2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Unified Behavioral Data-Driven Performance Analysis
A Generalized Plant Approach

L.M. Spin, C. Verhoek, W.P.M.H. Heemels, N. van de Wouw, R. Tóth

***Abstract*— In this paper, we present a novel approach to combine data-driven non-parametric representations with model-based representations of dynamical systems. Based on a data-driven form of linear fractional transformations, we introduce a data-driven form of generalized plants. This form can be leveraged to accomplish performance characterizations, e.g., in the form of a mixed-sensitivity approach, and LMI-based conditions to verify finite-horizon dissipativity. In particular, we show how finite-horizon ℓ_2-gain under weighting filter-based general performance specifications can be verified for implemented controllers on systems for which only input-output data is available. The overall effectiveness of the proposed method is demonstrated by simulation examples.**

Index Terms— **Data-Driven Control, Dissipativity Analysis**

I. Introduction

Direct data-driven control is a generic term to categorize all control strategies that are based on measured data. The data is subsequently converted into control laws without system identification as an intermediate step. Motivated by the fact that, in general, there is no separation principle between identification and control, the indirect control approach (consisting of identifying a model followed by optimal control) might be suboptimal [1]. Direct data-driven control approaches do not suffer from this inherent principle. Moreover, high-tech systems in, e.g., the semiconductor, aerospace, and process industries, are becoming increasingly more complex, causing first-principles modeling to become more and more challenging to accurately describe the behavior of these systems in terms of compact models that can be used for control. This has motivated a trend towards data-driven control in recent years as an alternative for model-based approaches [2].

A cornerstone result in direct data-driven analysis and control for discrete-time *linear time-invariant* (LTI) systems is known as *Willems' Fundamental Lemma* [3]. This result relies on the behavioral system theory [4] and characterizes the finite-horizon behavior of an LTI system using measured input-output data trajectories. For a detailed overview of results based on Willems' Fundamental Lemma, see the survey [5]. These data-driven non-parametric representations of LTI systems have also been used to analyze system properties such as stability [6] and dissipativity [7], [8]. Tractable formulations for synthesizing controllers that are optimal for various performance metrics have also been derived, see [9], [10]. However, these approaches assume direct full-state measurements, which is rarely available in practice. The construction of an artificial extended state from input-output data is discussed in [11], where it is pointed out that the non-controllable nature of the extended system results in loss of feasibility of controller design.

For model-based controller design, a systematic framework that allows the direct incorporation of performance specifications is known as the generalized plant framework [12]. Generalized plants contain all aspects, including performance shaping filters, of a control design problem except the controller itself. In fact, the interconnection structure between the generalized plant and the controller is defined as a *Linear Fractional Representation* (LFR). One of the powerful shaping concepts for adding performance specifications to the generalized plant in the form of weighting filters is known as mixed-sensitivity shaping [13]. Appropriate choices of weighting filters can be used to characterize requirements related to bandwidth, overshoot, and noise attenuation, [14].

Currently, it is not clear how to combine a data-based description of the plant and model-based descriptions of weighting filters and controllers within the generalized plant framework. Preliminary results are provided in [15], where the feedback interconnection between a data-driven plant and a model-based controller is studied. The analysis is quite limited since it does not allow the formulation of a generalized plant and is restricted to the SISO case. In this paper, we propose to unify the interconnection between MIMO data-driven representations, based on Willems' Fundamental Lemma, and MIMO model-based representations of controllers and weighting filters to analyze the performance of the controllers via a mixed-sensitivity argument.

The main contributions of this paper are as follows:

C1) Combine MIMO model-based and data-driven representations in an LFR-based data-driven generalized plant;

C2) Derive tractable *linear matrix inequality* (LMI)-based methods to analyze the dissipativity-based performance of data-driven generalized plants;

C3) Demonstrate the strength of the new framework in two numerical case studies, where the performance of given controllers is analyzed.

This research was supported by the Eindhoven Artificial Intelligence Systems Institute, Eindhoven University of Technology, the Netherlands, the European Union within the framework of the National Laboratory for Autonomous Systems (RRF-2.3.1-21-2022-00002) and by the European Research Council under the Advanced ERC grant agreement PROACTHIS, no. 101055384.

L. M. Spin, C. Verhoek and R. Tóth are with the Control Systems Group, Eindhoven University of Technology, The Netherlands. R. Tóth is also with the Institute for Computer Science and Control, Budapest, Hungary. W. P. M. H. Heemels is with the Control Systems Technology Group, Eindhoven University of Technology, The Netherlands. N. van de Wouw is with the Dynamics and Control Group, Eindhoven University of Technology, The Netherlands. Corresponding author: L. M. Spin, email address: `l.m.spin@tue.nl`.

The remainder of this paper is structured as follows. Firstly, preliminaries on discrete-time LTI systems are discussed in Section II. The problem setting and approach are defined in Section III. In Section IV, we present the main results of this paper, i.e., a unified representation of data-driven and model-based dynamics into a generalized plant and a tractable dissipativity analysis of this unified representation. We finalize the paper with examples in Section V followed by the conclusions in Section VI.

A. Notation

The sets $\mathbb{C}, \mathbb{R}, \mathbb{Z}$, and $\mathbb{N}$ denote the set of complex numbers, real numbers, integers, and non-negative integers, respectively. The zero matrix with dimension $n \times m$ and the square identity matrix with dimension n are denoted by $0_{n\times m}$ and I_n, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}(A) = r \le \min\{n, m\}$, $A^\top \in \mathbb{R}^{m\times n}$ denotes its transpose and the nullspace is defined as $\operatorname{Ker}(A) := \{x \in \mathbb{R}^m | Ax = 0\}$. Furthermore, $A_\perp \in \mathbb{R}^{m\times m-r}$ denotes a matrix where the columns span $\operatorname{Ker}(A)$. The set of symmetric matrices of dimension $n \times n$ is denoted by $\mathbb{S}^n$. The notation $A \succ 0 (\succcurlyeq 0)$ and $A \prec 0 (\preccurlyeq 0)$ denotes that A is positive (semi)-definite and negative (semi)-definite, respectively. We denote by $\mathbb{R}^{n\times m}[\xi]$ the ring of polynomials with matrix coefficients where ξ is the indeterminate. For brevity, we write quadratic products of the form $x^\top Ax$ as $(*)^\top Ax$. Given a discrete-time signal $z \in (\mathbb{R}^{n_z})^{\mathbb{Z}}$, where $(\mathbb{R}^{n_z})^{\mathbb{Z}}$ defines the collection of all signals $z : \mathbb{Z} \to \mathbb{R}^{n_z}$, the restriction to the interval $[0, L-1] \cap \mathbb{Z}$ is denoted by $z|_L$. The concatenation of two signals z_1 and z_2 is denoted by $z = z_1 \wedge z_2$. For a given sequence of samples $\{z_k\}_{k=0}^{N-1}$ such that $z(k) \in \mathbb{R}^{n_z}$ for all $k \in [0, N-1] \cap \mathbb{N}$, the Hankel matrix $\mathcal{H}_L(z) \in \mathbb{R}^{Ln_z \times N-L+1}$ with depth L is defined as

$$\mathcal{H}_L(z) = \begin{bmatrix} z(0) & z(1) & \dots & z(N-L) \\ z(1) & z(2) & \dots & z(N-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ z(L-1) & z(L) & \dots & z(N-1) \end{bmatrix},$$

for $0 \le L \le N-1$. Lastly, $A \otimes B$ denotes the Kronecker product of two matrices A and B.

II. Preliminaries

A discrete-time dynamical system Σ is defined by the triple $(\mathbb{Z}_{\ge 0}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{Z}_{\ge 0}$ is the time-axis, $\mathbb{W} \subseteq \mathbb{R}^{n_w}$ is a vector space in which signals $w : \mathbb{Z}_{\ge 0} \to \mathbb{W}$ take their value and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{Z}_{\ge 0}}$ is the behavior, which describes the trajectories w allowed by the systems dynamics. The system Σ is said to be linear, if $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{Z}_{\ge 0}}$ and is said to be time-invariant, if it is invariant with respect to the forward shift operator q, i.e., $q\mathfrak{B} \subseteq \mathfrak{B}$, where $(qw)(t) = w(t+1)$. It is known that every LTI system admits a kernel representation that characterizes the behavior

$$\mathfrak{B} := \left\{ w \in \mathbb{W}^{\mathbb{Z}_{\ge 0}} \mid P(q)w = 0 \right\}. \tag{1}$$

Here, $P(\xi) \in \mathbb{R}^{g\times n_w}[\xi]$ is a matrix-valued element of the polynomial ring. The polynomial $P(\xi)$ is defined as

$$P(\xi) = \begin{bmatrix} p_1(\xi) \\ \vdots \\ p_g(\xi) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\ell_1} p_1^{(i)} \xi^i \\ \vdots \\ \sum_{i=0}^{\ell_g} p_g^{(i)} \xi^i \end{bmatrix}. \tag{2}$$

$P(q)w = 0$ is said to be a minimal kernel representation, if the number of equations, i.e., $g \in \mathbb{N}$, is minimal among all equivalent kernel representations. Moreover, the degree of $P(\xi)$ is defined as $\deg(P) = \max_{1\le i\le g} \ell_i$. If the degree of P is minimal among all minimal kernel representations, then P is called a *minimal lag* representation where the lag is defined as

$$\ell(\mathfrak{B}) = \deg(P). \tag{3}$$

We assume that an input-output partitioning $w = (u, y)$ is available where u is maximally free, i.e., u is the input. For a minimal kernel representation, we define the following structured indices $\mathbf{p}(\mathfrak{B}) = n_y = g$ and $\mathbf{m}(\mathfrak{B}) = n_u = n_w - \mathbf{p}(\mathfrak{B})$. Moreover, let the quadruple (A, B, C, D) denote a state-space representation. We denote $\mathbf{n}(\mathfrak{B}) = \dim(A)$ if (A, B, C, D) is a minimal state-space representation. For a partitioned input-output signal w we define $P(\xi) = \begin{bmatrix} N(q) & -D(q) \end{bmatrix} \in \mathbb{R}^{\mathbf{p}(\mathfrak{B})\times(\mathbf{m}(\mathfrak{B})+\mathbf{p}(\mathfrak{B}))}[\xi]$, such that $D(q)y = N(q)u$ defines an input-output representation. The system Σ represented by $P(\xi)$ is said to be controllable if the polynomial matrices $D(\xi)$ and $N(\xi)$ are coprime over $\mathbb{R}[\xi]$, see [16].

III. Approach

In this paper, we consider a general control configuration consisting of discrete-time LTI systems. Specifically, we consider measured input-output data of the system $\mathcal{G}$ instead of a model-based representation. All other components of the control configuration, such as the controller $\mathcal{K}$ and weighting filters, have a model-based representation available. Next, we provide the problem statement.

Problem statement: *Consider a given control configuration including the controller $\mathcal{K}$, closed-loop performance specifications in terms of frequency-domain weighting filters and the unknown system $\mathcal{G}$. If a measured data set is available from $\mathcal{G}$, determine if the closed-loop will satisfy the given performance specifications.*

The following assumption on the available data is necessary to specify the approach towards providing a solution to the problem statement.

Assumption 1. *A measured input-output trajectory $\mathcal{D}_N = \{u_k^d, y_k^d\}_{k=0}^{N-1}$ (data-dictionary) is available of the controllable LTI system $\mathcal{G}$. The lag $\ell(\mathfrak{B}^{\mathcal{G}})$ and order $\mathbf{n}(\mathfrak{B}^{\mathcal{G}})$ of $\mathcal{G}$ are unknown, but upper bounded by ν and n_G, respectively.*

To provide a non-parametric data-driven representation of $\mathcal{G}$, based on $\mathcal{D}_N$, we require the following notion of persistence of excitation.

Definition 1 (Persistence of Excitation)**.** A sequence $\{u_k\}_{k=0}^{N-1}$, where $u(k) \in \mathbb{R}^{n_u}$ for all $k \in \{0, \cdots, N-1\}$, is said to be persistently exciting (PE) of order L, if $\mathcal{H}_L(u)$ has full row rank. □

Under a PE condition on the input in $\mathcal{D}_N$, we can represent LTI systems using *only* the data-set $\mathcal{D}_N$ [3], which, in [17], is formalized for controllable LTI systems by the following lemma.

Lemma 1 (Fundamental Lemma, [17]). *Given a data-dictionary $\mathcal{D}_N$ of $\mathcal{G}$ satisfying Assumption 1 for which the input sequence $\{u_k^d\}_{k=0}^{N-1}$ is PE of order $\nu+n_G$, the sequence $\{u_k, y_k\}_{k=0}^{L-1}$ is a trajectory of $\mathcal{G}$ if and only if there exists a $g \in \mathbb{R}^{N-L+1}$ such that*

$$\begin{bmatrix} \mathcal{H}_L(u^d) \\ \mathcal{H}_L(y^d) \end{bmatrix} g = \begin{bmatrix} u|_L \\ y|_L \end{bmatrix}. \tag{4}$$

This result requires $N \geq (n_{\mathrm{u}}+1)(\nu+n_G)-1$ data points and has been generalized in [18]. In particular, it is shown that the finite-horizon (FH) behavior of $\mathcal{G}$, defined as

$$\mathfrak{B}^{\mathcal{G}}|_L := \{(u|_L, y|_L) \in (\mathbb{R}^{n_{\mathrm{u}}} \times \mathbb{R}^{n_{\mathrm{y}}})^L \mid (u,y) \in \mathfrak{B}^{\mathcal{G}}\}, \tag{5}$$

is equal to the image of $\begin{bmatrix} \mathcal{H}_L(u^d) \\ \mathcal{H}_L(y^d) \end{bmatrix}$ if and only if

$$\operatorname{rank}\left(\begin{bmatrix} \mathcal{H}_L(u^d) \\ \mathcal{H}_L(y^d) \end{bmatrix}\right) = \mathbf{m}(\mathfrak{B}^{\mathcal{G}})L + \mathbf{n}(\mathfrak{B}^{\mathcal{G}}), \tag{6}$$

is satisfied for $L \geq \ell(\mathfrak{B}^{\mathcal{G}})$. The Hankel matrices provide a data-driven non-parametric system representation if this rank condition is satisfied.

Besides the data-driven representation of $\mathcal{G}$ given by (4), all other systems in the closed loop have model-based representations. By pulling $\mathcal{G}$ out of the closed loop, the LFR-based interconnection shown in Figure 1 is obtained, where the data-driven representations are separated from the model-based representations. Here $\mathcal{N}$ denotes the generalized controller and contains the signal routing, controller $\mathcal{K}$, and weighting filters. The minimal lag input-output representation of $\mathcal{N}$ is given by

$$\mathcal{N}: \begin{cases} D_{\mathrm{u}}(q)u = N_{\mathrm{uy}}(q)y + N_{\mathrm{uw}}(q)w, \\ D_{\mathrm{z}}(q)z = N_{\mathrm{zy}}(q)y + N_{\mathrm{zw}}(q)w. \end{cases} \tag{7}$$

The behavior $\mathfrak{B}^{\mathcal{N}}$ of $\mathcal{N}$ is defined as the collection of signals (y,w,u,z) such that (7) is satisfied. In the next section, we will provide a unified representation of the closed loop as the LFR-based interconnection between the data-driven representation $\mathcal{G}$, given by (4), and the model-based representation $\mathcal{N}$, given by (7), defined on a finite horizon.

Performance of the closed loop is evaluated through dissipativity, which can be used to characterize, for example, the ℓ_2 gain of the system through appropriate choice of the supply function. This system gain is equivalent to the $\mathcal{H}_\infty$-norm which is used to characterize performance in the mixed-sensitivity setting. The dissipativity notion introduced in [19] is defined for input-output representations of dynamical systems. In particular, this notion does not require a storage function and is equivalent to the notion of dissipativity of [20].

Definition 2 (Dissipativity, [19]). A discrete-time LTI system $\mathcal{G}$ with behavior $\mathfrak{B}$ is said to be dissipative with respect to the supply rate $\Pi \in \mathbb{S}^{(n_{\mathrm{u}}+n_{\mathrm{y}})\times(n_{\mathrm{u}}+n_{\mathrm{y}})}$ if

$$\sum_{k=0}^{r} (*)^\top \Pi \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \leq 0, \quad \text{for all } r \in \mathbb{N}, \tag{8}$$

holds for all trajectories $(u,y) \in \mathfrak{B}$ with zero initial condition. □

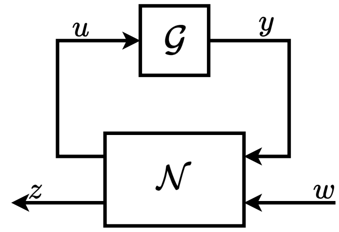

Fig. 1: LFR-based control interconnection between the generalized controller $\mathcal{N}$ and plant $\mathcal{G}$

In [6], the classical dissipativity notion has been reformulated for finite-horizon trajectories, which has been used for data-driven control based on the Fundamental Lemma [21].

Definition 3 (L-Dissipativity [6]). A discrete-time LTI system $\mathcal{G}$ with finite horizon behavior $\mathfrak{B}|_L$ is said to be L-dissipative with respect to the supply rate $\Pi \in \mathbb{S}^{(n_{\mathrm{u}}+n_{\mathrm{y}})\times(n_{\mathrm{u}}+n_{\mathrm{y}})}$ if

$$\sum_{k=0}^{L-1} (*)^\top \Pi \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \leq 0, \tag{9}$$

holds for all trajectories $(u|_L, y|_L) \in \mathfrak{B}|_L$ with zero initial condition. □

It has been shown in [22] that, under mild conditions, L-dissipativity is asymptotically equivalent to classical dissipativity. In the next section, we provide a tractable condition, based on L-dissipativity, to verify whether the closed-loop system satisfies the performance specifications.

IV. Main Results

In this section, we will formulate a representation of the closed-loop interconnection between a data-driven representation of the system $\mathcal{G}$ and a model-based representation of the generalized controller $\mathcal{N}$. A dissipativity analysis result will subsequently be derived for the closed-loop representation.

Through the Fundamental Lemma, we only have a FH representation of $\mathcal{G}$. Hence, the LFR-based interconnection of $\mathcal{G}$ and $\mathcal{N}$ can only be defined on the same finite horizon. Towards this end, the FH behavior $\mathfrak{B}^N|_L$ of $\mathcal{N}$ is defined as all sequences $(y|_L, w|_L, u|_L, z|_L)$ that satisfy

$$\begin{aligned} \mathscr{T}_L(D_{\mathrm{u}})u|_L &= \mathscr{T}_L(N_{\mathrm{uy}})y|_L + \mathscr{T}_L(N_{\mathrm{uw}})w|_L, \\ \mathscr{T}_L(D_{\mathrm{z}})z|_L &= \mathscr{T}_L(N_{\mathrm{zy}})y|_L + \mathscr{T}_L(N_{\mathrm{zw}})w|_L. \end{aligned} \tag{10}$$

Here the banded upper-triangular Toeplitz matrix $\mathscr{T}_L(D)$ with L block columns, related to the polynomial

$$D(\xi) = D_0 + D_1\xi + \cdots + D_\ell\xi^\ell \in \mathbb{R}^{n\times m}[\xi], \tag{11}$$

is introduced by [23] and is defined as

$$\mathscr{T}_L(D) = \begin{bmatrix} D_0 & \cdots & D_{\ell-1} & D_\ell & 0 & \cdots & 0 \\ 0 & D_0 & \cdots & D_{\ell-1} & D_\ell & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_0 & \cdots & D_{\ell-1} & D_\ell \end{bmatrix}. \tag{12}$$

Note that the interconnection between $\mathcal{G}$ and $\mathcal{N}$ eliminates the shared signals u and y. To define a well-posed interconnection, we require that the initial condition of the sequences $u|_L$ and $y|_L$ of both $\mathcal{G}$ and $\mathcal{N}$ align on the same finite horizon. For the dissipativity analysis, we require that this corresponds to a zero initial condition. To specify the *zero initial condition finite-horizon* (ZICFH) behaviors of both $\mathcal{G}$ and $\mathcal{N}$ we define the following subset of an arbitrary FH behavior $\mathfrak{B}|_{T+L}$:

$$\mathfrak{B}_\mu|_{T,L} := \big\{ \overbrace{(0,\cdots,0)}^{\mu} \wedge (u,y) \in (\mathbb{R}^{n_u} \times \mathbb{R}^{n_y})^{\mu+L} \mid \underbrace{(0,\cdots,0)}_{T} \wedge (u,y) \in \mathfrak{B}|_{T+L} \big\}. \quad (13)$$

By definition we have $\mathfrak{B}_0|_{0,L} = \mathfrak{B}|_L$. We denote the ZICFH behavior by $\mathfrak{B}_0|_L$ which is equal to $\mathfrak{B}_0|_{T,L}$ for all $T \geq \ell(\mathfrak{B})$. The observation in [22] leads to the following implicit characterization of $\mathfrak{B}_\mu|_{\mu,L}$ as a subspace of $\mathfrak{B}|_{\mu+L}$

$$\mathfrak{B}_\mu|_{\mu,L} := \mathfrak{B}|_{\mu+L} \cap \operatorname{Ker}\left(V_L^\mu(u,y)\right), \quad (14)$$

where

$$V_L^\mu(u,y) = \begin{bmatrix} I_{\mu n_u} & 0_{\mu n_u \times L n_u} & 0_{\mu n_u \times \mu n_y} & 0_{\mu n_u \times L n_y} \\ 0_{\mu n_y \times \mu n_u} & 0_{\mu n_y \times L n_u} & I_{\mu n_y} & 0_{\mu n_y \times L n_y} \end{bmatrix}. \quad (15)$$

Based on this subspace of $\mathfrak{B}|_{\mu+L}$, the ZICFH behavior can be defined as:

$$\mathfrak{B}_0|_L = \left\{ J_L^\mu(u,y) \begin{bmatrix} u|_{\mu+L} \\ y|_{\mu+L} \end{bmatrix} \middle| (u|_{\mu+L}, y|_{\mu+L}) \in \mathfrak{B}_\mu|_{\mu,L} \right\}, (16)$$

if $\mu \geq \ell(\mathfrak{B})$, where

$$J_L^\mu(u,y) = \begin{bmatrix} 0_{Ln_u \times \mu n_u} & I_{Ln_u} & 0_{Ln_u \times \mu n_y} & 0_{Ln_u \times L n_y} \\ 0_{Ln_y \times \mu n_u} & 0_{Ln_y \times L n_u} & 0_{Ln_y \times \mu n_y} & I_{Ln_y} \end{bmatrix}. \quad (17)$$

The following Lemma provides a computable basis for $\mathfrak{B}_0^{\mathcal{G}}|_L$ based on a data-dictionary $\mathcal{D}_N$ of $\mathcal{G}$.

Lemma 2. *Given a data dictionary $\mathcal{D}_N$ of $\mathcal{G}$ that satisfies Lemma 1, where additionally the input is PE of order at least $\nu + L + n_G$. The ZICFH behavior of $\mathcal{G}$ is given by*

$$\mathfrak{B}_0^{\mathcal{G}}|_L = \operatorname{Im}\left(\begin{bmatrix} B_{0,L}^{\mathrm{u}} \\ B_{0,L}^{\mathrm{y}} \end{bmatrix} \right) = \operatorname{Im}\left(J_L^\nu(u,y) \begin{bmatrix} B_{\nu,L}^{\mathrm{u}} \\ B_{\nu,L}^{\mathrm{y}} \end{bmatrix} \right), \quad (18)$$

where

$$\begin{bmatrix} B_{\nu,L}^{\mathrm{u}} \\ B_{\nu,L}^{\mathrm{y}} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{\nu+L}(u^d) \\ \mathcal{H}_{\nu+L}(y^d) \end{bmatrix} \left(V_L^\nu(u,y) \begin{bmatrix} \mathcal{H}_{\nu+L}(u^d) \\ \mathcal{H}_{\nu+L}(y^d) \end{bmatrix} \right)_\perp. \quad (19)$$

Proof. By the PE condition on the input and the Fundamental Lemma [3] we have that

$$\mathfrak{B}^{\mathcal{G}}|_{\nu+L} = \operatorname{Im}\left(\begin{bmatrix} \mathcal{H}_{\nu+L}(u^d) \\ \mathcal{H}_{\nu+L}(y^d) \end{bmatrix} \right).$$

Next, we note that the matrix $A(BA)_\perp$ is a basis for the space $\operatorname{Im}(\mathrm{A}) \cap \operatorname{Ker}(B)$, see [24]. Hence, we have that $\mathfrak{B}_\nu^{\mathcal{G}}|_{\nu,L} = \operatorname{Im}[(B_{\nu,L}^{\mathrm{u}})^\top \ (B_{\nu,L}^{\mathrm{y}})^\top]^\top$.Then the ZICFH behavior $\mathfrak{B}_0^{\mathcal{G}}|_L$ is simply defined as the image of the last L entries of the signals defined by $\mathfrak{B}_\nu^{\mathcal{G}}|_{\nu,L}$ which is exactly given by (18). ∎

The ZICFH behavior $\mathfrak{B}_0^{\mathcal{N}}|_L$ of $\mathcal{N}$ is defined as the collection of signals $(y|_L, w|_L, u|_L, z|_L)$ such that

$$\begin{aligned} \mathscr{T}_L^0(D_\mathrm{u})u|_L &= \mathscr{T}_L^0(N_\mathrm{uy})y|_L + \mathscr{T}_L^0(N_\mathrm{uw})w|_L, \\ \mathscr{T}_L^0(D_\mathrm{z})z|_L &= \mathscr{T}_L^0(N_\mathrm{zy})y|_L + \mathscr{T}_L^0(N_\mathrm{zw})w|_L, \end{aligned} \quad (20)$$

where $\mathscr{T}_L^0(D)$ is simply the lower right L-by-L block matrix of $\mathscr{T}_{\nu_N+L}(D)$ such that $\nu_N \geq \ell(\mathfrak{B}^{\mathcal{N}})$. We define the LFR-based closed-loop interconnection $\mathcal{T} = \mathcal{G} \star \mathcal{N}$ as the ZICFH behavioral interconnection denoted by $\mathfrak{B}_0^{\mathcal{T}}|_L = \mathfrak{B}_0^{\mathcal{G}}|_L \parallel_{(u,y)} \mathfrak{B}_0^{\mathcal{N}}|_L$. This behavioral interconnection eliminates the shared signals u and y from the closed-loop behavior and is defined as

$$\mathfrak{B}_0^{\mathcal{T}}|_L = \big\{ (w|_L, z|_L) \bigm| (u|_L, y|_L) \in \mathfrak{B}_0^{\mathcal{G}}|_L \text{ and } (y|_L, w|_L, u|_L, z|_L) \in \mathfrak{B}_0^{\mathcal{N}}|_L \big\}. \quad (21)$$

Next, we provide the main result of this paper.

Theorem 1. *Consider a given data-dictionary $\mathcal{D}_N$ of $\mathcal{G}$ that satisfies Lemma 2 and a minimal-lag input-output representation of $\mathcal{N}$. The L-long trajectory $(w|_L, z|_L)$ is a trajectory of the closed-loop system $\mathcal{T}$, if there exists a $g \in \mathbb{R}^\bullet$ such that*

$$\begin{aligned} \mathscr{T}_L^0(D_\mathrm{u})B_{0,L}^{\mathrm{u}}g &= \mathscr{T}_L^0(N_\mathrm{uy})B_{0,L}^{\mathrm{y}}g + \mathscr{T}_L^0(N_\mathrm{uw})w|_L, \\ \mathscr{T}_L^0(D_\mathrm{z})z|_L &= \mathscr{T}_L^0(N_\mathrm{zy})B_{0,L}^{\mathrm{y}}g + \mathscr{T}_L^0(N_\mathrm{zw})w|_L. \end{aligned} \quad (22)$$

Proof. By Lemma 2, $(u|_L, y|_L) \in \mathfrak{B}_0^{\mathcal{G}}|_L$ if and only if there exists a g such that

$$\begin{bmatrix} u|_L \\ y|_L \end{bmatrix} = \begin{bmatrix} B_{0,L}^{\mathrm{u}} \\ B_{0,L}^{\mathrm{y}} \end{bmatrix} g.$$

Substituting this parametrization of $(u|_L, y|_L)$ in (20) gives (22) and completes the proof. ∎

Theorem 1 is the first contribution of the paper and it provides a unified closed-loop representation between general LFR-based interconnections between data-driven and model-based representations. In order to analyse dissipativity of the closed-loop system $\mathcal{T}$, for which the behavior is given by (22), we define the following inequality

$$(*)^\top \Pi_L \begin{bmatrix} w|_L \\ z|_L \end{bmatrix} = (*)^\top \begin{bmatrix} I_L \otimes Q & I_L \otimes S \\ I_L \otimes S^\top & I_L \otimes R \end{bmatrix} \begin{bmatrix} w|_L \\ z|_L \end{bmatrix} \leq 0. \quad (23)$$

The following theorem gives an LMI condition to verify the dissipativity of the interconnection between model-based and data-driven representations as an LFR.

Theorem 2. *The closed-loop system $\mathcal{T}$ with zero initial condition behavior given by* (22) *is L-dissipative with respect to the supply rate Π_L* (23)*, if*

$$(*)^\top \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_L \otimes Q & I_L \otimes S \\ 0 & I_L \otimes S^\top & I_L \otimes R \end{bmatrix} B_\perp \preceq 0, \quad (24)$$

where

$$B = \begin{bmatrix} \mathscr{T}_L^0(N_\mathrm{uy})B_0^{\mathrm{y}} - \mathscr{T}_L^0(D_\mathrm{u})B_0^{\mathrm{u}} & \mathscr{T}_L^0(N_\mathrm{uw}) & 0 \\ \mathscr{T}_L^0(N_\mathrm{zy})B_0^{\mathrm{y}} & \mathscr{T}_L^0(N_\mathrm{zw}) & -\mathscr{T}_L^0(D_\mathrm{z}) \end{bmatrix}. \quad (25)$$

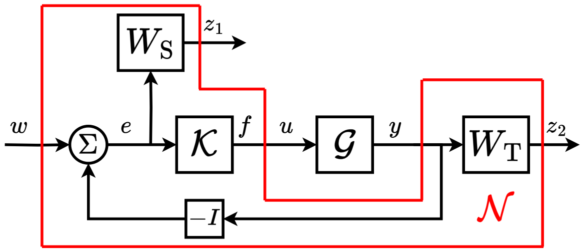

Fig. 2: Two-block reference tracking control configuration.

Proof. Note that (23) can be equivalently defined as

$$(*)^\top \begin{bmatrix} 0 & 0 \\ 0 & \Pi_L \end{bmatrix} \begin{bmatrix} g \\ w|_L \\ z|_L \end{bmatrix} \leq 0,$$

which only has to hold for $(g, w|_L, z|_L)$ such that the constraint, defined by the first equation in (22), is satisfied. This constraint can be formulated as

$$B \begin{bmatrix} g \\ w|_L \\ z|_L \end{bmatrix} = 0.$$

Then the LMI (24) is obtained by applying Finsler's Lemma to the dissipativity inequality and the constraint which states that $x^\top Q x \geq 0$ for x such that $Bx = 0$ is equivalent to $(*)^\top Q B_\perp \preceq 0$. ■

With the LMI (24) it is possible to verify dissipativity of general interconnections between model-based and data-driven representations of dynamical systems. The LMI-based dissipativity analysis provides a numerical tractable performance evaluation for the closed-loop system $\mathcal{T}$ through a mixed-sensitivity argument, which provides the second contribution of this paper.

V. Examples

We now demonstrate the applicability of the result on two simulation examples. The first example shows that Theorem 2 can be used to approximate the $\mathcal{H}_\infty$-norm. The second example shows how the analysis results can be used to design a controller with weighting filter specifications.

A. Two-block mixed-sensitivity example

In this example, we consider SISO reference tracking as shown in Fig. 2 with the tracking error e and plant output y as performance channels to shape the sensitivity and complementary sensitivity of the closed-loop behavior. We consider a two-mass-spring-damper system for $\mathcal{G}$ with state-space representation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{d_1+d_2}{m_1} & \frac{k_2}{m_1} & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{d_2}{m_2} & -\frac{k_2}{m_2} & -\frac{d_2}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u, \qquad (26)$$

$$y = \begin{bmatrix} 1 \; 0 \; 0 \; 0 \end{bmatrix} x,$$

which is discretized with a sampling time $h = 0.1s$. The parameters have the following values: $m_1 = 10$ kg, $m_2 = 0.5$ kg, $d_1 = 200$ Ns/m, $d_2 = 10$ Ns/m, $k_1 = 3000$ N/m, and $k_2 = 1000$ N/m. The generalized plant $\mathcal{P}$ is defined by the rational matrix

$$P(z) = \begin{bmatrix} W_S(z) & -W_S(z)G(z) \\ 0 & W_T(z)G(z) \\ 1 & -G(z) \end{bmatrix}, \qquad (27)$$

where z denotes the Laplace transform of the forward shift operator q. The generalized plant is used to compute an $\mathcal{H}_\infty$-optimal controller $K(z)$, with model-based synthesis. The weighting filters $W_S(z)$ and $W_T(z)$ are chosen as

$$W_S(z) = \tfrac{0.7741z - 0.7641}{z - 0.9998}, \quad W_T(z) = \tfrac{25.9z - 25.38}{z - 0.3333}, \qquad (28)$$

which specify a rise-time around 1 [s] and a bandwidth around 1.68 [rad/s]. The resulting $\mathcal{H}_\infty$-norm, computed with model-based analysis, is approximately equal to 0.9779. Next, we apply unit-variance white noise to the system $\mathcal{G}$ and collect 400 input-output data points. We extract the data-based representation of $\mathcal{G}$ into the upper block shown in Figure 2 such that $\mathcal{N}$ is given by the rational matrix

$$N(z) = \begin{bmatrix} -K(z) & K(z) \\ -W_S(z) & W_S(z) \\ W_T(z) & 0 \end{bmatrix}. \qquad (29)$$

The rational matrix representation $N(z)$ is converted to an input-output representation with the algorithm presented in [25]. The finite-horizon ℓ_2-gain calculated with Theorem 2 is compared to the model-based gain for $\nu = 15$ and a depth L ranging from ν to 60. This comparison is shown in the bottom plot of Fig. 3, which shows that both computed finite-horizon ℓ_2-gains are equivalent and converge to the $\mathcal{H}_\infty$-norm.

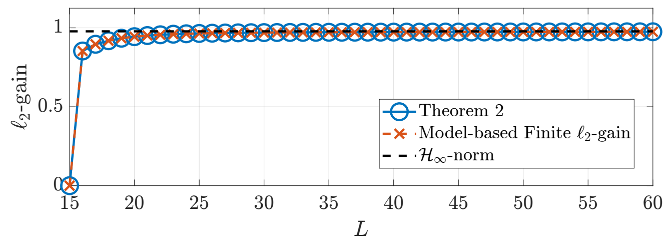

Fig. 3: Finite-horizon ℓ_2-gain computed with the data-driven approach of Theorem 2 compared to the model-based finite-horizon ℓ_2-gain.

B. Controller design example

In this example, we consider a discrete-time LTI system $\mathcal{G}$ for which a model-based representation is not assumed to be known, but for illustration, the Bode plot of $\mathcal{G}$ is shown in Figure 4. We apply unit-variance white noise as input and collect 2000 data samples from an open-loop experiment. Based only on the measured data of $\mathcal{G}$, we aim to design a controller that satisfies the performance criteria specified by the weighting filters shown in Figure 4. The generalized controller $\mathcal{N}$ is given by the following input-output representation

$$\left[\begin{array}{cc|c|c} D_u & 0 & N_{uy} & N_{uw} \\ \hline 0 & D_z & N_{zy} & N_{zw} \end{array}\right] = \left[\begin{array}{c|ccc|c} D_K & 0 & 0 & -N_K & N_K \\ \hline 0 & D_S & 0 & -N_S & N_S \\ 0 & 0 & D_T & N_T & 0 \end{array}\right], \qquad (30)$$

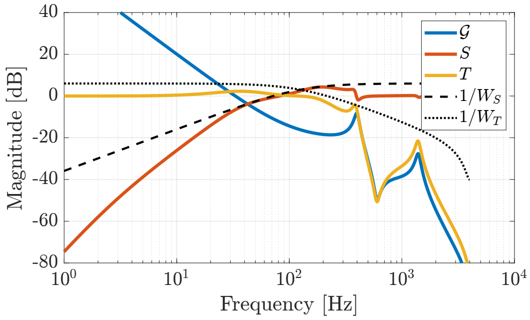

Fig. 4: Shaped closed-loop sensitivity functions with weighting filters.

where the explicit dependence on the shift operator q is left out. Here the pairs $(D_K, N_K), (D_S, N_S)$, and (D_T, N_T) denote the input-output representations of the controller $\mathcal{K}$, the sensitivity weighting filter W_S, and the complementary sensitivity weighting filter W_T, respectively. In this example, we consider the following parametrization of the controller:

$$K(z) = c\frac{(z-0.98)(z-r_1)(z^2-2r_2 z\cos(\theta_1)+r_2^2)}{(z-1)(z-r_3)(z^2-2r_4 z\cos(\theta_2)+r_4^2)}. \quad (31)$$

This controller consists of an integrator and a general third-order filter. In order to ensure that the third-order filter is stable and minimum-phase we enforce the constraints $|r_i| < 1$. We use Particle Swarm Optimization [26] to find the controller parameters $(r_1, r_2, r_3, r_4, \theta_1, \theta_2, c)$, which minimize the finite-horizon ℓ_2-gain based on Theorem 2. We have used $\nu = 14$ and $L = 180$ to design the controller $\mathcal{K}$, for which the resulting sensitivity S and complementary sensitivity T are shown in Figure 4. The finite-horizon ℓ_2-gain is approximately equal to 0,907, which indicates that the performance criteria are met. This is verified by visual inspection of the sensitivity shapes in Figure 4.

VI. Conclusions

In this paper, we have given a unified LFR representation form of data-driven and model-based representations on a finite horizon. This LFR structure is used to define arbitrary interconnection between data-driven representations of systems, for which only measured data is available, and model-based representations of controllers and weighting filters. The incorporation of weighting filters into the unified framework allows for direct data-driven performance analysis of the closed-loop system without the need to identify a model for the system. Moreover, tractable LMI-based finite-horizon dissipativity analysis results have been derived for these LFR-based unified representations. The importance of these results is that specifications of frequency-domain and time-domain performance of designed controllers, for data-driven representations, can be verified. For future work, we aim to extend this framework toward convex controller synthesis and applicability with noise-corrupted measured data.

Acknowledgement

The authors would like to thank Julian Berberich for a fruitful discussion and his insightful comments regarding finite-horizon representations.

References

[1] F. Dörfler, "Data-driven control: Part two of two: Hot take: Why not go with models?" *IEEE Control Systems Magazine*, vol. 43, no. 6, pp. 27–31, 2023.

[2] Z.-S. Hou and Z. Wang, "From model-based control to data-driven control: Survey, classification and perspective," *Information Sciences*, vol. 235, pp. 3–35, 2013.

[3] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[4] J. C. Willems and J. W. Polderman, *Introduction to mathematical systems theory: a behavioral approach*. Springer Science & Business Media, 1997, vol. 26.

[5] I. Markovsky and F. Dörfler, "Behavioral systems theory in data-driven analysis, signal processing, and control," *Annual Reviews in Control*, vol. 52, pp. 42–64, 2021.

[6] T. Maupong, J. C. Mayo-Maldonado, and P. Rapisarda, "On lyapunov functions and data-driven dissipativity," *In Proc. of 20th IFAC World Congress*, pp. 7783–7788, 2017.

[7] A. Romer, J. Berberich, J. Köhler, and F. Allgöwer, "One-shot verification of dissipativity properties from input–output data," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 709–714, 2019.

[8] A. Romer, J. M. Montenbruck, and F. Allgöwer, "Determining dissipation inequalities from input-output samples," *In Proc. of 20th IFAC World Congress*, pp. 7789–7794, 2017.

[9] C. De Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality, and robustness," *IEEE Transactions on Automatic Control*, vol. 65, no. 3, pp. 909–924, 2019.

[10] H. J. van Waarde and M. Mesbahi, "Data-driven parameterizations of suboptimal LQR and H2 controllers," *In Proc. of 21st IFAC World Congress*, pp. 4234–4239, 2020.

[11] M. Alsalti, V. G. Lopez, and M. A. Müller, "Notes on data-driven output-feedback control of linear mimo systems," *arXiv preprint arXiv:2311.17484*, 2023.

[12] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice hall Upper Saddle River, NJ, 1998, vol. 104.

[13] S. Skogestad and I. Postlethwaite, *Multivariable feedback control: analysis and design*. John Wiley & Sons, 2005.

[14] M. M. Seron, J. H. Braslavsky, and G. C. Goodwin, *Fundamental limitations in filtering and control*. Springer Science & Business Media, 2012.

[15] N. Wieler, J. Berberich, A. Koch, and F. Allgöwer, "Data-driven controller design via finite-horizon dissipativity," in *Proc. of the 3rd Conference on Learning for Dynamics and Control*, 2021, pp. 287–298.

[16] J. C. Willems and Y. Yamamoto, "Behaviors defined by rational functions," *Linear algebra and its applications*, vol. 425, no. 2-3, pp. 226–241, 2007.

[17] J. Berberich and F. Allgöwer, "A trajectory-based framework for data-driven system analysis and control," in *Proceedings of the European Control Conference*, 2020, pp. 1365–1370.

[18] I. Markovsky and F. Dörfler, "Identifiability in the behavioral setting," *IEEE Transactions on Automatic Control*, 2022.

[19] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: Basic input-output and state properties," *Journal of the Franklin Institute*, vol. 309, no. 5, pp. 327–357, 1980.

[20] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Archive for rational mechanics and analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[21] J. Berberich, A. Koch, C. W. Scherer, and F. Allgöwer, "Robust data-driven state-feedback design," in *Proc. of the American Control Conference*. IEEE, 2020, pp. 1532–1538.

[22] A. Koch, J. Berberich, J. Köhler, and F. Allgöwer, "Determining optimal input–output properties: A data-driven approach," *Automatica*, vol. 134, p. 109906, 2021.

[23] I. Markovsky, "Closed-loop data-driven simulation," *International journal of control*, vol. 83, no. 10, pp. 2134–2139, 2010.

[24] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.

[25] P. J. Antsaklis and A. N. Michel, *A linear systems primer*. Springer Science & Business Media, 2007.

[26] J. Kennedy and R. Eberhart, "Particle swarm optimization," in *Proceedings of international conference on neural networks*, 1995, pp. 1942–1948.