take $\hat{\xi}_1(0) = \eta_0$, which means that $\|\epsilon_\xi(0)\| = \|\epsilon(0)\|$. Finally, since $\|\epsilon(t)\| \leq \|\epsilon_\xi(t)\|, \forall t \geq 0$, then the bound (22) is inferred. This ends the proof of Theorem 1. ■

IV. Conclusion and future work

In this paper, we established an estimation approach for continuous-time nonlinear systems affected with state and measurement noises. We proved that the proposed observer guarantees the Input-to-State Stability (ISS) property through a novel LMI condition. Thus, reducing the effect of the measurement noise in the state observer is made possible, enhancing the accuracy and reliability of the estimated states.

Future work will be dedicated to exploring the practical applicability of this state estimation method across various real-world scenarios of autonomous systems, mainly in the domain of precision agriculture. In this frame, a deeper version of this work will address an application to a monitoring drone operating in an Adaptive Vertical Farm (AVF), a new vertical farming concept presented in [14], [15], to showcase the efficacy and versatility of our proposed methodology through detailed comparison with existing methods in the literature.

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