2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Clock Steering Techniques for Atomic Clocks of Arbitrary Order

Priyanka Dey[1], Sathishkumar Murugesan[1], Takahiro Kawaguchi[2], Yuichiro Yano[3], Yuko Hanado[3], and Takayuki Ishizaki[1]

Abstract— **In this paper, we have explored the possibility to exploit control theory to introduce new steering techniques for atomic clocks of arbitrary order. Current studies show that most of the well-known steering methods are developed for the two-state atomic clock model and not much attention is paid to find efficient steering methods for higher-order atomic clocks. We introduce two novel clock steering methods for atomic clocks: one based on output stabilization problem and the other based on frequency deviation regulation problem. First, we prove the existence of control laws for the solvability of these two problems for atomic clocks. Second, to find a suitable control law for both steering problems, we outlined efficient procedures to obtain gain matrices and used the Kalman filter for state estimates. A few numerical examples have been given to analyze the performance of the proposed control approaches.**

Keywords: Atomic clock, Control theory, Clock steering, Time scale, State-space model.

I. Introduction

The steering of frequency standards (atomic clocks) plays a crucial role in the time and frequency community. Depending on the task, steering is used to synchronize the master clock in the national timing laboratory of each country to the coordinated universal time (UTC) [1], to obtain the physical realisation of the paper clock time scale [2], to steer global navigation satellite system time to the local representation of UTC [3], referred as UTC(k), where k stands for the specific national laboratory, for instance k=BIPM for France, k=USNO for USA. The chief purpose of steering a time scale or clock is to minimize its deviation with regard to the reference time scale, or clock in terms of phase and/or frequency value while improving its frequency stability characteristics, which is usually evaluated by means of Allan variance/deviation [4], [5].

Several works have proposed various steering methods in the literature. On the one hand, most of the works are based on classical heuristic principles [6], [7]. For instance, signal based open-loop method is being utilized for compensating for the frequency offset [8]. On the other hand, some of the recent algorithms rely on various optimal control techniques [9]. The control techniques which are generally used for steering are linear quadratic Gaussian (LQG) [10], [11] and pole placement (PP) [12], respectively. Recently, a comparative study of the overall performance of LQG and PP for steering an oven-controlled quartz oscillator (OCXO) towards a rubidium clock is performed by means of simulation as well as in hardware setup [13].

Over the years, based on the experimental evidences, the time and frequency community researchers have obtained reliable models of atomic clocks. Indeed, it has been shown by experimental evidences that the time deviation (also called phase deviation) from the ideal clock behaviour can be modeled as a stochastic process. For example, a two-state model of atomic clocks with states as time deviation and frequency deviation was introduced in [14]. In [15], the two-state model was generalized to a three-state model by adding an additional state called frequency drift, which is useful for modelling rubidium clocks. The well-known control techniques such as the LQG control and the PP method which are extensively used for steering a time scale to another are only applied to two-state model of the clock. This is because the system model for three-state atomic clock is neither controllable nor stabilizable. Furthermore, to the best of our knowledge, there are no clock steering methods yet presented in the literature which could handle the case where both the reference clock and the clock to be controlled have arbitrary order but with an equal number of states. Thus, it would be interesting if we could propose some efficient steering methods for arbitrary order atomic clocks, which is the main objective of this work.

In this paper, we begin with showing that the state-space model of atomic clock used for steering of one time scale to another is neither controllable nor stabilizable; see Subsection III-A for more details. We propose two novel steering techniques by using tools from control theory for atomic clocks of arbitrary order. Both the techniques involve to find suitable control laws for output stabilization and for frequency deviation regulation, respectively. Firstly, we show that these two problems are solvable for atomic clocks. Secondly, for each problem, we outline a procedure to obtain a suitable gain matrix with some restriction on its entries for the control law to accomplish the desired objective of output stabilization or frequency deviation regulation.

Moreover, the two steering methods are compared through numerical simulations of two basic clock steering problems. The first problem involves steering a cesium clock, which may represent UTC(k) time scale against a reference time scale such as UTC. Second problem deals with steering a

*This work is supported by the Ministry of Internal Affairs and Communications (MIC) under its "Research and Development for Expansion of Radio Resources (JPJ000254)" program.

[1]P. Dey, S. Murugesan, and T. Ishizaki are with the Department of Systems and Control Engineering, Tokyo Institute of Technology, Meguro, Tokyo 152-8552 Japan (Emails: dey.p.aa@m.titech.ac.jp, murugesan.s.aa@m.titech.ac.jp, ishizaki@sc.e.titech.ac.jp).

[2]T. Kawaguchi is with the Division of Electronics and Informatics, Gunma University, Kiryu, Gunma 371-8510 Japan (Email: kawaguchi@gunma-u.ac.jp).

[3]Y. Yano and Y. Hanado are with the National Institute of Information and Communications Technology, Koganei, Tokyo 184-0015 Japan (Emails: y-yano@nict.go.jp, yuko@nict.go.jp).

hydrogen maser against a reference cesium clock. The simulation results highlights the advantages and disadvantages of the two proposed steering methods.

The rest of the paper unfolds as follows. The continuous-time and discrete-time standard model of an atomic clock of arbitrary order is introduced in Section II. We explain the basic state-space model for clock steering which we will use in this paper, and also discuss about its controllability and stabilizability in Section III. We introduce the two steering methods relying on the ideas of output stabilization and frequency deviation regulation in Section IV and Section V, respectively. In Section VI, we give several examples to exhibit the usefulness of the techniques developed. Section VII concludes the paper.

Notations: The set of complex numbers, real numbers, positive real numbers, and positive integers are denoted by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}^\star$, respectively. For $r \in \mathbb{N}^\star$, we let $[r] = \{1, 2, 3, \ldots, r\}$. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes its transpose. Furthermore, the kernel matrix A is denoted by $\operatorname{Ker} A$. We denote a diagonal matrix of dimension n with diagonal entries as a_1, a_2, ..., a_n as $\operatorname{diag}(a_1, a_2, \ldots, a_n)$, and I_n denotes the identity matrix of dimension n. Finally, $\mathbb{E}[\cdot]$ represents the expected value.

II. Background

A. System Description of Atomic Clock Model

An atomic clock is an independent oscillator which generates sinusoidal signals and displays clock reading $f(t)$ which differs from the ideal clock reading $f_0(t) = t$ of the ideal clock at time t. The time difference between the clock reading and the ideal clock reading is called time deviation (also referred as phase deviation) and is given by $\Delta f(t) = f(t) - f_0(t)$. It has been shown [16] from experimental observations that the time deviation can be modelled as a stochastic process

$$\begin{aligned}\Delta f(t) =& \sum_{i=1}^{n} \frac{\alpha_i t^{i-1}}{(i-1)!} \\ &+ \sum_{i=1}^{n} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-1}} \zeta_i(t_i) dt_i \cdots dt_2 dt_1,\end{aligned} \tag{1}$$

where n is the order of the clock, $\alpha_i \in \mathbb{R}_+$ for $i \in [n]$ denotes constant parameters corresponding to the clock's initial state, and $\zeta_i(t)$ for $t \geq 0$ represents a white Gaussian process with

$$\begin{aligned}\mathbb{E}[\zeta_i(t)] &= 0,\\ \mathbb{E}[\zeta_i(t)\zeta_i(t+\tau)] &= \sigma_i^2 \delta(\tau),\\ \mathbb{E}[\zeta_i(t)\zeta_j(t+\tau)] &= 0, \quad \text{for all } \tau \in \mathbb{R}, \quad i \neq j,\end{aligned}$$

for some non-negative parameters $\sigma_i \geq 0$, $i \in [n]$. For example, the order $n = 3$ is used for rubidium clocks.

The stochastic signal $\Delta f(t)$ in (1) can be characterized by n stochastic differential equation as

$$\begin{aligned}\dot{x}_i(t) &= x_{i+1}(t) + \zeta_i(t), \quad \text{for all} \quad i \in [n-1]\\ \dot{x}_n(t) &= \zeta_n(t),\end{aligned}$$

where $x_1(t)$ is the time deviation $\Delta f(t)$ with the initial state being $x_i(0) = \alpha_i$, $i \in [n]$. Consequently, the state equation of an atomic clock is

$$\dot{x}(t) = Ax(t) + v(t), \tag{2}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top \in \mathbb{R}^n$, $v(t) = (\zeta_1(t), \zeta_2(t), \ldots, \zeta_n((t)) \in \mathbb{R}^n$, and the state matrix $A \in \mathbb{R}^{n\times n}$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & & \ddots & \ddots & \vdots\\ \vdots & & & 0 & 1\\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{3}$$

The process noise v is a white Gaussian noise with $\mathbb{E}[v(t)] = 0$, and $\mathbb{E}[v(t)v^\top(t+\tau)] = Q\delta(\tau)$, for all $t \geq 0$, where the covariance matrix is $Q = \operatorname{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$.

B. Discrete-time State-space Model

Consider a time sequence $\{t_0, t_1, \ldots\}$ with the difference between any two adjacent times taken as a constant τ, i.e., $t_{k+1} - t_k = \tau$ for $k = 0, 1, 2, \ldots$. From the solution of the continuous-time model in (2), we obtain its discrete-time counterpart as

$$x(k+1) = A(\tau)x(k) + v(k), \tag{4}$$

where the discrete-time state $x(k) = x(t_k)$ and the discrete time state matrix $A(\tau) = e^{A\tau}$. Furthermore, the process noise $v(k)$, $k = 0, 1, \ldots$ is a Gaussian process which is independent and identically distributed with

$$\mathbb{E}[v(k)] = 0, \quad \mathbb{E}[v(k)v^\top(k)] = Q(\tau), \quad \text{for all} \quad k = 0, 1, \ldots.$$

where the covariance matrix is given as

$$Q(\tau) = \int_0^\tau e^{At} Q e^{A^\top t} dt. \tag{5}$$

In other words, $v(k) \sim \mathcal{N}(0, Q(\tau))$. In the clock model (2) (resp. (4)), the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$ (resp. $x_1(k)$, $x_2(k)$, $x_3(k)$) represent the time deviation, frequency deviation, and frequency drift, respectively.

III. Clock Steering Problem Setup

Consider two clocks 1 and 2 independent of each other. Let clock 1 be the reference clock and clock 2 be the clock to be controlled. The state equation of clock 1 is given by

$$x^{\text{ref}}(k+1) = A(\tau)x^{\text{ref}}(k) + v^{\text{ref}}(k), \tag{6}$$

where $x^{\text{ref}}(k) \in \mathbb{R}^n$, $v^{\text{ref}}(k) \sim \mathcal{N}(0, Q^{\text{ref}}(\tau))$.

In order to keep the time deviation and/or frequency deviation of clock 2 close to the reference clock, a control input $u(k)$ is applied to clock 2. Thus, the state equation of clock 2 is

$$x^{\text{c}}(k+1) = A(\tau)x^{\text{c}}(k) + Bu(k) + v^{\text{c}}(k), \tag{7}$$

where $x^{\text{c}}(k) \in \mathbb{R}^n$, $v^{\text{c}}(k) \sim \mathcal{N}(0, Q^{\text{c}}(\tau))$.

Even though the internal state of the atomic clocks can not be measured, the difference between the time deviation of clock 1 and clock 2 is measurable using a particular technique such as DMTD (Dual Mixer Time Difference). Let the discrete time output $y(k)$ be the measured time deviation difference between the two clocks, i.e.,

$$y(k) = x_1^{\text{c}}(k) - x_1^{\text{ref}}(k) + w(k), \tag{8}$$

where $y(k) \in \mathbb{R}$, $w(k) \sim \mathcal{N}(0, r^2)$ is a Gaussian process representing the measurement noise, where $r \in \mathbb{R}_+$.

The output equation (8) can be rewritten as

$$y(k) = C\Big(x^{\mathrm{c}}(k) - x^{\mathrm{ref}}(k)\Big) + w(k),$$

where $\mathbb{R}^{1\times n} \ni C = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$, $w(k) \sim \mathcal{N}(0, r^2)$.

Define

$$x^u(k) = x^{\mathrm{c}}(k) - x^{\mathrm{ref}}(k)$$

which describes the behaviour of the difference between clocks 1 and 2. Then, by using (6) and (7), the dynamics of $x^u(k)$ is given by

$$x^u(k+1) = A(\tau)x^u(k) + Bu(k) + v^u(k), \tag{9}$$

where $x^u(k) \in \mathbb{R}^n$, $A(\tau) = e^{A\tau}$, $v^u(k)$ is a white Gaussian noise with zero mean and covariance matrix equal to $Q^{\mathrm{ref}}(\tau) + Q^{\mathrm{c}}(\tau)$, i.e., $v^u(k) \sim \mathcal{N}(0, Q^{\mathrm{ref}}(\tau) + Q^{\mathrm{c}}(\tau))$.

The output equation (8) becomes

$$y(k) = Cx^u(k) + w(k), \tag{10}$$

where $\mathbb{R}^{1\times n} \ni C = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$, $w(k) \sim \mathcal{N}(0, r^2)$.

In this paper, we will be working with the state-space model (9)-(10) for our analysis. We call (9)-(10) as n-clock model.

A. Types of Actuators

In this subsection, we briefly discuss about three types of actuators which are commonly used for clock steering; see [17] for more details regarding these actuators.

1) **Phase Microstepper**: It is a device that adds frequency step, i.e., $u(k) = \Delta f$ at each time instant for correcting the phase of the controlled clock [9]. The input matrix corresponding to the phase microstepper is given by $B = \begin{bmatrix}\tau & 0 & 0 & \ldots & 0\end{bmatrix}^\top \in \mathbb{R}^n$.
2) **Breakiron-Koppang Actuator**: This actuator was introduced by Breakiron and Koppang for implementing LQG control for atomic clocks [10], [11]. For this actuator, the input matrix B is given by $B = \begin{bmatrix}\tau & 1 & 0 & \ldots & 0\end{bmatrix}^\top \in \mathbb{R}^n$. It is mostly used for clock model of order $n = 2$ with $B = \begin{bmatrix}\tau & 1\end{bmatrix}^\top$.
3) **Bang-Bang Actuator**: This actuator is used by the bang-bang steering method that is used for steering the global positioning system time to UTC [18]. The input matrix B is taken as $B = \begin{bmatrix}\tau^2/2 & \tau & 0 & \ldots & 0\end{bmatrix}^\top \in \mathbb{R}^n$.

Next, we make some observations regarding the controllability and stabilizability of the state-space representation (9)-(10) of the atomic clock for different types of actuators.

Lemma III.1. *The n-clock model (9)-(10) such that $n \geq 3$ is neither controllable nor stabilizable with input matrices $B = \begin{bmatrix}\tau & 0 & 0 & \ldots & 0\end{bmatrix}^\top$, $B = \begin{bmatrix}\tau & 1 & 0 & \ldots & 0\end{bmatrix}^\top$, and $B = \begin{bmatrix}\tau^2/2 & \tau & 0 & \ldots & 0\end{bmatrix}^\top$, respectively.*

Proof: Due to space constraints, the proof is omitted.

In recent years, among the three types of actuators discussed above, the one proposed by Breakiron and Koppang has been extensively used. With this actuator, two clock steering methods presented in the literature are namely the LQG control and the PP method [11], [12]. However, as mentioned in introduction, these two methods are used only for n-clock model (9)-(10) with $n = 2$, i.e., the model has only two states with $A(\tau) = \begin{bmatrix}1 & \tau \\ 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}\tau & 1\end{bmatrix}^\top$. By Lemma III.1, it follows that the n-clock model (9)-(10) with $n \geq 3$ and $B = \begin{bmatrix}\tau & 1 & 0 & \ldots & 0\end{bmatrix}^\top \in \mathbb{R}^n$ is uncontrollable as well as not stabilizable. So, the steering techniques LQG and PP cannot be applied to this model. Thus, in this work, the main purpose is to propose clock steering techniques which are applicable to n-clock model with $n = 2$ as well as $n \geq 3$. In the next two sections, we propose two new time scale steering approaches for the n-clock model (9)-(10) with $B = \begin{bmatrix}\tau & 1 & 0 & \ldots & 0\end{bmatrix}^\top$ based on output stabilization and frequency deviation regulation.

IV. Steering Via Output Stabilization

The chief purpose of steering a time scale or a clock is to keep its time deviation and/or frequency deviation closer to the reference clock and to ensure that the controlled clock has desirable frequency stability characteristics, evaluated in terms of the Allan variance/deviation. Motivated by this, we propose to find a control law that ensures that the output $y(k)$ in (10) goes to zero as $k \to \infty$.

Note that, in Sections IV and V, the control laws are obtained in terms of the state vector $x^u(k)$. Since the vector $x^u(k)$ is not measurable, it is being replaced by its estimate $\hat{x}^u(k)$. Since the pair $(A(\tau), C)$ is observable, an estimate $\hat{x}^u(k)$ of $x^u(k)$ is obtained by using a Kalman filter [19].

Initially, to obtain the control law, we neglect the process noise $v^u(k)$ and the measurement noise $w(k)$ present in the n-clock model (9)-(10), however, they are reintroduced later in the Kalman filter estimation. Thus, the n-clock model without process and measurement noises is given by

$$\begin{aligned} x^u(k+1) &= A(\tau)x^u(k) + Bu(k), \\ y(k) &= Cx^u(k), \end{aligned} \tag{11}$$

where $x^u(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$, $B = \begin{bmatrix}\tau & 1 & 0 & \ldots & 0\end{bmatrix}^\top$ is the input matrix, and $C = \begin{bmatrix}1 & 0 & 0 & \ldots & 0\end{bmatrix}$ is the output matrix, respectively. The state matrix $A(\tau)$ is given by

$$A(\tau) = e^{A\tau} = \begin{bmatrix} 1 & \tau & \frac{\tau^2}{2} & \cdots & \frac{\tau^{n-1}}{(n-1)!} \\ 0 & 1 & \tau & \cdots & \frac{\tau^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & 1 & \tau \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}. \tag{12}$$

With slight abuse of notation, we refer to (11) as the n-clock model.

The stabilization problem, we consider, is stated as follows:

Problem IV.1. *Given the n-clock model (11), find a state feedback $u(k) = Fx^u(k)$ such that $y(k) \to 0$ as $k \to \infty$ for every initial condition $x^u(0)$. We refer to this problem as* output stabilization problem (OSP) for atomic clocks.

Before moving ahead, we recall a few terms which will be used later and are taken from [20] for a linear system.

∘ We partition the complex plane $\mathbb{C} = \mathbb{C}_g \sqcup \mathbb{C}_b$, where

$$\begin{aligned}\mathbb{C}_g &= \{\lambda \mid \lambda \in \mathbb{C},\ \ |\lambda| < 1\},\\ \mathbb{C}_b &= \{\lambda \mid \lambda \in \mathbb{C},\ \ |\lambda| \geq 1\},\end{aligned} \tag{13}$$

and $\sqcup$ represent disjoint union.

Let the minimal polynomial of $A(\tau)$ be $\alpha(\lambda)$, and factor $\alpha(\lambda) = \alpha_g(\lambda)\alpha_b(\lambda)$, where the zeros in $\mathbb{C}$ of α_g (resp. α_b) belong to $\mathbb{C}_g$ (resp. $\mathbb{C}_b$). Then we have $\mathbb{R}^n = \mathcal{X}_g(A(\tau)) \oplus \mathcal{X}_b(A(\tau))$, where $\mathcal{X}_g(A(\tau)) = \operatorname{Ker} \alpha_g(A(\tau))$, $\mathcal{X}_b(A(\tau)) = \operatorname{Ker} \alpha_b(A(\tau))$, and $\oplus$ denotes the direct sum.

It has been shown in [20, Chapter 4], [21] that for a linear time-invariant system having state, input, and output matrices as A, B, and C, respectively, the output $y(k)$ goes to zero as $k \to \infty$ by state feedback $u(k) = Fx(k)$ if and only if $\mathcal{X}_b(A+BF) \subset \operatorname{Ker} C$. Note that the n-clock model (11) is a linear time-invariant system. Thus, $y(k) \to 0$ as $k \to \infty$ by a state feedback $u(k) = Fx^u(k)$ for the n-clock model (11) if and only if $\mathcal{X}_b(A(\tau) + BF) \subset \operatorname{Ker} C$. Next, we discuss about the solvability of OSP for atomic clocks.

Theorem IV.2. *Consider the n-clock model (11). The OSP for atomic clocks is solvable.*

Proof: Due to space constraints, the proof is omitted.

As a consequence of Theorem IV.2, there exists a feedback matrix F such that $\mathcal{X}_b(A(\tau)+BF) \subset \operatorname{Ker} C$. Our next goal is to find such a feedback matrix F. Let the matrix $F = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$ where a_i, $i \in [n]$ are the constants which are to be selected. We have $A(\tau) + BF$ as

$$\begin{bmatrix} 1 + a_1\tau & \tau(1+a_2) & a_3\tau + \frac{\tau^2}{2} & \cdots & a_n\tau + \frac{\tau^{n-1}}{(n-1)!} \\ a_1 & 1 + a_2 & a_3 + \tau & \cdots & a_n + \frac{\tau^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & 1 & \tau \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{14}$$

Lemma IV.3. *Fix the sampling interval $\tau \in \mathbb{R}_+$. Choose $a_1 \neq 0$ such that $|1 + a_1\tau| < 1$. Choose $a_2, a_3, \ldots, a_n$ such that all the entries of the first row of $A(\tau) + BF$ in (14) becomes zero except the first entry $1 + a_1\tau$ then $\mathcal{X}_b(A(\tau) + BF) \subset$ Ker C, where* Ker $C = \{x \in \mathbb{R}^n \mid x_1 = 0\}$.

Proof: Due to space constraints, the proof is omitted.

V. Steering Via Frequency Deviation Regulation

In this section, we seek to find a control law to minimize the frequency deviation of the controlled clock relative to the reference clock. To accomplish this goal, we add an additional expression of a variable $z(k)$ to regularize the second state of $x^u(k)$ in the n-clock model (11). So, the modified n-clock model is given by

$$\begin{aligned} x^u(k+1) &= A(\tau)x^u(k) + Bu(k),\\ y(k) &= Cx^u(k),\\ z(k) &= Dx^u(k),\end{aligned} \tag{15}$$

where $x^u(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$ is the directly measured phase deviation difference between the two clocks, and $z(k)$ is the variable to be regulated, which is taken as the difference of frequency deviation of the two clocks in our setting. The matrices $A(\tau) = e^{A\tau}$, $B = \begin{bmatrix} \tau & 1 & 0 & \ldots & 0 \end{bmatrix}^\top$, $C = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{1\times n}$, respectively.

We consider the following regularization problem:

Problem V.1. *Given the modified n-clock model (15), find a state feedback $u(k) = Fx^u(k)$ such that $z(k) \to 0$ as $k \to \infty$ for every initial state $x^u(0)$. We refer to this problem as* frequency regulation problem (FRP) for atomic clocks.

Consider a linear time-invariant system whose state, input, and output matrices are A, B, and C, respectively. Let $z(k) = Dx(k)$ is the variable to be regulated. Then, it is shown in [20, Chapter 6] that $z(k) \to 0$ as $k \to \infty$ by a state feedback $u(k) = Fx(k)$ if and only if $\eta_{(A,C)} \subset \operatorname{Ker} F$ and $\mathcal{X}_b(A + BF) \subset \operatorname{Ker} D$, where $\eta_{(A,C)}$ is the unobservable subspace of the pair (A, C). The modified n-clock model (15) is a linear time-invariant system containing an additional variable $z(k)$ which is to be regulated. Thus, $z(k) \to 0$ as $k \to \infty$ by a state feedback $u(k) = Fx^u(k)$ if and only if $\eta_{(A(\tau),C)} \subset \operatorname{Ker} F$ and $\mathcal{X}_b(A(\tau) + BF) \subset \operatorname{Ker} D$, where $\eta_{(A(\tau),C)}$ is the unobservable subspace of $(A(\tau), C)$. Next, we discuss about the solvability of FRP for atomic clocks.

Theorem V.2. *Consider the modified n-clock model (15). The FRP for atomic clocks is solvable.*

Proof: Due to space constraints, the proof is omitted.

Theorem V.2 confirms the existence of a feedback matrix F such that $\eta_{(A(\tau),C)} \subset \operatorname{Ker} F$ and $\mathcal{X}_b(A(\tau) + BF) \subset \operatorname{Ker} D$. Since $\eta_{(A(\tau),C)} = \{0\}$, the condition $\eta_{(A(\tau),C)} \subset \operatorname{Ker} F$ is always satisfied. So, our next objective is find a matrix F such that $\mathcal{X}_b(A(\tau) + BF) \subset \operatorname{Ker} D$. Recall $F = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n \end{bmatrix}$ and the matrix $A(\tau) + BF$ is given in (14).

Lemma V.3. *Fix the sampling interval $\tau \in \mathbb{R}_+$. Choose $a_2 \neq 0$ such that $|1 + a_2| < 1$. Choose $a_1, a_3, \ldots, a_n$ such that all the entries of the second row of $A(\tau) + BF$ in (14) becomes zero except the second entry $1 + a_2$ then $\mathcal{X}_b(A(\tau) + BF) \subset$ Ker D, where* Ker $D = \{x \in \mathbb{R}^n \mid x_2 = 0\}$.

Proof: Due to space constraints, the proof is omitted.

As mentioned in beginning of Section IV, in the control law, we use the estimate of $x^u(k)$ computed by a Kalman filter for the n-clock model (9)-(10). Thus, the control law applied to (9)-(10) is $u(k) = F\hat{x}^u(k)$, where $\hat{x}^u(k)$ is the estimate of $x^u(k)$ and F is obtained by using Lemma IV.3 and Lemma V.3, respectively according to the objective.

VI. Simulations

In this section, we consider two examples to showcase the effectiveness of our results. In both examples, a third-order atomic clock model containing three states is considered, i.e., $n = 3$. The effectiveness of the proposed clock steering techniques are evaluated in terms of its capability to reduce the difference of the time deviation and the frequency deviation between the two clocks and in terms of Allan deviation.

A. Steering a Cesium Clock to a Stable Reference Clock

In the first example, we take the scenario of controlling a cesium clock versus a reference time scale which has

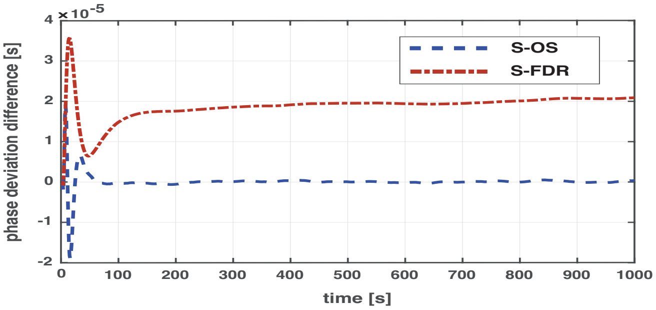

Fig. 1: Phase deviation of the cesium clock controlled by steering via output stabilization (shown in blue dashed line) and via frequency deviation regulation (shown in red dash-dot line) with respect to the reference clock.

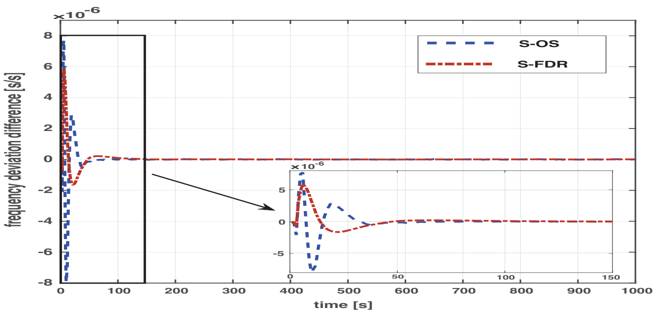

Fig. 2: Frequency deviation of the cesium clock controlled by steering via output stabilization (shown in blue dashed line) and via frequency deviation regulation (shown in red dash-dot line) with respect to the reference clock.

higher accuracy and stability performance.[1] Let the sampling interval be $\tau = 1$. Let the clock noise parameters of the cesium clock be $\sigma_1^{Cs} = 3 \times 10^{-9}$, $\sigma_2^{Cs} = 5 \times 10^{-10}$, and $\sigma_3^{Cs} = 0$. We consider the initial state of the cesium clock as $x^{\mathrm{c}}(0) = \begin{bmatrix} 0 & 10^{-8} & 0 \end{bmatrix}^\top$. The reference clock's standard deviations are considered ten times lower than that of σ_1^{Cs}, σ_2^{Cs}, σ_3^{Cs} and its initial state is $x^{\mathrm{ref}}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. The measurement noise parameter is given by $r = 1 \times 10^{-12}$.

The two control techniques presented in Section IV and Section V are applied and compared with each other.

1) Steering via output stabilization: By using Lemma IV.3, we choose the feedback matrix as $F = \begin{bmatrix} a_1 & -1 - \frac{\tau}{2} \end{bmatrix}$, where a_1 can be varied such that $a_1 \neq 0$ and $|1 + a_1\tau| < 1$. The constant a_1 is chosen in order to optimize the control performance. Thus, we choose $a_1 = -0.5$.
2) Steering via frequency deviation regulation: In accordance to Lemma V.3, we take F as $F = \begin{bmatrix} 0 & a_2 & -\tau \end{bmatrix}$, where a_2 is chosen in a way that $a_2 \neq 0$ and $|1 + a_2| < 1$. In our simulation, we take $a_2 = -0.1$.

The plot of phase deviation and frequency deviation of the controlled cesium clock relative to the reference clock are depicted in Fig. 1 and Fig. 2, respectively. The Allan deviation plots of the controlled cesium clock and reference clock are depicted in Fig. 3.

[1]Note that by stability we do not mean Lyapunov stability. In the time and frequency community, stability is judged by Allan deviation [4].

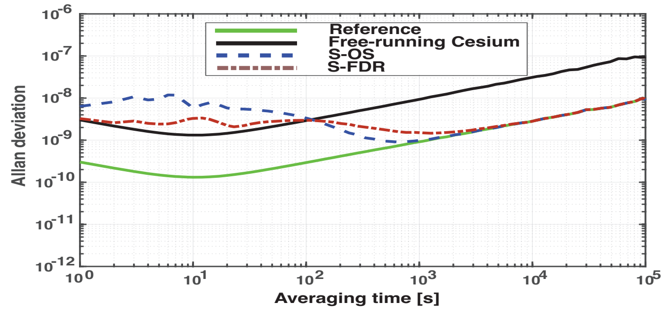

Fig. 3: Allan deviation of the phase deviations of the reference clock (shown in green), of the free-running cesium clock (shown in black), and the cesium clock controlled by steering via output stabilization (shown in blue dashed line) and via frequency deviation regulation (shown in red dash-dot line).

On the basis of the results obtained, it is clear that steering via output stabilization is more effective in reducing the time deviation and the frequency deviation between the reference clock and the controlled clock. It is also observed from the Allan deviation plots that, at large values of the averaging time, the controlled clock's stability approaches reference clock's stability. Thus, among the two techniques proposed, steering via output stabilization shows better performance.

B. Steering a Hydrogen Maser to Cesium Clock

In this example, we consider steering a hydrogen maser clock to a reference time scale which is taken as cesium clock. This case exhibits the example of steering one type of clock to another type of clock. Let $\tau = 1$. Let the noise parameters for the hydrogen maser be $\sigma_1^{Hm} = 3 \times 10^{-11}$, $\sigma_2^{Hm} = 1.7 \times 10^{-10}$, $\sigma_3^{Hm} = 1 \times 10^{-19}$. We take the initial conditions of the hydrogen maser as $x^{\mathrm{c}}(0) = \begin{bmatrix} 0 & 10^{-9} & 10^{-10} \end{bmatrix}^\top$. For the reference cesium clock, the standard deviations are chosen same as given in subsection VI-A, i.e., $\sigma_1^{Cs} = 3 \times 10^{-9}$, $\sigma_2^{Cs} = 5 \times 10^{-10}$, and $\sigma_3^{Cs} = 0$ and its initial state is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. Let $r = 1 \times 10^{-12}$.

1) Steering via output stabilization: Again, in accordance to Lemma IV.3, we chose feedback matrix $F = \begin{bmatrix} a_1 & -1 & -\frac{\tau}{2} \end{bmatrix}$, where $a_1 \neq 0$ and $|1 + a_1\tau| < 1$. For our simulation, we vary a_1 to obtain good control performance. In this way, we choose $a_1 = -0.1$.
2) Steering via frequency deviation regulation: As mentioned in the Subsection VI-A, we choose feedback matrix as $F = \begin{bmatrix} 0 & a_2 & -\tau \end{bmatrix}$, where a_2 is chosen in a way that $|1 + a_2| < 1$. We take $a_2 = -0.5$ in our simulation.

The phase deviation and frequency deviation plots of the controlled hydrogen maser relative the reference cesium clock are depicted in Fig. 4 and Fig. 5, respectively. The Allan deviation plot is shown in Fig. 6.

From Fig. 6, it is evident that the reference cesium clock has better long-term stability while the free-running hydrogen maser has better short-term stability. The results show that steering via output stabilization gives better results as compared to steering via frequency deviation regulation in terms of reducing the difference between the time deviation and the frequency deviation of the two clocks. The Allan de-

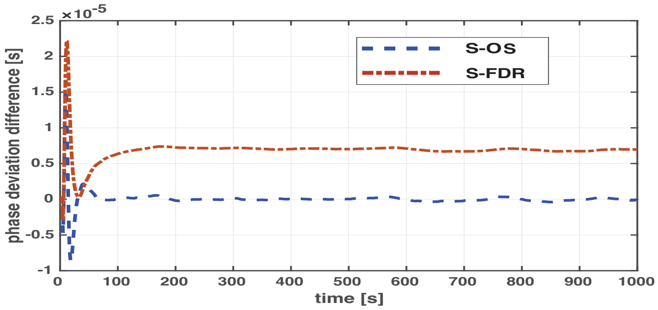

Fig. 4: Phase deviation of the hydrogen maser clock controlled by steering via output stabilization (shown in blue dashed line) and via frequency deviation regulation (shown in red dash-dot line) with respect to the reference cesium clock.

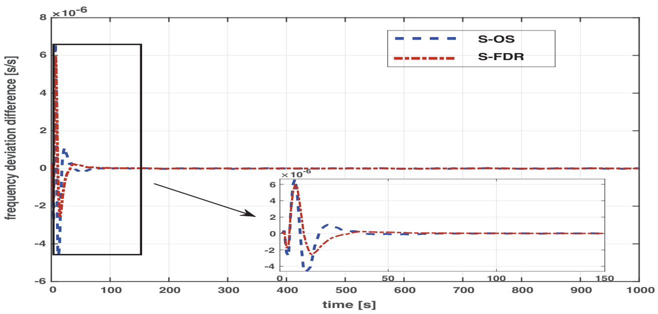

Fig. 5: Frequency deviation of the hydrogen maser clock controlled by steering via output stabilization (shown in blue dashed line) and via frequency deviation regulation (shown in red dash-dot line) with respect to the reference cesium clock.

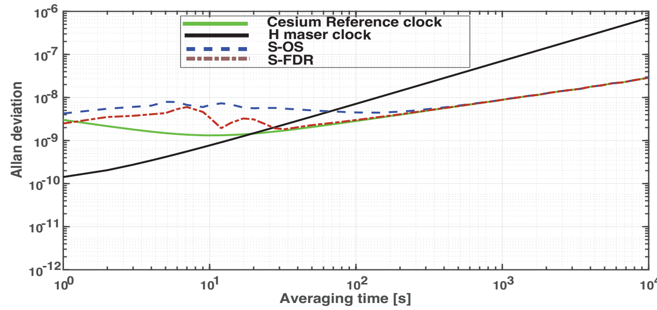

Fig. 6: Allan deviation of the phase deviations of the reference cesium clock (shown in green), of the free-running hydrogen maser (shown in black), and the hydrogen maser clock controlled by steering via output stabilization (shown in blue dashed line) and via frequency deviation regulation (shown in red dash-dot line).

viation plots show that proposed control techniques degrade the short-term stability of the controlled hydrogen maser, but its stability follows the reference cesium clock's stability for medium and large values of averaging time.

VII. Concluding Remarks

In this paper, we proposed two clock steering techniques, namely steering via output stabilization and steering via frequency deviation regulation. We see that the first technique is quite effective in reducing both the time deviation and the frequency deviation of the controlled clock relative to the reference clock in most common simulation cases. The techniques developed are also effective in improving the long-term stability of the controlled clock with reference to minimizing the Allan deviation. Lastly, we want to mention that the feedback matrices that we obtained by using Lemma IV.3 and Lemma V.3 are quite restrictive since we can only vary one parameter with some constraints whereas the rest of the parameters are fixed. It may be possible to obtain better Allan deviation plots of the controlled clock when there is some more flexibility in choosing the feedback matrices, which is considered as a natural future research direction.

References

[1] P. Tavella, "Statistical and mathematical tools for atomic clocks," *Metrologia*, vol. 45, no. 6, pp. S183–S192, 2008.

[2] C. Trainotti, T. Schmidt, and J. Furthner, "Simulating the realization of a mixed clock ensemble," in *2019 Joint Conference of the IEEE International Frequency Control Symposium and European Frequency and Time Forum (EFTF/IFC)*, 2019, pp. 1–11.

[3] P. Whibberley, J. Davis, and S. Shemar, "Local representations of UTC in national laboratories," *Metrologia*, vol. 48, no. 4, pp. S154–S164, 2011.

[4] D. Allan, "Statistics of atomic frequency standards," *Proceedings of the IEEE*, vol. 54, no. 2, pp. 221–230, 1966.

[5] T. Ishizaki, T. Ichimura, T. Kawaguchi, Y. Yano, and Y. Hanado, "Higher-order allan variance for atomic clocks of arbitrary order: mathematical foundation," *Metrologia*, vol. 61, no. 1, p. 015003, 2023.

[6] L. Bernier and G. Dudle, "Practical performance of the UTC(CH.R) real time realization of UTC(CH) and prospects for improvement," in *2004 18th European Frequency and Time Forum*, 2004, pp. 170–174.

[7] L. Bernier, "A prediction method applicable to steered timescales," in *Proceedings of the European Time and Frequency Forum*, 2005, pp. 74–78.

[8] P. Tavella and C. Thomas, "Comparative study of time scale algorithms," *Metrologia*, vol. 28, no. 2, pp. 57–63, 1991.

[9] M. Farina, S. Bittanti, P. Tavella, and L. Galleani, "Control of clock signals," *Journal of the Franklin Institute*, vol. 346, no. 5, pp. 449–469, 2009.

[10] L. Breakiron and P. Koppang, "Frequency steering of hydrogen masers," in *Proceedings of 1996 IEEE International Frequency Control Symposium*, 1996, pp. 1113–1122.

[11] P. Koppang and R. Leland, "Linear quadratic stochastic control of atomic hydrogen masers," *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, vol. 46, no. 3, pp. 517–522, 1999.

[12] T. D. Schmidt, M. Gödel, and J. Furthner, "Investigation of pole placement technique for clock steering," in *Proceedings of the 49th Annual Precise Time and Time Interval Systems and Applications Meeting*, 2018, pp. 22–29.

[13] M. Gödel, T. Schmidt, and J. Furthner, "Comparison between simulation and hardware realization for different clock steering techniques," *Metrologia*, vol. 56, no. 3, p. 035001, 2019.

[14] L. Galleani, "A tutorial on the two-state model of the atomic clock noise," *Metrologia*, vol. 45, no. 6, pp. S175–S182, 2008.

[15] C. Zucca and P. Tavella, "The clock model and its relationship with the allan and related variances," *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, vol. 52, no. 2, pp. 289–296, 2005.

[16] L. Galleani and P. Tavella, "Time and the Kalman filter," *IEEE Control Systems Magazine*, vol. 30, no. 2, pp. 44–65, 2010.

[17] M. Farina, L. Galleani, P. Tavella, and S. Bittanti, "A control theory approach to clock steering techniques," *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, vol. 57, no. 10, pp. 2257–2270, 2010.

[18] L. Huser and S. Hutsell, "1998 GPS time transfer performance," in *30th Annu. Precise Time and Time Interval (PTTI) Meeting*, 1998, pp. 125–134.

[19] M. Grewal and A. Andrews, *Kalman filtering: Theory and Practice*. New York, NY:Prentice-Hall, 2014.

[20] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*. Berlin ; New York: Springer-Verlag, 1974.

[21] S. Bhattacharyya, J. Pearson, and W. Wonham, "On zeroing the output of a linear system," *Information and Control*, vol. 20, no. 2, pp. 135–142, 1972.