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# Robust Sliding Manifold Design for Uncertain Linear Systems

Edoardo Vacchini, Nikolas Sacchi, Michele Cucuzzella, and Antonella Ferrara

***Abstract*— This paper proposes a novel approach for the design of stabilizing sliding manifolds for linear systems affected by model uncertainties and external disturbances. In classical sliding mode control approaches, rejecting model uncertainties and external disturbances often relies on designing a discontinuous control law with a suitable gain. Specifically, the greater the uncertainty, the larger the control gain. However, this approach might be detrimental to the plant. Instead, the proposed technique deals with this problem by focusing on the design of a suitable sliding manifold, where stability is guaranteed despite model uncertainties. This approach exhibits several benefits such as not needing any further identification process and designing a smaller control gain.**

***Index Terms*— sliding mode control, sliding manifold design, uncertain systems**

## I. INTRODUCTION

Sliding Mode Control (SMC) is a well-known robust nonlinear control technique which owes its popularity to its straightforward implementation and proven robustness with respect to external disturbances [1]–[3]. In particular, SMC aims to enforce the stability of a given system by means of the design of a suitable *sliding manifold*, on which the so-called equivalent dynamics exhibits desired convergence properties. Once the manifold is designed, a control law is designed to guarantee finite-time convergence of the system trajectory onto the designed manifold.

Although different SMC approaches have been proposed over the years, e.g. Integral SMC [4], Sub-optimal Second Order SMC [5], SMC with optimal reaching [6], event-based SMC [7], less attention has been devoted to the design of the sliding manifold, focusing only on the development of new control algorithms.

Only in the latest years the problem of designing a stabilizing manifold has become more and more relevant and several works in the literature explore different design techniques in the nominal case, i.e., considering no model uncertainties. For instance, [8] reduces the sliding manifold design to a deterministic free-cost Linear Quadratic Regulation (LQR) problem, [9] relates the design of the equivalent dynamics on the manifold to the Ackermann's formula, which is iteratively used to achieve the desired placement of the poles of the equivalent system.

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The problem of the sliding surface design in presence of unmatched disturbances is tackled in [10] and [11]. In [10] the effect of the disturbance is minimized by choosing the manifold as the transposed of the control effectiveness matrix, while in [11] the manifold is modified with an integral term and the disturbance is estimated with an observer and treated separately.

However, two main flaws arise in the cited works. Specifically, either the unmatched uncertainty is disregarded, which is a classical issue in SMC theory, or it is assumed to fully know the system dynamics, which in practice is unrealistic. Although the (matched) model uncertainty can be gathered in the (matched) disturbance term and therefore rejected once the manifold is attained, this comes at the cost of a larger control effort compared to the one required with an effective sliding manifold design.

To cope with this problem, data-driven approaches for the design of sliding surface in the case of systems with uncertain dynamics have been proposed in the literature (see, e.g., [12] and [13]). However, such data-driven solutions require gathering a large amount of data in order to either satisfy the persistent excitation assumption and effectively design the sliding manifold, or to train and validate the used models.

From this point of view, if one correctly designs the sliding manifold despite the model uncertainty, it is possible to ignore them when designing the gain of the SMC controller, while still having a stable controlled system.

This paper proposes a sliding surface design approach for linear time-invariant (LTI) systems which guarantees that the controlled system is stable even in presence of model uncertainty and external disturbances, without relying on any additional identification procedure. In particular, both matched and unmatched disturbances are considered, and it is proven that, besides rejecting matched disturbances, the controlled system is input-to-state stable with respect to unmatched disturbances.

The paper is organized as follows: in Section II we introduce the fundamental concepts upon which the main result is built. In Section III the main result of the paper is presented, while in Section IV the proposed sliding manifold design technique is assessed on an academic example. Finally, in Section V conclusions are drawn.

*Notation:* Given a matrix $M \in \mathbb{R}^{n\times n}$, $M \prec 0$ denotes that $M$ is Hurwitz. Furthermore, we denote by $\overline{\lambda}_M$ the maximum eigenvalue of $M$, and by $\underline{\lambda}_M$ its smallest eigenvalue. Given two matrices $M, N \in \mathbb{R}^{n\times n}$, $M \not\prec N$ indicates the fact that the matrix $M - N$ is not Hurwitz.

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## II. Preliminaries and Problem Statement

Consider a perturbed and uncertain LTI system

$$\begin{aligned}\dot{\xi}(t) &= \bar{A}\xi(t) + \bar{B}u(t) + h(t) \\ &= \left(\hat{\bar{A}} + \Delta\bar{A}\right)\xi(t) + \left(\hat{\bar{B}} + \Delta\bar{B}\right)u(t) + h(t), \qquad (1)\end{aligned}$$

where $\xi(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the control input, and $h(t) \in \mathbb{R}^n$ the disturbance. Moreover, $\bar{A}, \hat{\bar{A}}, \Delta\bar{A} \in \mathbb{R}^{n\times n}$, and $\bar{B}, \hat{\bar{B}}, \Delta\bar{B} \in \mathbb{R}^{n\times m}$. In particular, $\Delta\bar{A}$ and $\Delta\bar{B}$ represent the model uncertainty.

As described in [3], if matrix $\bar{B}$ is full column rank, then there exists a change of variables $x(t) = T\xi(t)$, with $T \in \mathbb{R}^{n\times n}$ non singular, that allows to rewrite system (1) in the so-called *canonical reduced* (or *regular*) *form* characterized by the matrices $A := T\bar{A}T^{-1}$ and $B := T\bar{B}$ and the disturbance in the new coordinates, i.e., $d(t) := Th(t)$.

In this form, the state $x(t)$ and the disturbance $d(t)$ can be split, respectively, into two vectors, i.e., $x_1(t), d_1(t) \in \mathbb{R}^{n-m}$ and $x_2(t), d_2(t) \in \mathbb{R}^m$, such that (1) can be rewritten as

$$\dot{x}_1(t) = A_{1,1}x_1(t) + A_{1,2}x_2(t) + d_1(t) \qquad (2a)$$

$$\dot{x}_2(t) = A_{2,1}x_1(t) + A_{2,2}x_2(t) + Bu(t) + d_2(t), \qquad (2b)$$

where $A_{1,1} \in \mathbb{R}^{(n-m)\times(n-m)}$, $A_{1,2} \in \mathbb{R}^{(n-m)\times m}$, $A_{2,1} \in \mathbb{R}^{m\times(n-m)}$ $A_{2,2} \in \mathbb{R}^{m\times m}$, and $B \in \mathbb{R}^{m\times m}$. Note that from now, the dependence on time $t$ is neglected when obvious from the context.

The uncertainty on the matrices $\bar{A}$ and $\bar{B}$ is transferred to the matrices $A_{i,j}$ and $B$ as follows

$$A_{i,j} = \hat{A}_{i,j} + \Delta_{i,j}, \ \forall (i,j), \qquad (3a)$$

$$B = \hat{B} + \Delta_B, \qquad (3b)$$

with $\hat{A}_{i,j}$ and $\hat{B}$ being the nominal part of matrix $A_{i,j}$ and $B$ respectively, and matrices $\Delta_{i,j}$ $\Delta_B$ being the corresponding uncertainty.

*Assumption 1:* The unmatched disturbance $d_1(t)$ and the matched one $d_2(t)$ are bounded, i.e.,

$$\|d_1(t)\|_2 \leq \bar{d}_1 \qquad \|d_2(t)\|_2 \leq \bar{d}_2,$$

where $\bar{d}_1$ and $\bar{d}_2$ are known positive constants.

*Assumption 2:* The model uncertainty on $B$ satisfies the following inequality

$$\left\|\hat{B}\right\|_2 > \|\Delta_B\|_2 .$$

Also, the model uncertainties satisfy

$$\|\Delta_{i,j}\|_2 \leq \delta_{i,j} \qquad \|\Delta_B\|_2 \leq \delta_B,$$

where $\delta_{i,j}$ and $\delta_B$ are known positive constants.

Note that the first inequality in Assumption 2 is needed to guarantee that the control direction is not affected by the uncertainty on $B$, which is a key assumption to design the gain of the discontinuous controller.

For the linear system (2), it is possible to design the sliding variable $\sigma \in \mathbb{R}^m$ as the linear combination of the states, i.e.,

$$\sigma := \Lambda x = \Lambda_1 x_1 + \Lambda_2 x_2, \qquad (4)$$

where $\Lambda \in \mathbb{R}^{m\times n}$, $\Lambda_1 \in \mathbb{R}^{m\times(n-m)}$, $\Lambda_2 \in \mathbb{R}^{m\times m}$. Without loss of generality, it is possible to define $\Lambda_2 = I_m$ and $\Lambda_1 = K$, so that when the system is in sliding mode, i.e., when $\sigma = 0$, $x_2$ can be written as a function $x_1$, i.e.,

$$\sigma = Kx_1 + x_2 = 0 \iff x_2 = -Kx_1. \qquad (5)$$

Note that, since the sliding variable $\sigma$ is regarded as the system output, then, in order to design a first-order SMC, the control input $u$ must appear in the first time derivative of $\sigma$. This is straightforward to verify in our case. Indeed, by virtue of the choice of $\Lambda$, $\sigma$ in (4) satisfies

$$\begin{aligned}\dot{\sigma} &= \Lambda_1\dot{x}_1 + \Lambda_2\dot{x}_2 \\ &= K\dot{x}_1 + \dot{x}_2 \\ &= K\left(A_{1,1}x_1 + A_{1,2}x_2 + d_1\right) + \\ &\quad + A_{2,1}x_1 + A_{2,2}x_2 + Bu + d_2\end{aligned}$$

along the solution of (2), where $B$ is by assumption a full rank matrix. Moreover, one can notice that the choice of $K$ does not affect the relative degree of the system, which in this case is said to be *strictly equal to 1* according to the definition given in [14].

The motion of a system in closed-loop with a sliding mode controller consists of two phases. The first one is the so-called *reaching* phase, in which the state trajectory converges in a finite time $t_r$ onto the sliding manifold $\sigma = 0$, while the second one is the *sliding* phase, in which the system trajectory remains on the manifold $\sigma = 0$.

The system in sliding can be written as an autonomous system referred to as the *equivalent system*, which can be obtained by imposing $\sigma = \dot{\sigma} = 0$. For the canonical form (2) the equivalent dynamics is described by $n-m$ differential equations and $m$ algebraic equations, i.e.,

$$\begin{cases}\dot{x}_1 = \left(A_{1,1} - A_{1,2}K\right)x_1 + d_1 \\ x_2 = -Kx_1\end{cases} \qquad (6)$$

Then, if the pair $(A_{1,1}, A_{1,2})$ is controllable, and $K$ in (5) is well designed, it is possible to establish the input-to-state stability of the dynamics of the equivalent system in (6) with respect to the unmatched disturbance $d_1$. Asymptotic stability can be established for linear systems only if $d_1$ is constant.

*Definition 1 (Input-to-State Stability [15]):* The dynamical system $\dot{\xi} = f(\xi, \nu)$ is input-to-state stable (ISS) with respect to the input $\nu$ if and only if there exist a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, three class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, and a class $\mathcal{K}$ function $\alpha_4(\cdot)$ such that

$$\alpha_1\left(\|\xi\|\right) \leq V(\xi) \leq \alpha_2\left(\|\xi\|\right) \qquad (7a)$$

$$\dot{V}(\xi) \leq -\alpha_3\left(\|\xi\|\right) + \alpha_4\left(\|\nu\|\right). \qquad (7b)$$

Then, $V$ is referred to as an ISS-Lyapunov function, and the system is said to be ISS with respect to $\nu$.

A necessary condition for the system's input-to-state stability, is that the matrix $K$ has to be designed so that the equivalent system without $d_1$ is asymptotically stable [16, Section 2]. However, model uncertainties can make the system unstable since there exist different choices of $K$

that stabilize the nominal system but not the uncertain one, i.e., in general $K$ might satisfy $\hat{A}_{1,1} - \hat{A}_{1,2}K \prec 0$ and $A_{1,1} - A_{1,2}K \not\prec 0$.

For this reason, given the LTI system (2), we seek for a condition that allows to design a matrix $K$ on the basis of the nominal system and that stabilizes the system despite model uncertainties. In the next section we will present such a condition, derived from Lyapunov stability analysis on the equivalent system.

Note that in this paper we focus only on the design of the sliding manifold, i.e., on the *sliding* phase, while the design of the *reaching* phase is standard and can be enforced via classical techniques [1]–[3].

## III. Proposed Manifold Design Technique

In this Section we aim to solve the problem formulated in the previous section, i.e., we propose a systematic procedure to design a sliding function $\sigma$ given by (4), such that on the manifold $\sigma = 0$ the system (2) exhibits desired stability properties. More precisely, considering that the pair $(\hat{A}, \hat{B})$ is the only known information about the plant, we aim to provide a condition on $K$ that allows to asymptotically stabilize the system by making sure that the nominal part of the equivalent dynamics is dominant with respect to the uncertain one.

*Theorem 1 (Robust Stability of the Manifold):* Consider the uncertain linear system (2) and let Assumptions 1 and 2 hold. Given a sliding variable $\sigma$ as in (4), let $\hat{P}$ and $\hat{Q}$ be symmetric positive definite (SPD) matrices satisfying the the Lyapunov equation associated with the nominal dynamics of the equivalent system (6), i.e.,

$$\left(\hat{A}_{1,1} - \hat{A}_{1,2}K\right)^\top \hat{P} + \hat{P}\left(\hat{A}_{1,1} - \hat{A}_{1,2}K\right) = -\hat{Q}. \quad (8)$$

If the following inequality is satisfied

$$\frac{\underline{\lambda}_{\hat{Q}}}{\overline{\lambda}_{\hat{P}}} > 2\left(\delta_{1,1} + \delta_{1,2}\left\|K\right\|_2\right) + \gamma, \quad (9)$$

with $\gamma \in \mathbb{R}_{>0}$ arbitrarily small, then (6) is ISS with respect to the unmatched disturbance $d_1$.

*Proof:* Consider a candidate ISS-Lyapunov function associated with the equivalent dynamics (6), i.e.,

$$V(x_1) \coloneqq x_1^\top \hat{P} x_1. \quad (10)$$

To comply with the condition (7a), let $\varepsilon \in \mathbb{R}_{>0}$ be an arbitrarily small positive constant. Then, since $\hat{P}$ is SPD, its eigenvalues are all positive and it is possible to choose $\alpha_1 = \underline{\lambda}_{\hat{P}} - \varepsilon > 0$ and $\alpha_2 = \overline{\lambda}_{\hat{P}} + \varepsilon > 0$ such that $V$ in (10) satisfies

$$\alpha_1\|x_1\|_2^2 \le V(x_1) \le \alpha_2\|x_1\|_2^2,$$

where $\alpha_1\|x_1\|_2^2$ and $\alpha_2\|x_1\|_2^2$ are class $\mathcal{K}_\infty$ functions, since they are radially unbounded, strictly increasing in $\|x_1\|_2^2$, and equal to 0 when $\|x_1\|_2^2 = 0$. Thus, $V$ in (10) is a candidate ISS-Lyapunov function and satisfies

$$\begin{aligned}\dot{V} &= x_1^\top \hat{P}\dot{x}_1 + \dot{x}_1^\top \hat{P} x_1 \\ &= x_1^\top \hat{P}\left(A_{1,1} - A_{1,2}K\right)x_1 + \\ &\quad + x_1^\top \left(A_{1,1} - A_{1,2}K\right)^\top \hat{P} x_1 + \\ &\quad + x_1^\top \hat{P} d_1 + d_1^\top \hat{P} x_1.\end{aligned}$$

Using (3a), it yields

$$\begin{aligned}\dot{V} = x_1^\top \hat{P}&\left(\hat{A}_{1,1} - \hat{A}_{1,2}K\right)x_1 + \\ &+ x_1^\top\left(\hat{A}_{1,1} - \hat{A}_{1,2}K\right)^\top \hat{P}x_1 + \\ &+ x_1^\top \hat{P}\left(\Delta_{1,1} - \Delta_{1,2}K\right)x_1 + \\ &+ x_1^\top\left(\Delta_{1,1} - \Delta_{1,2}K\right)^\top \hat{P}x_1 + \\ &+ x_1^\top \hat{P} d_1 + d_1^\top \hat{P} x_1.\end{aligned}$$

Now, using (8) we obtain

$$\begin{aligned}\dot{V} = &-x_1^\top \hat{Q} x_1 + x_1^\top \hat{P}\left(\Delta_{1,1} - \Delta_{1,2}K\right)x_1 + \\ &+ x_1^\top\left(\Delta_{1,1} - \Delta_{1,2}K\right)^\top \hat{P}x_1 + 2x_1^\top \hat{P} d_1. \end{aligned} \quad (11)$$

As for the cross term in (11), it is possible to apply Young's inequality, as it exists $\gamma \in \mathbb{R}_{>0}$ such that

$$x_1^\top \hat{P} d_1 \le \frac{\gamma\|x_1\hat{P}\|_2^2}{2} + \frac{\|d_1\|_2^2}{2\gamma},$$

thus allowing the following upper bound for equation (11)

$$\begin{aligned}\dot{V} \le &-x_1^\top \hat{Q} x_1 + x_1^\top \hat{P}\left(\Delta_{1,1} - \Delta_{1,2}K\right)x_1 + \\ &+ x_1^\top\left(\Delta_{1,1} - \Delta_{1,2}K\right)^\top \hat{P}x_1 + \gamma\|x_1\hat{P}\|_2^2 + \frac{\|d_1\|_2^2}{\gamma}.\end{aligned}$$

Since $\hat{Q}$ is SPD, the corresponding quadratic form is bounded from above and below by its biggest and smallest eigenvalue, respectively. Then, it yields

$$\|x_1\|_2^2\underline{\lambda}_{\hat{Q}} \le x_1^\top \hat{Q} x_1 \le \|x_1\|_2^2\overline{\lambda}_{\hat{Q}}.$$

Now, since also $\hat{P}$ is SPD, we have

$$\|\hat{P}\|_2 = \sqrt{\overline{\lambda}\left(\hat{P}^\top\hat{P}\right)} = \sqrt{\overline{\lambda}_{\hat{P}}^2} = \overline{\lambda}_{\hat{P}}.$$

Then, by virtue of Assumption 2, we obtain

$$\begin{aligned}\dot{V} &\le -\|x_1\|_2^2\underline{\lambda}_{\hat{Q}} + 2\|x_1\|_2^2\|\hat{P}\|_2\left\|\Delta_{1,1} - \Delta_{1,2}K\right\|_2 + \\ &\quad + \gamma\|x_1\hat{P}\|_2^2 + \frac{\|d_1\|_2^2}{\gamma} \\ &\le -\|x_1\|_2^2\underline{\lambda}_{\hat{Q}} + 2\|x_1\|_2^2\overline{\lambda}_{\hat{P}}\left\|\Delta_{1,1} - \Delta_{1,2}K\right\|_2 + \\ &\quad + \gamma\overline{\lambda}_{\hat{P}}\|x_1\|_2^2 + \frac{\|d_1\|_2^2}{\gamma} \\ &\le \|x_1\|_2^2\left(2\overline{\lambda}_{\hat{P}}\left(\frac{\gamma}{2} + \left\|\Delta_{1,1} - \Delta_{1,2}K\right\|_2\right) - \underline{\lambda}_{\hat{Q}}\right) + \\ &\quad + \frac{\|d_1\|_2^2}{\gamma} \\ &\le \|x_1\|_2^2\left(2\overline{\lambda}_{\hat{P}}\left(\frac{\gamma}{2} + \delta_{1,1} + \delta_{1,2}\left\|K\right\|_2\right) - \underline{\lambda}_{\hat{Q}}\right) + \\ &\quad + \frac{\|d_1\|_2^2}{\gamma}.\end{aligned}$$

Now, let us define $\alpha_3, \alpha_4 \in \mathbb{R}$ as

$$\alpha_3 \coloneqq 2\overline{\lambda}_{\hat{P}} \left(\frac{\gamma}{2} + \delta_{1,1} + \delta_{1,2} \|K\|_2\right) - \underline{\lambda}_{\hat{Q}}$$
$$\alpha_4 \coloneqq \frac{1}{\gamma}.$$

Then, if (9) is satisfied, both $\alpha_3$ and $\alpha_4$ are positive, and we obtain

$$\dot{V} \leq -\alpha_3 \|x_1\|_2^2 + \alpha_4 \|d_1\|_2^2.$$

Since $\alpha_3 \|x_1\|_2^2$ is a class $\mathcal{K}_\infty$ function and $\alpha_4 \|d_1\|_2^2$ is a class $\mathcal{K}$ function, we can conclude that the equivalent dynamics (6) is ISS with respect to the unmatched disturbance $d_1$. ■

*Remark 3.1 (Asymptotic Stability):* Note that the ISS property of (6) implies that when $d_1$ is constant or equal to zero, then (6) is asymptotically stable. In particular the proof in the case of $d_1 = 0$ is straightforward, while for the case in which $d_1 = d_1^\star \in \mathbb{R}^{n-m}$, it is enough to first shift the system with respect to the equilibrium point corresponding to the constant $d_1^\star$.

*Remark 3.2 (Ultimately Bounded Set):* It is possible to compute the ultimately bounded set of the state of the equivalent system (6) by using the upperbound of the unmatched disturbance given in Assumption 1. Specifically, from the inequality

$$\dot{V} \leq -\alpha_3 \|x_1\|_2^2 + \alpha_4 \bar{d}_1^2,$$

the ultimately bounded set $\Omega_1$ for the state $x_1$ is defined as

$$\Omega_1 \coloneqq \left\{ x_1 \in \mathbb{R}^{n-m} : \|x_1\| \leq \sqrt{\tfrac{\alpha_4}{\alpha_3}} \bar{d}_1 \right\}. \tag{12}$$

It is clear that, by virtue of the enforcement of a sliding mode on the manifold $\sigma = 0$, where $x_2 = -K x_1$, also the state $x_2$ is ultimately bounded and the corresponding set $\Omega_2$ can be computed straightforwardly.

*Remark 3.3:* Let $\mathcal{S}$ and $\hat{\mathcal{S}}$ denote the sets of matrices $K \in \mathbb{R}^{m \times (n-m)}$ making, respectively, the full and nominal equivalent dynamics Hurwitz, i.e.,

$$\mathcal{S} \coloneqq \{K : (A_{1,1} - A_{1,2}K) \prec 0\}$$
$$\hat{\mathcal{S}} \coloneqq \left\{K : \left(\hat{A}_{1,1} - \hat{A}_{1,2}K\right) \prec 0\right\}.$$

Moreover, let $\mathcal{S}_r$ denote the set of matrices $K \in \hat{\mathcal{S}}$ that satisfy (9), then one has that $\mathcal{S}_r \subseteq (\mathcal{S} \cap \hat{\mathcal{S}})$.

*Remark 3.4:* Note that Theorem 1 holds even if the uncertainties $\Delta_{i,j}(x)$ and $\Delta_B(x)$ are nonlinear functions of the state as long as Assumption 2 is satisfied.

## IV. Simulation Results

To assess the proposed technique, in this section we consider as an academic example the RLC circuit depicted in Fig. 1 and described by the Kirchhoff equations

$$C\dot{V}_C = -\frac{V_C}{R_2} + I_L - h_1$$
$$L\dot{I}_L = -V_C - R_1 I_L + u + h_2,$$

where $h_1$ and $h_2$ are unmatched and matched disturbances, respectively.

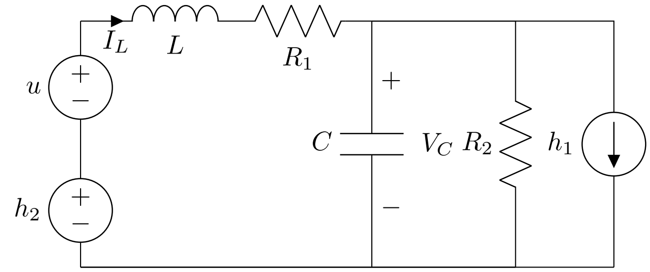


Fig. 1. RLC case study circuit described in (13).

The aim of the control problem is to impose a given reference voltage $V_C^*$ on the load, which will correspond to that on the capacitor $V_C$, while imposing a certain reference current $I_L^*$ on the inductor. In the following simulations, without loss of generality, the reference voltage is chosen as $V_C^* = 0V$, and thus the reference current is set to $I_L^* = 0A$. Let $\tilde{V}_C \coloneqq V_C - V_C^*$ and $\tilde{I}_L \coloneqq I_L - I_L^*$ denote the voltage and current error. Then, the error system can be expressed as

$$\dot{\tilde{V}}_C = -\frac{1}{CR_2}\tilde{V}_C + \frac{1}{C}\tilde{I}_L - \frac{1}{C}h_1 \tag{13a}$$

$$\dot{\tilde{I}}_L = -\frac{1}{L}\tilde{V}_C - \frac{R_1}{L}\tilde{I}_L + \frac{1}{L}(\tilde{u} + h_2). \tag{13b}$$

Given a sliding variable as in (4), i.e.,

$$\sigma = k\tilde{V}_C + \tilde{I}_L,$$

with $k \in \mathbb{R}$, then the equivalent dynamics of (13) on the sliding manifold $\sigma = 0$ can be expressed as

$$\dot{\tilde{V}}_C = -\frac{1}{C}\left(\frac{1}{R_2} + k\right)\tilde{V}_C - \frac{1}{C}h_1,$$

while the nominal one can be written as

$$\dot{\tilde{V}}_{\hat{C}} = -\frac{1}{\hat{C}}\left(\frac{1}{\hat{R}_2} + k\right)\tilde{V}_C - \frac{1}{\hat{C}}h_1.$$

In the case of the considered RLC circuit, the Lyapuonv equation is scalar. In particular, since $\hat{P}$, $\hat{Q} \in \mathbb{R}$, then such matrices coincide with their eigenvalues, and thus we obtain

$$-2\frac{1}{\hat{C}}\left(\frac{1}{\hat{R}_2} + k\right)\lambda_{\hat{P}} = -\lambda_{\hat{Q}}. \tag{14}$$

Then, by using (14) in (9), it yields

$$\frac{1}{\hat{C}}\left(\frac{1}{\hat{R}_2} + k\right) > \delta_{1,1} + \delta_{1,2}|k| + \frac{\gamma}{2}.$$

The parameters used in the simulation are reported in Table I, while the disturbances $h_1$ and $h_2$ are defined as

$$h_1(t) \coloneqq 0.05\sin\left(10^3 t\right) \qquad h_2(t) \coloneqq 0.1\cos\left(10^3 t\right).$$

The bounds $\delta_{1,1}$ and $\delta_{1,2}$ are calculated on the basis of the maximum possible variation of the parameters, i.e.,

$$\max_{\Delta R_2, \Delta C}\left\{\left|\frac{1}{(\hat{R}_2 + \Delta R_2)(\hat{C} + \Delta C)} - \frac{1}{\hat{R}_2\hat{C}}\right|\right\} \leq 260.1$$

$$\max_{\Delta C}\left\{\left|\frac{1}{\hat{C} + \Delta C} - \frac{1}{\hat{C}}\right|\right\} \leq 214.3$$

TABLE I

PARAMETERS OF THE RLC CIRCUIT IN FIGURE 1.

| Parameter Symbol | True Value | Nominal Value | Uncertainty Percentage | Unit of Measure |
|---|---|---|---|---|
| $C$ | $1.8 \times 10^{-3}$ | $1.4 \times 10^{-3}$ | 30% | F |
| $L$ | $2.2 \times 10^{-3}$ | $2.9 \times 10^{-3}$ | 25% | H |
| $R_1$ | $5.0 \times 10^{-1}$ | $4.0 \times 10^{-1}$ | 25% | Ω |
| $R_2$ | 2.0 | $1.5 \times 10^{2}$ | 41% | Ω |

Therefore, $\delta_{1,1} = 260.1$ and $\delta_{1,2} = 214.3$. The simulation results depicted in Fig. 2 are obtained by setting $k = 1$. Indeed it is easy to verify that

$$1224.5 = \frac{1}{\hat{C}}\left(\frac{1}{\hat{R}_2} + k\right) > \delta_{1,1} + \delta_{1,2}\,|k| = 474.4. \quad (15)$$

To calculate the ultimately bounded set, one first has to calculate the quantity denoted as $\bar{d}_1$ in equation (12), i.e.,

$$\bar{d}_1 := \sup_{t,\Delta C}\left\{\frac{h_1(t)}{\hat{C} + \Delta C}\right\} \approx \frac{5 \times 10^{-2}}{10^{-3}} = 50.$$

Then, it is possible to derive that, for $k = 1$, the corresponding term $\gamma$ appearing in equation (9) can be at most $1.4 \times 10^3$. This implies that for the considered case study, the error bounds due to the unmatched disturbance are

$$\left|\tilde{V}_C\right| \leq \sqrt{\frac{\alpha_4}{\alpha_3}}\bar{d}_1 \approx 3 \times 10^{-3}\text{V} \quad (16a)$$

$$\left|\tilde{I}_L\right| = k\left|\tilde{V}_C\right| \leq 3 \times 10^{-3}\text{A}, \quad (16b)$$

defining the ultimately bounded sets as $\Omega_1 = \{|V_C| \leq 3 \times 10^{-3}V\}$ and $\Omega_2 = \{|I_L| \leq 3 \times 10^{-3}A\}$. The control law is obtained as a classical relay type switching controller as described in classical literature, i.e.,

$$u = -\rho\,\mathrm{sign}\,(\sigma)\,.$$

The gain can be determined according to the reaching condition $\sigma\dot{\sigma} \leq -\eta\,|\sigma|$, with $\eta \in \mathbb{R}_{>0}$ being an arbitrary constant. In particular, in the simulation depicted in Fig. 2, the chosen gain is $\rho = 2$, which is able to enforce a sliding mode in finite time for the given initial condition. Moreover, Fig. 2 also shows the ultimately bounded set $\Omega_1$ for $V_C$ and $\Omega_2$ for $I_L$ defined in (16). Finally, Fig. 3 shows an equivalent representation of the sets $\mathcal{S}$, $\hat{\mathcal{S}}$, and $\mathcal{S}_r$ defined in Remark 3.3. More precisely, we show in Fig. 3 the areas on the plane $\tilde{V}_C$-$\tilde{I}_L$ where the line $\tilde{I}_L = -k\tilde{V}_C$ lies for all $k \in \hat{\mathcal{S}}, \mathcal{S}$ and $\mathcal{S}_r$, respectively. It is then clear that choosing $k$ on the basis of the nominal system, i.e., $k \in \hat{\mathcal{S}}$, could lead to an unstable system, due to the uncertainty. The same Figure shows also that the set satisfying the robustness condition (15), i.e., $\mathcal{S}_r$, is a subset of the truly stabilizing set $\mathcal{S}$.

## V. CONCLUSIONS

In this paper, we develop a novel procedure to design a stabilizing sliding manifold for uncertain and perturbed linear systems. In particular, a robustness condition based on the Lyapunov equation associated with the nominal dynamics of the controlled system is provided to design a manifold where stability is guaranteed despite model uncertainties, and both matched and unmatched disturbances. Finally, the proposed approach has been assessed in simulation on an electric RLC circuit, showing satisfactory performance.

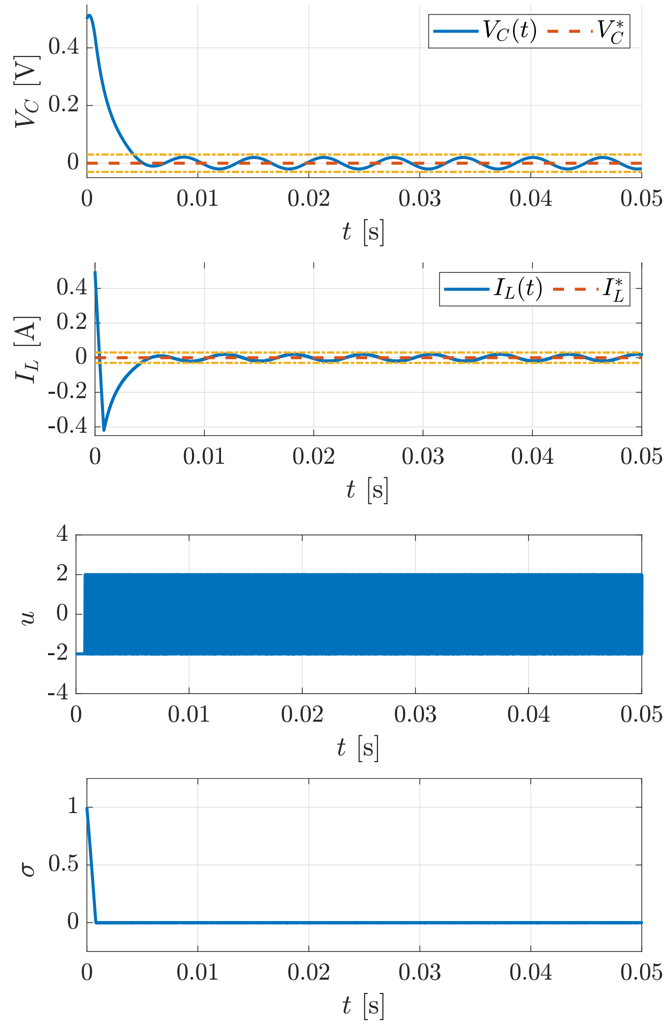


Fig. 2. Plots of voltage $V_C$, current $I_L$, control input $u$, and sliding variable $\sigma$ for the considered RLC circuit. The solid blue line are the state, input and sliding variable, the dashed red lines are the reference values, while the dash-dotted yellow line indicates the ultimately bounded set $\Omega_1$ and $\Omega_2$ given by (16).

## REFERENCES


[1] V. I. Utkin, *Sliding modes in control and optimization*. Springer Science & Business Media, 2013.

[2] A. Ferrara, G. P. Incremona, and M. Cucuzzella, *Advanced and optimization based sliding mode control: Theory and applications*. SIAM, 2019.

[3] C. Edwards and S. Spurgeon, *Sliding mode control: theory and applications*. Crc Press, 1998.

[4] V. Utkin and J. Shi, "Integral sliding mode in systems operating under uncertainty conditions," in *Proceedings of 35th IEEE Conference on Decision and Control*, vol. 4, 1996, pp. 4591–4596 vol.4.

[5] A. Ferrara and M. Rubagotti, "A sub-optimal second order sliding mode controller for systems with saturating actuators," *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1082–1087, 2009.

[6] F. Dinuzzo and A. Ferrara, "Higher order sliding mode controllers with optimal reaching," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2126–2136, 2009.

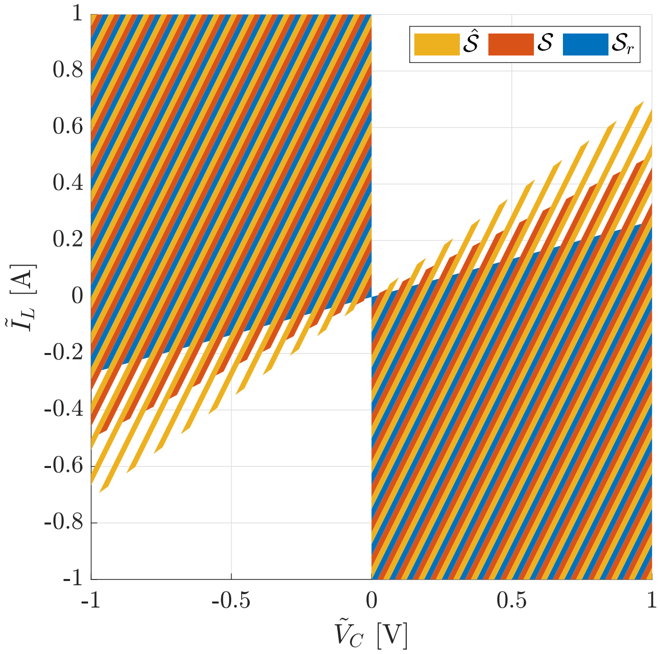


Fig. 3. Plane $\tilde{V}_C$-$\tilde{I}_L$, where areas delimited by the yellow, orange, and blue stripes correspond to the areas where the line $\tilde{I}_L = -k\tilde{V}_C$ lies for $k \in \hat{\mathcal{S}}$, $\mathcal{S}$, and $\mathcal{S}_r$, respectively.


[7] A. K. Behera, B. Bandyopadhyay, M. Cucuzzella, A. Ferrara, and X. Yu, "A Survey on Event-Triggered Sliding Mode Control," *IEEE Journal of Emerging and Selected Topics in Industrial Electronics*, vol. 2, no. 3, pp. 206–217, July 2021.

[8] I. Castillo, M. Jimenez-Lizarraga, and E. Ibarra, "Higher order sliding modes manifold design via singular lq control," *Journal of the Franklin Institute*, vol. 352, no. 7, pp. 2810–2830, 2015.

[9] J. Ackermann and V. Utkin, "Sliding mode control design based on ackermann's formula," *IEEE transactions on automatic control*, vol. 43, no. 2, pp. 234–237, 1998.

[10] M. Rubagotti, A. Estrada, F. Castaños, A. Ferrara, and L. Fridman, "Integral sliding mode control for nonlinear systems with matched and unmatched perturbations," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2699–2704, 2011.

[11] J. Yang, S. Li, and X. Yu, "Sliding-mode control for systems with mismatched uncertainties via a disturbance observer," *IEEE Transactions on industrial electronics*, vol. 60, no. 1, pp. 160–169, 2012.

[12] Y. Weng and X. Gao, "Data-driven sliding mode control of unknown MIMO nonlinear discrete-time systems with moving PID sliding surface," *Journal of the Franklin Institute*, vol. 354, no. 15, pp. 6463–6502, 2017.

[13] M. A. Hussain and P. Y. Ho, "Adaptive sliding mode control with neural network based hybrid models," *Journal of Process Control*, vol. 14, no. 2, pp. 157–176, 2004.

[14] M. Mueller, "Normal form for linear systems with respect to its vector relative degree," *Linear algebra and its applications*, vol. 430, no. 4, pp. 1292–1312, 2009.

[15] A. A. Agrachev, A. S. Morse, E. D. Sontag, H. J. Sussmann, V. I. Utkin, and E. D. Sontag, "Input to state stability: Basic concepts and results," *Nonlinear and optimal control theory: lectures given at the CIME summer school held in Cetraro, Italy June 19–29, 2004*, pp. 163–220, 2008.

[16] E. D. Sontag *et al.*, "On the input-to-state stability property." *Eur. J. Control*, vol. 1, no. 1, pp. 24–36, 1995.