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Constructive Synchronous Observer Design for Inertial Navigation with Delayed GNSS Measurements

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***Abstract*— Inertial Navigation Systems (INS) estimate a vehicle's navigation states (attitude, velocity, and position) by combining measurements from an Inertial Measurement Unit (IMU) with other supporting sensors, typically including a GNSS and a magnetometer. Recent nonlinear observer designs for INS provide powerful stability guarantees but do not account for some of the real-world challenges of INS. One of the key challenges is to account for the time-delay characteristic of GNSS measurements. This paper addresses this question by extending recent work on synchronous observer design for INS. The delayed GNSS measurements are related to the state at the current time using recursively-computable delay matrices, and this is used to design correction terms that leads to almost-globally asymptotic and locally exponential stability of the error. Simulation results verify the proposed observer and show that the compensation of time-delay is key to both transient and steady-state performance.**

I. Introduction

Inertial Navigation Systems (INS) are algorithms that fuse measurements from Inertial Measurement Units (IMUs), consisting of a gyroscope and accelerometer, to estimate a vehicle's attitude, velocity, and position with respect to a fixed reference frame. Typically, INS is supported by additional sensors, including a GNSS and a magnetometer, to counteract the build-up of error resulting from integration of noisy MEMS IMU devices [1]. INS solutions are a vital part of many navigation and control systems across application domains in aerospace, maritime, and robotics engineering [1], [2], [3], [4].

The industry-standard approach to INS is the multiplicative extended Kalman filter (MEKF) [5]. Recently, more advanced alternatives such as the Invariant EKF [6] and Equivariant Filter (EqF) [7], [8] have been shown to significantly improve accuracy and robustness. However, this class of solutions provides only local and trajectory-dependent guarantees of convergence. Authors in the nonlinear observers community have proposed alternative solutions to the INS problem with greatly improved domains of convergence [2], [9], [3], [10], [4]. Due to the nonlinearity of attitude and its coupling with the velocity and position dynamics, these observer designs typically exhibit semiglobal exponential stability rather than the almost-global stability characteristic of earlier work on attitude estimation [11], [12]. Recently, the authors have developed a new approach to INS [13], [14], [15] that exploits a 'group-affine' property of the system dynamics to yield a synchronous error with almost-global asymptotic and local exponential stability [16]. However, GNSS position and velocity measurements are typically delayed in time due to the cross correlation process required to extract the timing signals received from each satellite. These time-delays can cause significant errors if not compensated in an observer design [17]. There is some prior work in the nonlinear observer community that addresses this issue. Khosravian *et al.* [18] proposed an *observer-predictor* approach to handle delayed measurements for mixed-invariant systems on Lie groups, building on an existing observer design to inherit its stability properties. Hansen *et al.* [17] extended the semiglobally exponentially stable observer proposed in [3] to consider delayed GNSS measurements by estimating the delayed state and then propagating this forward in time using stored IMU measurements.

In this paper, we consider the INS problem with magnetometer and delayed GNSS measurements. We build on the observer architecture proposed in [15] to obtain synchronous error dynamics [16], and we use a predictor structure similar to [18] to relate the delayed GNSS measurements of the position and velocity at time $t-\delta$ to the state at the current time t. Thanks to the synchrony property of the error, we are able to design separate correction terms for the delayed position and velocity measurements and easily combine them in the final design. This yields (to the authors' knowledge) the first INS solution for delayed GNSS measurements with almost-globally asymptotic and locally exponential stability of the error dynamics. Our simulation results compare the performance of an observer with and without delay compensation providing a clear demonstration of the impact of the proposed methodology.

II. Preliminaries

For any $\omega \in \mathbb{R}^3$, the skew matrix is defined as $\omega^\times \in \mathbb{R}^{3\times 3}$ such that $\omega^\times v = \omega \times v$ for all $v \in \mathbb{R}^3$. The $n \times n$ symmetric positive definite (semi-definite) matrices are denoted $\mathbb{S}_+(n)$ (resp. $\mathbb{S}_{\geq 0}(n)$). A set of time-varying vectors $\mu_1(t), ..., \mu_n(t) \in \mathbb{R}^3$ is said to be *persistently exciting* [19] if there exist $\delta, T > 0$ such that, for all $t \geq 0$,

$$-\int_t^{t+T} \sum_{i=1}^{n} (\mu_i(\tau)^\times \mu_i(\tau)^\times) \mathrm{d}\tau > \delta I_3. \tag{1}$$

A number of Lie groups and their Lie algebras are used throughout the paper.

The special orthogonal group:

$$\mathbf{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \mid RR^\top = I_3, \det(R) = 1 \right\},$$
$$\mathfrak{so}(3) = \left\{ \Omega^\times \mid \Omega \in \mathbb{R}^3 \right\}.$$

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The extended special Euclidean group [20]:

$$\mathbf{SE}_2(3) = \left\{ \begin{pmatrix} R & V \\ 0_{2\times 3} & I_2 \end{pmatrix} \in \mathbb{R}^{5\times 5} \;\middle|\; R \in \mathbf{SO}(3), V \in \mathbb{R}^{3\times 2} \right\},$$

$$\mathfrak{se}_2(3) = \left\{ \begin{pmatrix} \Omega^\times & W \\ 0_{2\times 3} & 0_{2\times 2} \end{pmatrix} \in \mathbb{R}^{5\times 5} \;\middle|\; \Omega \in \mathbb{R}^3, W \in \mathbb{R}^{3\times 2} \right\}.$$

The extended similarity transformation group [15]:

$$\mathbf{SIM}_2(3) = \left\{ \begin{pmatrix} R & V \\ 0_{2\times 3} & A \end{pmatrix} \;\middle|\; R \in \mathbf{SO}(3), V \in \mathbb{R}^{3\times 2}, \det(A) \neq 0 \right\},$$

$$\mathfrak{sim}_2(3) = \left\{ \begin{pmatrix} \Omega^\times & W \\ 0_{2\times 3} & S \end{pmatrix} \;\middle|\; \Omega \in \mathbb{R}^3, W \in \mathbb{R}^{3\times 2}, S \in \mathbb{R}^{2\times 2} \right\}.$$

III. Problem Formulation

We consider the problem of estimating the navigation states of a vehicle equipped with an IMU, a GNSS, and a magnetometer. For simplicity, we will identify the body-frame of the vehicle with the axes of the IMU. Let $R \in \mathbf{SO}(3), v, p \in \mathbb{R}^3$ be the vehicle attitude, velocity, and position, respectively, all with respect to a reference frame fixed to the Earth's surface. Let $\Omega \in \mathbb{R}^3$ denote the angular velocity measured by the gyroscope, let $a \in \mathbb{R}^3$ denote the specific acceleration measured by the accelerometer, and let $g \in \mathbb{R}^3$ denote the gravity vector as measured in the reference frame (typically $g \approx 9.81\mathbf{e}_3$ ms^{-2}). Then the system dynamics are given by

$$\dot{R} = R\Omega^\times, \qquad \dot{v} = Ra + g, \qquad \dot{p} = v. \tag{2}$$

The GNSS is modelled as providing measurements of the position and velocity of the vehicle, delayed by a constant offset $\delta \geq 0$. The measured position y_p^δ and velocity y_v^δ at a time t are given by

$$y_p^\delta(t) = p(t-\delta), \qquad y_v^\delta(t) = v(t-\delta). \tag{3}$$

The superscript δ is used to emphasise the delay. The (undelayed) magnetometer measurement is given by

$$y_m(t) = R(t)^\top \mathring{y}_m,$$

where $\mathring{y}_m \in \mathbb{R}^3$ is the reference magnetic field direction. Our goal is to design an observer for the states at the time t using only the IMU, GNSS, and magnetometer measurements available at t.

IV. Lie Group Interpretation

The system dynamics (2) may be interpreted as group-affine dynamics on the extended pose group $\mathbf{SE}_2(3)$ [14]. Specifically, let $X \in \mathbf{SE}_2(3)$ so that

$$X = \begin{pmatrix} R & V \\ 0_{2\times 3} & I_2 \end{pmatrix}, \qquad V = \begin{pmatrix} v & p \end{pmatrix} \in \mathbb{R}^{3\times 2}. \tag{4}$$

Then the system dynamics (2) may be written as

$$\dot{X} = (G+N)X + X(U-N), \tag{5}$$

where

$$G = \begin{pmatrix} 0_{3\times 3} & W_G \\ 0_{2\times 3} & 0_{2\times 2} \end{pmatrix}, \qquad W_G = \begin{pmatrix} g & 0_{3\times 1} \end{pmatrix},$$
$$U = \begin{pmatrix} \Omega^\times & W_U \\ 0_{2\times 3} & 0_{2\times 2} \end{pmatrix}, \qquad W_U = \begin{pmatrix} a & 0_{3\times 1} \end{pmatrix}, \tag{6}$$
$$N = \begin{pmatrix} 0_{3\times 3} & 0_{3\times 2} \\ 0_{2\times 3} & S_N \end{pmatrix}, \qquad S_N = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

This interpretation of the dynamics leads us to the observer architecture proposed in [16], and has been applied to INS with undelayed GNSS measurements in [14], [15].

The measurements may also be interpreted through the Lie group formalism. One observes that

$$X \begin{pmatrix} 0_{3\times 1} \\ C \end{pmatrix} = \begin{pmatrix} R & V \\ 0_{2\times 3} & I_2 \end{pmatrix} \begin{pmatrix} 0_{3\times 1} \\ C \end{pmatrix} = \begin{pmatrix} VC \\ C \end{pmatrix}, \tag{7}$$

for any $C \in \mathbb{R}^2$. In particular, $C = \mathbf{e}_1$ for velocity measurements and $C = \mathbf{e}_2$ for position measurements. This matrix form of the measurements is powerful for studying the effect of time-delay in the sequel.

A. Time-Delay Matrices

In this section, we show how one can construct state-independent *delay matrices* $Y_L^\delta, Y_R^\delta \in \mathbf{SIM}_2(3)$ that capture the effect of a delay δ on a trajectory of the system (5).

Lemma 4.1: Let $U(t)$ be an input signal constructed from IMU measurements and G and N be matrices defined above (6). Fix $t \geq 0$ and let $Y_L^\delta(t), Y_R^\delta(t) \in \mathbf{SIM}_2(3)$ satisfy the differential equations

$$\frac{\partial}{\partial\delta} Y_L^\delta(t) = -(G+N)Y_L^\delta(t), \qquad Y_L^0(t) = I_5, \tag{8a}$$
$$\frac{\partial}{\partial\delta} Y_R^\delta(t) = -Y_R^\delta(t)(U(t-\delta)-N), \quad Y_R^0(t) = I_5, \tag{8b}$$

for all $\delta \in [0,t]$. Then, for any system trajectory $X(t) \in \mathbf{SE}_2(3)$ with dynamics (5),

$$X(t-\delta) = Y_L^\delta(t) X(t) Y_R^\delta(t), \tag{9}$$

for any $\delta \in [0,t]$.

Proof: The result is shown using the uniqueness of ODE solutions. Let $X^\delta := X(t-\delta)$ for a convenient shorthand. Then, for a fixed $t \in [0,\infty)$ and recalling (5) on has

$$\frac{\partial}{\partial\delta} X^\delta = -\left((G+N)X^\delta + X^\delta(U(t-\delta)-N)\right)$$

for $X^0 = X(t)$ when $\delta = 0$.

Now (9) holds if and only if

$$(Y_L^\delta)^{-1} X^\delta (Y_R^\delta)^{-1} = X(t).$$

Thus, taking the partial derivative with respect to δ of the left-hand side should yield zero:

$$\begin{aligned} &\frac{\partial}{\partial\delta}(Y_L^\delta)^{-1} X^\delta (Y_R^\delta)^{-1} \\ &= (Y_L^\delta)^{-1}(G+N)X^\delta(Y_R^\delta)^{-1} \\ &\quad - (Y_L^\delta)^{-1}((G+N)X^\delta + X^\delta(U(t-\delta)-N))(Y_R^\delta)^{-1} \\ &\quad + (Y_L^\delta)^{-1}X^\delta(U(t-\delta)-N)(Y_R^\delta)^{-1} \\ &= 0. \end{aligned}$$

As for the initial condition,

$$(Y_L^0)^{-1}X(t-0)(Y_R^0)^{-1} = X(t),$$

since $Y_L^0 = Y_R^0 = I_5$ by definition. Hence (9) is satisfied for all δ. ■

The solutions of these time-delay matrices $Y_L^\delta(t)$ and $Y_R^\delta(t)$ can also be propagated through time. In fact, since the matrices G and N are constants, the solution to Y_L^δ is also constant for any δ.

Lemma 4.2: Let $Y_L^\delta(t), Y_R^\delta(t) \in \mathbf{SIM}_2(3)$ be defined as in Lemma 4.1. Then, for any fixed $\delta \geq 0$, the left delay matrix is constant and given by

$$Y_L^\delta(t) = \exp(-\delta(G+N)), \tag{10}$$

for all $t \in [0, \infty)$. The time-derivative of the right delay matrix is given by

$$\dot{Y}_R^\delta(t) = -(U(t)-N)Y_R^\delta(t) + Y_R^\delta(t)(U(t-\delta)-N), \tag{11}$$

for all $t \geq \delta$.

Proof: For the left delay matrix, simply observe that the exponential $\exp(-\delta(G+N))$ is exactly the solution to the defining equation (8a), independent of t. In particular, this also means that

$$(G+N)Y_L^\delta = Y_L^\delta(G+N). \tag{12}$$

As for the right delay matrix, it must be that (9) holds for all $t \geq \delta$ by Lemma 4.1. We use the shorthand notations $X^\delta := X(t-\delta)$ and $U^\delta := U(t-\delta)$. Rewriting (9), one has

$$Y_R^\delta = X^{-1}(Y_L^\delta)^{-1}X^\delta.$$

Differentiating this with respect to t yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Y_R^\delta &= -((U-N)X^{-1} + X^{-1}(G+N))(Y_L^\delta)^{-1}(X^\delta) \\
&\quad + X^{-1}(Y_L^\delta)^{-1}((G+N)X^\delta + X^\delta(U^\delta - N)) \\
&= -(U-N)X^{-1}(Y_L^\delta)^{-1}(X^\delta) \\
&\quad + X^{-1}(Y_L^\delta)^{-1}X^\delta(U^\delta - N) \\
&= -(U(t)-N)Y_R^\delta(t) + Y_R^\delta(t)(U(t-\delta)-N),
\end{aligned}$$

where the second-last line follows from (12). ■

Let Y_R^δ, Y_L^δ be delay matrices as defined in Lemma 4.1. Expanding the formula (10) yields

$$Y_L^\delta = \exp\left(-\delta\begin{pmatrix}0_{3\times3} & W_G \\ 0_{2\times3} & S_N\end{pmatrix}\right) = \begin{pmatrix}I_3 & V_{Y_L^\delta} \\ 0_{2\times3} & A_{Y_L^\delta}\end{pmatrix},$$

$$V_{Y_L^\delta} = \begin{pmatrix}-\delta g & -\frac{\delta^2}{2}g\end{pmatrix}, \quad A_{Y_L^\delta} = \begin{pmatrix}1 & \delta \\ 0 & 1\end{pmatrix}. \tag{13}$$

The previous Lemma showed that the solution to $Y_L^\delta(t)$ is fixed in t and can be easily obtained for any δ. It also showed that the solution of Y_R^δ can be propagated through time. In practice this means that, following initialisation of $Y_R^\delta(t)$ for the period of $t \in [0, \delta)$ using (8b), the solution $Y_R^\delta(t)$ can be updated by recursively solving (11) for all $t \geq \delta$. The next Lemma shows how the delay matrices can be used to modify the GNSS measurements by using their matrix form (7).

Lemma 4.3: Let $X(t) \in \mathbf{SE}_2(3)$ be a trajectory of the system (5) as in (4), and define the delay matrices $Y_L^\delta, Y_R^\delta \in \mathbf{SIM}_2(3)$ as in Lemma 4.1. Fix an arbitrary $C \in \mathbb{R}^2$, and consider a delayed measurement $y^\delta := V(t-\delta)C$. Then

$$\mu^\delta = R\mathring{\mu}^\delta + VC^\delta, \tag{14}$$

where

$$\mu^\delta := y^\delta - V_{Y_L^\delta}C^\delta, \quad \mathring{\mu}^\delta := V_{Y_R^\delta}C, \quad C^\delta := A_{Y_L^\delta}^{-1}C.$$

Proof: Using the matrix form (7) of the measurement,

$$\begin{pmatrix}y^\delta \\ C\end{pmatrix} = \begin{pmatrix}V(t-\delta)C \\ C\end{pmatrix} = X(t-\delta)\begin{pmatrix}0_{3\times1} \\ C\end{pmatrix}.$$

Recall the delay equation (9) and the expanded form of Y_L (13). Then multiply both sides with $(Y_L^\delta)^{-1}$ to obtain

$$\begin{aligned}
(Y_L^\delta)^{-1}\begin{pmatrix}y^\delta \\ C\end{pmatrix} &= (Y_L^\delta)^{-1}X(t-\delta)\begin{pmatrix}0_{3\times1} \\ C\end{pmatrix}, \\
\begin{pmatrix}I_3 & -V_{Y_L^\delta}A_{Y_L^\delta}^{-1} \\ 0_{2\times3} & A_{Y_L^\delta}^{-1}\end{pmatrix}\begin{pmatrix}y^\delta \\ C\end{pmatrix} &= (Y_L^\delta)^{-1}Y_L^\delta XY_R^\delta\begin{pmatrix}0_{3\times1} \\ C\end{pmatrix}, \\
\begin{pmatrix}y^\delta - V_{Y_L^\delta}A_{Y_L^\delta}^{-1}C \\ A_{Y_L^\delta}^{-1}C\end{pmatrix} &= X\begin{pmatrix}R_{Y_R^\delta} & V_{Y_R^\delta} \\ 0_{2\times3} & A_{Y_R^\delta}\end{pmatrix}\begin{pmatrix}0_{3\times1} \\ C\end{pmatrix}, \\
\begin{pmatrix}y^\delta - V_{Y_L^\delta}A_{Y_L^\delta}^{-1}C \\ A_{Y_L^\delta}^{-1}C\end{pmatrix} &= \begin{pmatrix}R & V \\ 0_{2\times3} & I_2\end{pmatrix}\begin{pmatrix}V_{Y_R^\delta}C \\ A_{Y_R^\delta}C\end{pmatrix}, \\
\begin{pmatrix}y^\delta - V_{Y_L^\delta}A_{Y_L^\delta}^{-1}C \\ A_{Y_L^\delta}^{-1}C\end{pmatrix} &= \begin{pmatrix}RV_{Y_R^\delta}C + VA_{Y_R^\delta}C \\ A_{Y_R^\delta}C\end{pmatrix}, \\
\begin{pmatrix}\mu^\delta \\ C^\delta\end{pmatrix} &= \begin{pmatrix}R\mathring{\mu}^\delta + VC^\delta \\ C^\delta\end{pmatrix},
\end{aligned}$$

where the last line follows from the fact that $A_{Y_R^\delta} = A_{Y_L^\delta}^{-1}$ as another consequence of the delay equation (9). ■

V. Observer Design

We will use the observer architecture proposed in [14, Section 3.1], and then provide a general way to incorporate measurements of the form (14). The result of Lemma 4.3 is the key to using the delayed measurements. Recall the system dynamics (5) and consider the observer architecture [14]

$$\dot{\hat{X}} = (G+N)\hat{X} + \hat{X}(U-N) + \mathrm{Ad}_Z(\Delta)\hat{X}, \tag{15a}$$

$$\dot{Z} = (G+N)Z - Z\Gamma, \tag{15b}$$

where $\hat{X} \in \mathbf{SE}_2(3)$ is the *state estimate*, $Z \in \mathbf{SIM}_2(3)$ is the *auxiliary state*, and $\Delta \in \mathfrak{se}_2(3)$ and $\Gamma \in \mathfrak{sim}_2(3)$ are correction terms that we have yet to design. The *observer error* is defined to be

$$\bar{E} := Z^{-1}X\hat{X}^{-1}Z \in \mathbf{SE}_2(3), \tag{16}$$

and is a synchronous error [15]. For further details about the rationale of this error definition, please see [15].

We may simplify the auxiliary state dynamics (15b) by choosing the rotation correction $\Omega_\Gamma \equiv 0$ and the initial condition $R_Z(0) = I_3$. The result is the $\dot{R}_Z \equiv 0$ for all time, and thus $R_Z \equiv I_3$. This is possible as the uncorrected dynamics $\dot{Z} = (G + N)Z$ do not involve the rotation component R_Z. We will use this simplification through the remainder of the paper.

Let $R_{\bar{E}} \in \mathbf{SO}(3)$ and $V_{\bar{E}} \in \mathbb{R}^{3\times 2}$ denote the rotational and translational components of $\bar{E}$, respectively, then

$$\begin{aligned} R_{\bar{E}} &= R\hat{R}^{-1} = R\hat{R}^\top, && (17a)\\ V_{\bar{E}} &= R(\hat{R}^\top V_Z - \hat{R}^\top \hat{V} A_Z) + (V - V_Z A_Z^{-1})A_Z, \\ &= (VA_Z - V_Z) - R_{\bar{E}}(\hat{V}A_Z - V_Z). && (17b)\end{aligned}$$

Lemma 5.1 (Synchrony [14]): The dynamics of the observer error $\bar{E}$ defined in (16) are

$$\dot{\bar{E}} = \Gamma\bar{E} - \bar{E}(\Gamma + \Delta). \tag{18}$$

In particular, they are independent of the inputs, and are zero when the correction terms Δ and Γ are nullified.

One of the useful consequences of synchrony is its application to modular observer design (cf. [15, Theorem 5.4]). This property is central to the final observer design we will propose. In the sequel, we will consider the positive-definite cost function $\mathcal{L} : \mathbf{SE}_2(3) \to \mathbb{R}^+$ defined by

$$\mathcal{L}(\bar{E}) := \operatorname{tr}(I_3 - R_{\bar{E}}) + |V_{\bar{E}}|^2. \tag{19}$$

We restate Theorem 5.4 from [15].

Theorem 5.2 (Modularity Theorem): Let $\mathcal{L} : \mathbf{SE}_2(3) \to \mathbb{R}^+$ be a uniformly continuous positive-definite cost function. Let $(\Delta_i, \Gamma_i) \in \mathfrak{se}_2(3) \times \mathfrak{sim}_2(3)$ be a collection of correction terms and define the component total-derivatives of $\mathcal{L}$ by

$$\dot{\mathcal{L}}^i := \mathrm{D}_{\bar{E}}\mathcal{L}(\bar{E})[\Gamma_i\bar{E} - \bar{E}\Gamma_i - \bar{E}\Delta_i],$$

for each $i = 1, \dots, n$. Suppose that $\dot{\mathcal{L}}^i \leq 0$ and that Δ_i and Γ_i are uniformly continuous in time. Define the correction terms

$$\Delta = \sum_{i=1}^n \alpha_i\Delta_i, \qquad \Gamma = \sum_{i=1}^n \alpha_i\Gamma_i,$$

for some (possibly time-varying) uniformly continuous gains $\alpha_i \geq \underline{\alpha} > 0$. Then the cost function $\mathcal{L}$ is a Lyapunov function for the dynamics of $\bar{E}$, in the sense that $\dot{\mathcal{L}} \leq 0$, and its set of equilibria is exactly the intersection $\mathcal{E} := \cap_{i=1}^n \mathcal{E}_i$, where

$$\mathcal{E}_i := \left\{\bar{E} \in \mathbf{SE}_2(3) \;\middle|\; \dot{\mathcal{L}}^i \equiv 0\right\}.$$

The following Lemma provides a way to choose correction terms based on any vector-type measurement of the form $\mu = R\mathring{\mu} + VC$. As we have already seen in Section IV-A, the delayed GNSS position and velocity measurements are of this form with $\mathring{\mu} = 0_{3\times 1}$ and $\mu = v, C = \mathbf{e}_1$ for velocity and $\mu = p, C = \mathbf{e}_2$ for position. In fact, the magnetometer measurement can also be written in the same form, with $\mathring{\mu} = y_m$, $\mu = \mathring{y}_m$, and $C = 0_{2\times 1}$. Combining this insight with Theorem 5.2 allows us to easily create an observer that incorporates all three measurement types.

Lemma 5.3: Suppose $\mu \in \mathbb{R}^3$ is a measurement of the system state $X = (R, V) \in \mathbf{SE}_2(3)$ of the form

$$\mu = R\mathring{\mu} + VC,$$

where $\mathring{\mu} \in \mathbb{R}^3$ and $C \in \mathbb{R}^2$ are known. Consider the observer architecture (15), the observer error (16), and the cost function (19). Let $\hat{\mu} = \hat{R}\mathring{\mu} + \hat{V}C$ and $\mu_Z = V_Z A_Z^{-1}C$. Choose gain $k_R, k_V \geq 0$ and $K_q \in \mathbb{S}_{\geq 0}(2)$, and define the correction terms by

$$\begin{aligned} \Omega_\Delta &= 4k_R(\hat{\mu} - \mu_Z) \times (\mu - \mu_Z), \\ W_\Delta &= (k_V + k_R)(\mu - \hat{\mu})C^\top A_Z^{-\top}, \\ \Omega_\Gamma &= 0, \\ W_\Gamma &= (k_V + k_R)(\mu_Z - \mu)C^\top A_Z^{-\top}, \\ S_\Gamma &= -\frac{k_V}{2}A_Z^{-1}CC^\top A_Z^{-\top} + \frac{1}{2}A_Z^\top K_q A_Z. \end{aligned}$$

Then the cost function derivative satisfies

$$\begin{aligned}\dot{\mathcal{L}} \leq &-2k_R\left(|(R_{\bar{E}}^2 - I_3)(\mu - \mu_Z)| - |V_{\bar{E}}A_Z^{-1}C|\right)^2 \\ &- 2k_V|V_{\bar{E}}A_Z^{-1}C|^2 - \lambda_{\min}(K_q)|V_{\bar{E}}A_Z^\top|^2,\end{aligned}$$

where $\lambda_{\min}(K_q)$ is the minimum eigenvalue of K_q.

Proof: Expanding the error dynamics (18) into their component parts yields

$$\begin{aligned} \dot{R}_{\bar{E}} &= R_{\bar{E}}\Omega_\Delta^\times, && (20a)\\ \dot{V}_{\bar{E}} &= -V_{\bar{E}}S_\Gamma + (I - R_{\bar{E}})W_\Gamma - R_{\bar{E}}W_\Delta. && (20b)\end{aligned}$$

Substituting the chosen correction terms into (20b) yields

$$\begin{aligned}\dot{V}_{\bar{E}} &= -V_{\bar{E}}S_\Gamma + W_\Gamma - R_{\bar{E}}(W_\Gamma + W_\Delta) \\ &= V_{\bar{E}}\frac{k_V}{2}A_Z^{-1}CC^\top A_Z^{-\top} - V_{\bar{E}}\frac{1}{2}A_Z^\top K_q A_Z \\ &\quad + (k_V + k_R)(\mu_Z - \mu)C^\top A_Z^{-\top} \\ &\quad - R_{\bar{E}}(k_V + k_R)(\mu_Z - \hat{\mu})C^\top A_Z^{-\top} \\ &= V_{\bar{E}}\frac{k_V}{2}A_Z^{-1}CC^\top A_Z^{-\top} - V_{\bar{E}}\frac{1}{2}A_Z^\top K_q A_Z \\ &\quad - (k_V + k_R)V_{\bar{E}}A_Z^{-1}CC^\top A_Z^{-\top} \\ &= -V_{\bar{E}}\left((k_R + \frac{k_V}{2})A_Z^{-1}CC^\top A_Z^{-\top} + V_{\bar{E}}\frac{1}{2}A_Z^\top K_q A_Z\right).\end{aligned}$$

As for the rotational component, observe that

$$\begin{aligned}&\mu - \mu_Z \\ &= R\mathring{\mu} + VC - \mu_Z \\ &= R_{\bar{E}}\hat{R}\mathring{\mu} + VC - \mu_Z \\ &= R_{\bar{E}}(\hat{\mu} - \hat{V}C) + VC - \mu_Z \\ &= R_{\bar{E}}(\hat{\mu} - \mu_Z) - R_{\bar{E}}(\hat{V}C - \mu_Z) + VC - \mu_Z \\ &= R_{\bar{E}}(\hat{\mu} - \mu_Z) \\ &\quad + \left((VA_Z - V_Z) - R_{\bar{E}}(\hat{V}A_Z - V_Z)\right)A_Z^{-1}C \\ &= R_{\bar{E}}(\hat{\mu} - \mu_Z) + V_{\bar{E}}A_Z^{-1}C.\end{aligned}$$

Therefore, by [15, Lemma A.1],

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}(I_3 - R_{\bar{E}}) = &-2k_R|(I_3 - R_{\bar{E}}^2)(\mu - \mu_Z)|^2 \\ &- 4k_R\langle(I_3 - R_{\bar{E}}^2)(\mu - \mu_Z), V_{\bar{E}}A_Z^{-1}C\rangle.\end{aligned}$$

These results combine to give the cost function derivative,

$$\begin{aligned}
\dot{\mathcal{L}} &= \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}(I_3 - R_{\bar{E}}) + |V_{\bar{E}}|^2 \\
&= -2k_R|(I_3 - R_{\bar{E}}^2)(\mu - \mu_Z)|^2 \\
&\quad - 4k_R\langle (I_3 - R_{\bar{E}}^2)(\mu - \mu_Z), V_{\bar{E}} A_Z^{-1} C\rangle \\
&\quad - (2k_R + k_V)\langle V_{\bar{E}}, V_{\bar{E}} A_Z^{-1} C C^\top A_Z^{-\top}\rangle \\
&\quad - \langle V_{\bar{E}}, V_{\bar{E}} A_Z^\top K_q A_Z\rangle \\
&\leq -2k_R|(I_3 - R_{\bar{E}}^2)(\mu - \mu_Z)|^2 \\
&\quad + 4k_R|(I_3 - R_{\bar{E}}^2)(\mu - \mu_Z)||V_{\bar{E}} A_Z^{-1} C| \\
&\quad - (2k_R + k_V)|V_{\bar{E}} A_Z^{-1} C|^2 - \lambda_{\min}(K_q)|V_{\bar{E}} A_Z^\top|^2 \\
&= -2k_R\left(|(R_{\bar{E}}^2 - I_3)(\mu - \mu_Z)| - |V_{\bar{E}} A_Z^{-1} C|\right)^2 \\
&\quad - k_V|V_{\bar{E}} A_Z^{-1} C|^2 - \lambda_{\min}(K_q)|V_{\bar{E}} A_Z^\top|^2.
\end{aligned}$$

This completes the proof. ∎

Finally, we are ready to state our main theorem. This theorem combines correction terms for delayed GNSS position and velocity, and (undelayed) magnetometer measurements.

Theorem 5.4: Let $X \in \mathbf{SE}_2(3)$ denote the system state as in (4), with dynamics given by (5). Let $y_p^\delta, y_v^\delta, y_m \in \mathbb{R}^3$ denote the delayed GNSS position (3), delayed GNSS velocity, and magnetometer measurements, respectively. Define $Y_L^\delta, Y_R^\delta \in \mathbf{SIM}_2(3)$ to be the delay matrices as in Lemma 4.1, and define

$$\mu_v^\delta := y_v^\delta - V_{Y_L^\delta} C_v^\delta, \quad \mathring{\mu}_v^\delta := V_{Y_R^\delta}\mathbf{e}_1, \quad C_v^\delta := A_{Y_L^\delta}^{-1}\mathbf{e}_1 \tag{21a}$$

$$\mu_p^\delta := y_p^\delta - V_{Y_L^\delta} C_p^\delta, \quad \mathring{\mu}_p^\delta := V_{Y_R^\delta}\mathbf{e}_2, \quad C_p^\delta := A_{Y_L^\delta}^{-1}\mathbf{e}_2 \tag{21b}$$

Define the observer state $\hat{X} \in \mathbf{SE}_2(3)$ and auxiliary state $Z \in \mathbf{SIM}_2(3)$ to have dynamics given by (15). Then define

$$\hat{\mu}_v^\delta = \hat{R}\mathring{\mu}_v^\delta + \hat{V} C_v^\delta, \qquad \mu_{Z,v}^\delta = V_Z A_Z^{-1} C_v^\delta, \tag{22a}$$

$$\hat{\mu}_p^\delta = \hat{R}\mathring{\mu}_p^\delta + \hat{V} C_p^\delta, \qquad \mu_{Z,p}^\delta = V_Z A_Z^{-1} C_p^\delta. \tag{22b}$$

Let the initial condition of the auxiliary rotation $R_Z(0) = I_3$, and choose gains $k_p, k_c > 0$, $K_q \in \mathbb{S}_+(2)$, and $k_v, k_d, k_m \geq 0$. Define the correction terms $\Delta \in \mathfrak{se}_2(3)$ and $\Gamma \in \mathfrak{sim}_2(3)$ to be

$$\begin{aligned}
\Omega_\Delta &= 4k_c(\hat{\mu}_p^\delta - \mu_{Z,p}^\delta) \times (\mu_p^\delta - \mu_{Z,p}^\delta) \\
&\quad + 4k_d(\hat{\mu}_v^\delta - \mu_{Z,v}^\delta) \times (\mu_v^\delta - \mu_{Z,v}^\delta) \\
&\quad + 4k_m(\hat{R} y_m) \times \mathring{y}_m,
\end{aligned} \tag{23a}$$

$$W_\Delta = (k_p + k_c)(\mu_p^\delta - \hat{\mu}_p^\delta){C_p^\delta}^\top A_Z^{-\top} + (k_v + k_d)(\mu_v^\delta - \hat{\mu}_v^\delta){C_v^\delta}^\top A_Z^{-\top}, \tag{23b}$$

$$\Omega_\Gamma = 0, \tag{23c}$$

$$W_\Gamma = -(k_p + k_c)(\mu_p^\delta - \mu_{Z,p}^\delta){C_p^\delta}^\top A_Z^{-\top} - (k_v + k_d)(\mu_v^\delta - \mu_{Z,v}^\delta){C_v^\delta}^\top A_Z^{-\top}, \tag{23d}$$

$$S_\Gamma = -\frac{k_p}{2} A_Z^{-1} C_p^\delta {C_p^\delta}^\top A_Z^{-\top} - \frac{k_v}{2} A_Z^{-1} C_v^\delta {C_v^\delta}^\top A_Z^{-\top} + \frac{1}{2} A_Z^\top K_q A_Z. \tag{23e}$$

Denote the error state $\bar{E} \in \mathbf{SE}_2(3)$ as defined in (16). Assume that the vectors $\sqrt{k_c}\mu_p^\delta, \sqrt{k_d}\mu_v^\delta, \sqrt{k_m} y_m$ are persistently exciting (1). Then

1) The translational error $V_{\bar{E}} \to 0$ globally exponentially.
2) The rotational error $R_{\bar{E}} \to I_3$ almost-globally asymptotically and locally exponentially, with the only stable equilibrium being $\bar{E} = I_5$, and the set unstable equilibria given by

$$\mathcal{E}_u = \left\{ \bar{E} \in \mathbf{SE}_2(3) \;\middle|\; V_{\bar{E}} = 0,\ \operatorname{tr}(R_{\bar{E}}) = -1 \right\}.$$

3) If the error $\bar{E}$ converges to I_5, then the estimated state $\hat{X}$ converges to the true state X in the sense that $|\hat{X} - X| \to 0$.

Proof: Before proving the individual items, observe that the choice $R_Z(0) = I_3$ and $\Omega_\Gamma \equiv 0$ mean that $R_Z \equiv I_3$ for all time.

Proof of Item 1: Recall from Lemma 4.3 that

$$\mu_p^\delta = R\mathring{\mu}_p^\delta + V C_p^\delta, \qquad \mu_v^\delta = R\mathring{\mu}_v^\delta + V C_v^\delta.$$

Thus, the correction terms are simply the sum of correction terms constructed according to Lemma 5.3. Note that S_Γ includes only one instance K_q, which is simply the sum of the term obtained from the individual corrections relating to position, velocity and magnetometer. Let $\mathcal{L}_V := |V_{\bar{E}}|^2$. Then, following the same computation as in the proof of Lemma 5.3 yields

$$\begin{aligned}
\dot{\mathcal{L}}_V &\leq -k_p|V_{\bar{E}} A_Z^{-1} C_p^\delta|^2 - k_v|V_{\bar{E}} A_Z^{-1} C_v^\delta|^2 \\
&\quad - \lambda_{\min}(K_q)|V_{\bar{E}} A_Z^\top|^2 \\
&< -\lambda_{\min}(K_q)\lambda_{\min}(A_Z A_Z^\top)|V_{\bar{E}}|^2.
\end{aligned}$$

We proceed to show that $\lambda(A_Z A_Z^\top)$ is bounded below. Let $P = A_Z A_Z^\top$, then computation yields

$$\begin{aligned}
\dot{P} &= (S_D A_Z - A_Z S_\Gamma) A_Z^\top + A_Z(S_D A_Z - A_Z S_\Gamma)^\top, \\
&= S_D P + P S_D^\top + k_p C_p^\delta {C_p^\delta}^\top + k_v C_v^\delta {C_v^\delta}^\top - P K_q P,
\end{aligned}$$

This is the continuous differential Riccati equation associated with the state dynamics and measurement matrices,

$$\left(-S_D^\top, \begin{pmatrix} k_p C_p^\top \\ k_v C_v^\top \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -\delta k_p & k_p \\ k_v & 0 \end{pmatrix}\right),$$

which is clearly observable for any delay δ, even if $k_v = 0$. Therefore the eigenvalues of $P = A_Z A_Z^\top$ are bounded above and below, and $\mathcal{L}_V = |V_{\bar{E}}|$ is exponentially decreasing to zero with its exponent lower-bounded by $\lambda_{\min}(K_q)\lambda_{\min}(P) > 0$.

Proof of Item 2: We apply Theorem 5.2. Since the proposed correction terms are the sum of corrections terms drawn from Lemma 5.3, the Lyapunov function $\mathcal{L}$ satisfies

$$\begin{aligned}
\dot{\mathcal{L}}(\bar{E}) &\leq -2k_c\left(|(R_{\bar{E}}^2 - I_3)(\mu_p^\delta - \mu_{Z,p}^\delta)| - |V_{\bar{E}} A_Z^{-1} C_p^\delta|\right)^2 \\
&\quad - 2k_d\left(|(R_{\bar{E}}^2 - I_3)(\mu_v^\delta - \mu_{Z,v}^\delta)| - |V_{\bar{E}} A_Z^{-1} C_v^\delta|\right)^2 \\
&\quad - 2k_m|(R_{\bar{E}}^2 - I_3)\mathring{y}_m|^2 + \dot{\mathcal{L}}_V
\end{aligned}$$

Since $\dot{\mathcal{L}}$ is the composition sum and product of uniformly continuous signals, it itself is uniformly continuous. Thus, as the cost $\mathcal{L}(\bar{E})$ is bounded above by its initial value and below by zero, and $V_{\bar{E}} \to 0$ by item 1, applying Barbalat's lemma [21, Lemma 4.2/4.3] yields

$$\begin{aligned}\dot{\mathcal{L}} \to &-2k_c|(R_{\bar{E}}^2 - I_3)(\mu_p^\delta - \mu_{Z,p}^\delta)|^2 - 2k_m|(R_{\bar{E}}^2 - I_3)\mathring{y}_m|^2 \\ &- 2k_d|(R_{\bar{E}}^2 - I_3)(\mu_v^\delta - \mu_{Z,v}^\delta)|^2 \\ &\to 0.\end{aligned}$$

Therefore, each of the three individual terms must also converge to zero. By persistence of excitation [19], $R_{\bar{E}}$ converges to either I_3 or to the set of rotation matrices

$$\mathcal{R}_u := \{R_{\bar{E}} \in \mathbf{SO}(3) \mid \operatorname{tr}(R_{\bar{E}}) = -1\}.$$

To see that the first case is locally exponentially stable, linearise $R_{\bar{E}} \approx I_3 + \bar{\varepsilon}_R^\times$ and differentiate to obtain

$$\dot{\bar{\varepsilon}} \approx 4\left(k_c(\mu_p^\delta - \mu_{Z,p}^\delta)^{\times 2} + k_d(\mu_v^\delta - \mu_{Z,v}^\delta)^{\times 2} + k_m \mathring{y}_m^{\times 2}\right)\bar{\varepsilon},$$

which is persistently exciting by assumption, and symmetric negative semi-definite. Uniform local exponential stability follows from [22, Theorem 1].

In the case that $R_{\bar{E}} \to \mathcal{R}_u$, or equivalently $\bar{E} \to \mathcal{E}_u$, we have left to show that any such equilibrium is unstable. It suffices to show that, if $R_{\bar{E}} \in \mathcal{R}_u$, then in any neighbourhood $\mathcal{U} \subset \mathbf{SO}(3)$ of $R_{\bar{E}}$ there exists $Q \in \mathcal{U}$ for which $\mathcal{L}_R(Q) < \mathcal{L}_R(R_{\bar{E}})$, where

$$\mathcal{L}_R(Q) := \operatorname{tr}(I_3 - R_{\bar{E}})$$

is the rotational component of the Lyapunov function (19). To this end, fix $R_{\bar{E}} \in \mathcal{R}_u$, then $R_{\bar{E}}$ has an eigenvalue equal to 1 associated with a unit vector $\omega \in \mathbb{R}^3$; i.e. $R_{\bar{E}}\omega = \omega$, $|\omega| = 1$. Define $Q(s) = R_{\bar{E}} \exp(s\omega^\times)$. Using a second-order Taylor expansion, one has

$$\begin{aligned}\mathcal{L}_R(Q(s)) &\approx \operatorname{tr}(I_3 - R_{\bar{E}}(I_3 + s\omega^\times + \frac{s^2}{2}(\omega^\times)^2)) \\ &= \operatorname{tr}(I_3 - R_{\bar{E}}) - s\operatorname{tr}(R_{\bar{E}}\omega^\times) - \frac{s^2}{2}\operatorname{tr}(R_{\bar{E}}(\omega^\times)^2) \\ &= \mathcal{L}_R(R_{\bar{E}}) - \frac{s^2}{2}\operatorname{tr}(R_{\bar{E}}(\omega\omega^\top - I_3)) \\ &= \mathcal{L}_R(R_{\bar{E}}) - \frac{s^2}{2}\operatorname{tr}(\omega\omega^\top - R_{\bar{E}}) \\ &= \mathcal{L}_R(R_{\bar{E}}) - s^2.\end{aligned}$$

Clearly, then, in any neighbourhood of $R_{\bar{E}} \in \mathcal{R}_u$ one can find s sufficiently small so that $\mathcal{L}_R(Q(s)) < \mathcal{L}_R(R_{\bar{E}})$. This shows that any equilibrium point $\bar{E} \in \mathcal{E}_u$ is indeed unstable.

Proof of Item 3: Since Z is bounded above and below as shown in the proof of item 1, the operation $E \mapsto ZEZ^{-1}$ is bounded also. Therefore, by [16, Lemma 5.3], $\bar{E} \to I_5$ if and only if $X\hat{X}^{-1} \to I_5$. Then, assuming boundedness of X, $\hat{X} \to X$ as required. This completes the proof. ■

VI. Simulation Results

The proposed observer was verified using a simulation of a flying vehicle equipped with a magnetometer and time-delayed GNSS flying in a circular trajectory of radius 50 m at a speed of 25 m/s. The true initial conditions $X = (R, (v, p)) \in \mathbf{SE}_2(3)$ and inputs $\Omega, a \in \mathbb{R}^3$ were set to

$$R(0) = I_3, \quad v(0) = \begin{pmatrix}0 & 25 & 0\end{pmatrix}^\top, \quad p(0) = \begin{pmatrix}50 & 0 & 0\end{pmatrix}^\top,$$

$$\Omega(t) = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^\top, \quad a(t) = -\frac{1}{4}R(t)^\top p(t) - R(t)^\top g,$$

where $g = 9.81\mathbf{e}_3 \in \mathbb{R}^3$. The magnetic reference and the GNSS delay were defined by

$$\mathring{y}_m = \mathbf{e}_1, \qquad \delta = 0.2.$$

The observer proposed in Theorem 5.2 was implemented with the following conditions. The estimated state $\hat{X} = (\hat{R}, (\hat{v}, \hat{p})) \in \mathbf{SE}_2(3)$ was initialised with an extreme initial attitude error,

$$\begin{aligned}\hat{R}(0) &= \exp(0.99\mathbf{e}_1^\times), \\ \hat{v}(0) &= \begin{pmatrix}2 & 27 & 2\end{pmatrix}^\top, \\ \hat{p}(0) &= \begin{pmatrix}70 & 20 & 20\end{pmatrix}^\top,\end{aligned}$$

and the auxiliary state $Z \in \mathbf{SIM}_2(3)$ was initialised by

$$R_Z(0) = I_3, \quad V_Z(0) = \hat{V}A_Z, \quad A_Z(0) = \operatorname{diag}(2, 10).$$

The gains were chosen to be

$$\begin{aligned}K_q &= \operatorname{diag}(10.0, 2.0), \quad & k_p &= 10.0, \quad & k_c &= 0.1, \\ k_v &= 10.0 & k_d &= 0.1, & k_m &= 2.0.\end{aligned}$$

Both the system and observer equations were implemented using Lie group Euler integration at 50 Hz for 20 s. Additionally, a second copy of the observer was implemented without correction for the GNSS delay; that is, the observer dynamics were implemented with $\delta = 0$ although the measurements received were still delayed.

The first observer requires the delay matrices described in Lemma 4.1 to implement the correction terms. Since the right-delay matrix $Y_R^\delta(t)$ cannot be constructed without access to the input $U(t-\delta)$, this matrix was initialised at the time $t = \delta$. For the period $t \in [0, \delta)$, the correction terms Δ and Γ were both set to zero. Thanks to the synchrony of the error, this meant $\dot{\bar{E}} = 0$ and hence $\mathcal{L}(\bar{E})$ is constant during this period.

Figure 1 shows the estimated and true states over time. Figure 2 shows the estimation errors and the Lyapunov function value. The estimates from the delay-compensated observer are shown to quickly converge to the true values, and this is reflected in the exponentially decreasing Lyapunov value. In contrast, the second observer (without delay compensation) is shown to converge more slowly during the transient phase, and fails to converge fully in the steady state with errors of approximately 3.5 deg in attitude, 2.5 m/s in velocity, and 5 m in position. This failure to converge is also seen in the Lyapunov value, which quickly plateaus. These results not only verify the proposed observer design, but also demonstrate the importance of delay compensation to ensure accurate estimation in INS.

VII. Conclusion

This paper studied the problem of INS with magnetometer and delayed GNSS measurements by extending recent work on synchronous observer design for INS [15]. Delay matrices that are recursively defined through differential equations

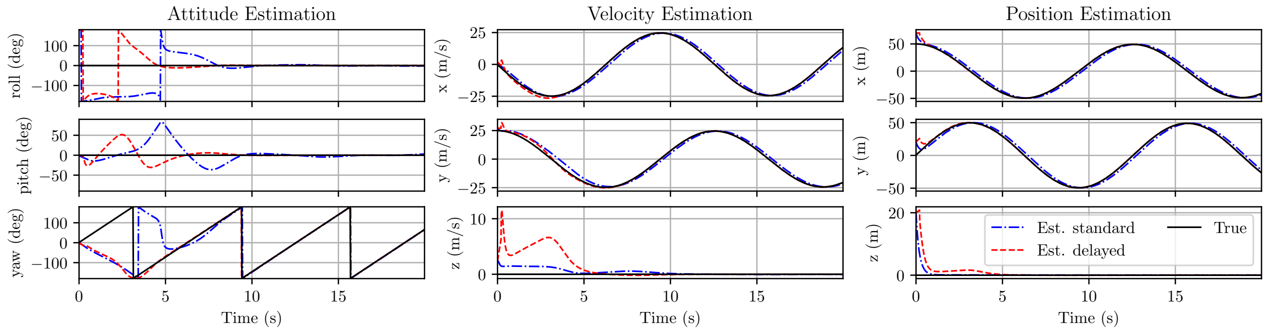

Fig. 1. The states estimated by an observer with (- - -) and without (-·-·) delay compensation compared to the true states (——).

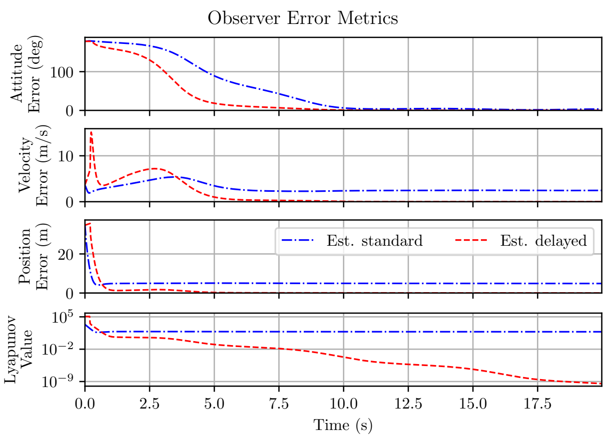

Fig. 2. The estimation errors of an observer with (- - -) and without (-·-·) delay compensation.

are shown to relate the delayed GNSS measurements to the present state of the system. Using this relationship, correction terms were proposed that yield almost-globally asymptotic and locally exponential stability of the observer error dynamic. Finally, the simulation results showed that the proposed observer design is able to converge from extreme initial error, and that the compensation of time-delay contributes to both transient and steady-state performance.

References

[1] O. J. Woodman, "An introduction to inertial navigation," University of Cambridge, Computer Laboratory, Tech. Rep. UCAM-CL-TR-696, 2007. [Online]. Available: https://www.cl.cam.ac.uk/techreports/UCAM-CL-TR-696.html?ref=https://githubhelp.com

[2] B. Vik and T. Fossen, "A nonlinear observer for GPS and INS integration," in *Proceedings of the 40th IEEE Conference on Decision and Control*, 2001, pp. 2956–2961.

[3] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, "Nonlinear observer for GNSS-aided inertial navigation with quaternion-based attitude estimation," in *2013 American Control Conference*, Jun. 2013, pp. 272–279.

[4] S. Berkane, A. Tayebi, and S. de Marco, "A nonlinear navigation observer using IMU and generic position information," *Automatica*, vol. 127, p. 109513, May 2021. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109821000339

[5] J. Sola, "Quaternion kinematics for the error-state Kalman filter," *arXiv preprint arXiv:1711.02508*, 2017.

[6] A. Barrau and S. Bonnabel, "The Invariant Extended Kalman Filter as a Stable Observer," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797–1812, Apr. 2017.

[7] P. van Goor, T. Hamel, and R. Mahony, "Equivariant Filter (EqF)," *IEEE Transactions on Automatic Control*, vol. 68, no. 6, pp. 3501–3512, Jun. 2023.

[8] A. Fornasier, Y. Ge, P. van Goor, R. Mahony, and S. Weiss, "Equivariant Symmetries for Inertial Navigation Systems," Sep. 2023. [Online]. Available: http://arxiv.org/abs/2309.03765

[9] A. Roberts and A. Tayebi, "On the attitude estimation of accelerating rigid-bodies using GPS and IMU measurements," in *2011 50th IEEE Conference on Decision and Control and European Control Conference*, Dec. 2011, pp. 8088–8093.

[10] M.-D. Hua, T. Hamel, and C. Samson, "Riccati nonlinear observer for velocity-aided attitude estimation of accelerated vehicles using coupled velocity measurements," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, Dec. 2017, pp. 2428–2433.

[11] R. Mahony, T. Hamel, and J. Pflimlin, "Nonlinear Complementary Filters on the Special Orthogonal Group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, Jun. 2008.

[12] M.-D. Hua, "Attitude estimation for accelerated vehicles using GPS/INS measurements," *Control Engineering Practice*, vol. 18, no. 7, pp. 723–732, Jul. 2010. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0967066110000249

[13] P. van Goor, T. Hamel, and R. Mahony, "Constructive Equivariant Observer Design for Inertial Velocity-Aided Attitude," *IFAC-PapersOnLine*, vol. 56, no. 1, pp. 349–354, Jan. 2023. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S2405896323002471

[14] P. van Goor, T. Hamel, and R. Mahony, "Constructive equivariant observer design for inertial navigation," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 2494–2499, 2023.

[15] P. van Goor, T. Hamel, and R. Mahony, "Synchronous Observer Design for Inertial Navigation Systems with Almost-Global Convergence," Nov. 2023. [Online]. Available: http://arxiv.org/abs/2311.02234

[16] P. van Goor and R. Mahony, "Autonomous Error and Constructive Observer Design for Group Affine Systems," in *2021 60th IEEE Conference on Decision and Control (CDC)*, Dec. 2021, pp. 4730–4737.

[17] J. M. Hansen, T. I. Fossen, and T. Arne Johansen, "Nonlinear observer design for GNSS-aided inertial navigation systems with time-delayed GNSS measurements," *Control Engineering Practice*, vol. 60, pp. 39–50, Mar. 2017. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0967066116302751

[18] A. Khosravian, J. Trumpf, R. Mahony, and T. Hamel, "State estimation for invariant systems on Lie groups with delayed output measurements," *Automatica*, vol. 68, pp. 254–265, Jun. 2016. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S000510981600025X

[19] J. Trumpf, R. Mahony, T. Hamel, and C. Lageman, "Analysis of Non-Linear Attitude Observers for Time-Varying Reference Measurements," *IEEE Transactions on Automatic Control*, vol. 57, no. 11, pp. 2789–2800, Nov. 2012.

[20] A. Barrau and S. Bonnabel, "Invariant particle filtering with application to localization," in *53rd IEEE Conference on Decision and Control*, Dec. 2014, pp. 5599–5605.

[21] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, N.J: Prentice Hall, 1991.

[22] A. P. Morgan and K. S. Narendra, "On the uniform asymptotic stability of certain linear nonautonomous differential equations," *SIAM Journal on Control and Optimization*, vol. 15, no. 1, pp. 5–24, 1977. [Online]. Available: https://doi.org/10.1137/0315002