2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Optimal Observer-based Controller Design for Linear Systems

Yogesh Kumar, Prasad Vilas Chanekar and Sayan Basu Roy

Abstract— **This paper presents a method to design an optimal controller-observer pair for a continuous linear time-invariant system with respect to a quadratic cost. First, we propose a novel generalized method that makes this otherwise complex problem solvable within the linear optimal control framework. Then, we derive a solution approach based on the augmented Lagrangian method to handle the inherent structural constraints associated with the problem. Finally, we show the utility of the proposed method through a numerical example.**

I. Introduction

Optimal control refers to finding a control input to the system that optimizes the given performance index while moving the state vector from an initial to a final value. One such problem for linear time-invariant (LTI) systems is widely studied in the optimal control literature as linear quadratic regulator (LQR) [1], [2]. This corresponds to designing a static optimal gain matrix in the full-state feedback setting that maps the system's states to the control input. However, measuring all the system states in general is not always possible or feasible. In such a scenario, the system can be controlled either by a static output-feedback control [3] or state observer-based full-state feedback, often termed as dynamic output feedback control [1].

Static output feedback is comparatively straightforward and computationally efficient from an implementation perspective than dynamic output feedback. However, dynamic output feedback is preferred due to its inherent robustness properties, especially when the system has high-order dynamics and multiple outputs [2]. Additionally, verification of the necessary and sufficient conditions for static output feedback stabilizability is a complex task [4], [5]. In contrast, the stabilizability of the dynamic output feedback can be verified by evaluating the system's stabilizability and detectability. Thus, dynamic state feedback is advantageous than static output feedback if one has enough computational resources to implement the observer in a sampled data setting [1].

The standard process of designing dynamic output feedback control within the optimal control framework is as follows. First, the full state feedback optimal gain is designed followed by a separate design of the observer gain [1], [6]–[9]. This is done by leveraging the so-called separation principle. For deterministic systems, the observer gain is selected such that for a fixed optimal control gain, the performance of the observer-based system is close to the optimal state-feedback setting. In contrast, for stochastic systems, the observer gains are selected based on minimizing yet another cost function to tune the variance of the steady state. The weight matrices are decided based on the process and measurement noise characteristics. Recent analytical results provide insights into unifying the dynamic output feedback problem for discrete deterministic systems to determine the controller and observer gain simultaneously corresponding to the standard linear quadratic performance index [10]. For continuous systems, a solution method for such formulation is presented in [11], [12] using a block pulse function-based nonlinear programming problem with satisfactory performance in the simulations. However, the performance index was modified by replacing the state penalty with the output penalty and adding an observation error term in the cost function. Considering the limited literature, this article discusses a technique to obtain the optimal observer-controller gain pair corresponding to the standard linear quadratic performance index for a deterministic continuous LTI system.

Paper contribution: In this work, we formulate the optimal observer-based control design problem for a continuous LTI system corresponding to the standard quadratic performance index. We define an extended linear system using state and observer dynamics and with a performance index consisting of controller and observer objectives. We show that this extended dynamics is stabilized by a extended gain matrix controller of specific structure and composed of control and observer gain of original dynamics. We later reformulate the time-dependent optimization problem into a time-independent optimization problem with matrix equality constraints. The structural constraints for the extended gain matrix are represented as a set of linear constraints in the optimization problem. The reformulated optimization problem is solved using the state-of-the-art augmented Lagrangian method [13]. At last, the results are presented to show the efficacy and utility of the proposed algorithm.

The paper is organized as follows: section II involves the preliminaries, followed by the problem formulation and methodology in Section III. Sections IV and V present the results, conclusion, and future directions, respectively.

II. Preliminaries

A. Notation

Let $\mathbb{R}$ denote set of real numbers. For a matrix $X \in \mathbb{R}^{m\times n}$ having $m-$rows and $n-$columns, we use $X^\top$ for its transpose, $\mathrm{tr}(X)$ for its trace and $\|X\|$ for its Frobenius norm. We use $\mathbf{0}$ and I to denote the matrix of zeros and identity matrix, respectively of appropriate sizes. For a matrix X, X_{mn} implies X has $m-$rows and $n-$columns. For $X \in \mathbb{R}^{m\times m}$, $X(\succeq) \succ 0$ denotes X is positive (semi-)definite. For a matrix X, $\mathscr{R}(X)$

YK was suported by by TCS Research, India under TCS Research Scholar Program FP-140. PVC was supported by SERB Start-up Research Grant - SRG/2023/001636. YK, PVC and SBR are with the Department of Electronics and Communication Engineering, Indraprastha Institute of Information Technology, New Delhi, India 110020, {yogeshk, prasad, sayan}@iiitd.ac.in

and $\mathcal{N}(X)$ denote its range space and null space respectively. For a scalar function f, notation $\nabla_X f$ denotes the gradient of f with respect to X. $\frac{\partial f}{\partial X}$ denotes the partial derivative of f with respect to X.

B. System Dynamics

Consider a LTI system with the following dynamics

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{r\times n}$ are the system matrices. Whereas, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^r$, $u \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$ denote the state, measured output, control input and some unknown initial state the system, respectively. We assume that the system (A,B) is stabilizable and (A,C) is detectable. We also assume that $\text{rank}(C) = r$ and $r < n$. Since all the states are not directly measurable, so by virtue of detectability, we can estimate the complete state by using a Luenberger observer for the system (1) [14]. The observer dynamics is defined by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \tag{2}$$

where $\hat{x}(t)$ is the estimate of $x(t)$ and $L \in \mathbb{R}^{n\times r}$ is the static observer gain. We use the state estimate to define the control input and the closed loop system as,

$$u = K\hat{x}, \quad \dot{x} = Ax + BK\hat{x}, \tag{3}$$

where $K \in \mathbb{R}^{m\times n}$ is the static controller gain. We quantify the system performance by the following quadratic objective,

$$f_c = \int_0^\infty [x^\top Q_1 x + u^\top R_1 u] \mathrm{d}t, \tag{4}$$

where $Q_1 \in \mathbb{R}^{n\times n}$, $Q_1 = Q_1^\top \succeq 0$ and $R_1 \in \mathbb{R}^{m\times m}$, $R_1 = R_1^\top \succ 0$ are weighing matrices. Note that f_c is a function of x and $\hat{x}$ (through u).

We intend to solve the problem of computing K, L for a given Q_1, R_1 such that the performance objective f_c is minimized and the dynamics (1) is stabilized. Mathematically the problem is expressed as,

$$\begin{aligned} \min_{K,L} \quad & f_c = \int_0^\infty [x^\top Q_1 x + u^\top R_1 u] \, \mathrm{d}t \\ \text{s.t.} \quad & \dot{x} = Ax + Bu, \quad x(0) = x_0, \\ & \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \quad \hat{x}(0) = \hat{x}_0 \\ & u = K\hat{x}, \\ & \hat{x} \to x \text{ and } x \to 0 \quad \text{as } t \to \infty \text{ asymptotically.} \end{aligned} \tag{5}$$

In (5), the controller K and the observer gains L are being simultaneously optimized. The constraint $\hat{x} \to x$ and $x \to 0$ as $t \to \infty$ ensures that the state estimates finally converge to the true state with time and guarantee the stability of the system. However, solving problem (5) poses challenges, as it requires more than a straightforward adaptation of established linear quadratic regulator theory [1], primarily due to the incorporation of observer dynamics. Subsequently, we delve into these obstacles in depth and propose a systematic solution approach for addressing (5).

III. Problem Formulation and Solution Method

We first present the general controller-observer design problem. Next based on Lyapunov stability theory [15] we present an equivalent reformulation and then propose a gradient-based solution procedure.

A. General controller-observer design problem

Consider the observer error $e \triangleq x - \hat{x}$. Using (1), (2) and (3) the error dynamics is

$$\dot{e} = (A - LC)e. \tag{6}$$

We introduce a new extended variable $z = [x^\top \quad e^\top]^\top$. The extended dynamics combining (1) and (6) is

$$\dot{z} = \overline{A}z + \overline{B}\,\overline{u}, \quad \overline{u} = \overline{K}z, \tag{7}$$

where,

$$\overline{A} = \begin{bmatrix} A & \mathbf{0}_{nn} \\ \mathbf{0}_{nn} & A \end{bmatrix}, \overline{B} = \begin{bmatrix} B & \mathbf{0}_{nn} \\ \mathbf{0}_{nm} & I \end{bmatrix}, \overline{K} = \begin{bmatrix} K & -K \\ \mathbf{0}_{nn} & -LC \end{bmatrix}.$$

To quantify the performance of the extended dynamics (7) we have the quadratic objective

$$J = \int_0^\infty [z^\top \overline{Q} z + \overline{u}^\top \overline{R}\,\overline{u}] \, \mathrm{d}t, \tag{8}$$

where

$$\overline{Q} = \begin{bmatrix} Q_1 & \mathbf{0}_{nn} \\ \mathbf{0}_{nn} & Q_2 \end{bmatrix}, \quad \overline{R} = \begin{bmatrix} R_1 & \mathbf{0}_{mn} \\ \mathbf{0}_{mn} & R_2 \end{bmatrix},$$

with $Q_2 \in \mathbb{R}^{n\times n}$, $Q_2 = Q_2^\top \succeq 0$ and $R_2 \in \mathbb{R}^{n\times n}$, $R_2 = R_2^\top \succeq 0$ are the weighing matrices. Notice that $\overline{K}$ is a static controller gain for an auxiliary input $\overline{u} = [u^\top \quad u_e^\top]^T = \overline{K}z$ to the extended dynamics (7). Here $u_e = -LCe$ can be conceptualized as the control input for the observer error dynamics (6). To optimally determine $\overline{K}$, we state the general controller-observer design problem as

$$\begin{aligned} \min_{\overline{K}} \quad & J = \int_0^\infty [z^\top \overline{Q} z + \overline{u}^\top \overline{R}\,\overline{u}] \, \mathrm{d}t \\ \text{s.t.} \quad & \dot{z} = \overline{A}z + \overline{B}\,\overline{u}, \quad z(0) = z_0, \quad \overline{u} = \overline{K}z, \\ & z \to 0 \quad \text{as } t \to \infty \text{ asymptotically.} \end{aligned} \tag{9}$$

The constraint $z \to 0$ as $t \to \infty$ ensures the stability of the closed loop system.

Remark III.1. *(Relation between* (5) *and* (9)*). It is worthwhile to note that if $Q_2 = \mathbf{0}_{nn}$ and $R_2 = \mathbf{0}_{nn}$, problems* (5) *and* (9) *are identical provided* $z(0) = [x_0^\top, x_0^\top - \hat{x}_0^\top]^\top$.

Note that we can view the performance index J in (8) as a combination of controller objective f_c in (4) and the observer objective $f_o = \int_0^\infty [e^\top Q_2 e + u_e^\top R_2 u_e] \mathrm{d}t$ where $u_e = -LCe$. So $Q_2 = \mathbf{0}_{nn}$ and $R_2 = \mathbf{0}_{nn}$ implies $f_o = 0$ or no cost for the observer objective. As (9) is still not in a computationally viable format, we subsequently present a solvable equivalent reformulation of (9).

Remark III.2. *In observer objective f_o, the weight matrices Q_2 and R_2 act as the state and control weight matrices for the observer error dynamics* (6)*. The user can choose a non-zero Q_2 and R_2 to tune the observer behavior to comply with user-defined sensitivity requirements to output noise and measurement errors [16].*

B. Reformulated controller-observer design problem

We use Lyapunov stability theory [15] to derive an equivalent reformulation of (9). First we define the following set of matrices,

$$\mathscr{K} := \left\{ X \;\middle|\; X \in \mathbb{R}^{(m+n)\times 2n}; \right. \tag{10}$$

$$\left. X = \begin{bmatrix} X_{mn} & -X_{mn} \\ \mathbf{0}_{nn} & X_{nn} \end{bmatrix}; \exists L \in \mathbb{R}^{n\times r} : X_{nn} = -LC \right\}$$

Remark III.3. *(Regarding structure of $\overline{K}$).* The subspace $\mathscr{K}$ encapsulates the lower left block of $\overline{K}$ to be zero, upper blocks being oppositely signed equal quantities, and other structural constraints arising from incorporating the C matrix as a gain multiplier. To identify an unique optimal observer gain, L from optimal gain $\overline{K}$, the gain component $-LC$ should be in the span of C. In the subsequent section, we will discuss these structural constraints. •

Now we show that $\overline{K} \in \mathscr{K}$ stabilizes the system $(\overline{A}, \overline{B})$.

Theorem III.4. *(Existence of stabilizing controller gain and reformulation of objective function). Consider the dynamics* (9) *with system* (A, B) *stabilizable and the performance objective defined in* (8)*. Then*

(i) There exists some $\overline{K} = \begin{bmatrix} K & -K \\ \mathbf{0}_{nn} & \Theta \end{bmatrix} \in \mathscr{K}$ *which stabilizes* $(\overline{A}, \overline{B})$.

(ii) Let $Z_0 = z_0 z_0^\top$ *then there exists a* $\overline{P} = \overline{P}^\top \in \mathbb{R}^{2n\times 2n}$, $\overline{P} \succ 0$ *such that*

$$(\overline{A}+\overline{B}\,\overline{K})^\top \overline{P} + \overline{P}(\overline{A}+\overline{B}\,\overline{K}) + \overline{Q} + \overline{K}^\top \overline{R}\,\overline{K} = \mathbf{0}$$

and $J = \mathrm{tr}(\overline{P} Z_0)$.

Proof. (i) We have the closed loop system

$$\overline{A}+\overline{B}\,\overline{K} = \begin{bmatrix} A+BK & -BK \\ \mathbf{0}_{nn} & A+\Theta \end{bmatrix}.$$

Now as (A, B) is given to be stabilizable and (A, I) is obviously stabilizable. So from stability theory of LTI dynamics [2] there must exists some K and Θ such that $A+BK$ and $A+\Theta$ are Hurwitz. As from structure of $\overline{A}+\overline{B}\,\overline{K}$, its eigenvalues are union of eigenvalues of $A+BK$ and $A+\Theta$. Hence $\overline{A}+\overline{B}\,\overline{K}$ is Hurwitz and $\overline{K} \in \mathscr{K}$ stabilizes $(\overline{A}, \overline{B})$.

(ii) Consider candidate Lyapunov function

$$V = z^\top \overline{P} z$$

with $\overline{P} \succ 0$. Differentating with respect to time and substituting the closed-loop dynamics we have

$$\dot{V} = \dot{z}^\top \overline{P} z + z^\top \overline{P} \dot{z} = z^\top [(\overline{A}+\overline{B}\,\overline{K})^\top \overline{P} + \overline{P}(\overline{A}+\overline{B}\,\overline{K})] z.$$

Integrating both sides with respect to t with limits 0 to ∞ and adding $J = \int_0^\infty [z^\top \overline{Q} z + \overline{u}^\top \overline{R}\,\overline{u}]\, \mathrm{d}t$ on both sides with $\overline{u} = \overline{K} z$ we have

$$\int_0^\infty \dot{V}\, \mathrm{d}t + J = \int_0^\infty z^\top [(\overline{A}+\overline{B}\,\overline{K})^\top \overline{P} + \overline{P}(\overline{A}+\overline{B}\,\overline{K}) + \overline{Q} + \overline{K}^\top \overline{R}\,\overline{K}]\, z\, \mathrm{d}t$$

Now $\int_0^\infty \dot{V}\, \mathrm{d}t = V(\infty) - V(0)$. As $\overline{A}+\overline{B}\,\overline{K}$ is Hurwitz so $z \to 0$ as $t \to \infty$ which implies $V(\infty) = 0$ using definition of V. Also from Lyapunov stability theory [2], [15] we must have $\overline{P} = \overline{P}^\top \succ 0$ such that

$$(\overline{A}+\overline{B}\,\overline{K})^\top \overline{P} + \overline{P}(\overline{A}+\overline{B}\,\overline{K}) + \overline{Q} + \overline{K}^\top \overline{R}\,\overline{K} = \mathbf{0}.$$

Thus $J = V(0) = z_0^\top \overline{P} z_0 = \mathrm{tr}(\overline{P} z_0 z_0^\top) = \mathrm{tr}(\overline{P} Z_0)$. □

Note that $\overline{K} \in \mathscr{K}$ has a specific structure. For a given $\overline{K} \in \mathscr{K}$ and its Θ an L satisfying $\Theta = -LC$ may not exist. Next result provides us conditions on general $\overline{K} \in \mathbb{R}^{(m+n)\times 2n}$ such that it also belongs to $\mathscr{K}$ and determination of a L for a given C is possible. First we state some definitions which will be used to prove the result.

$$\Pi_1 = \begin{bmatrix} I_{nn} & \mathbf{0}_{nn} \\ \mathbf{0}_{nn} & \mathbf{0}_{nn} \end{bmatrix},\ \Pi_2 = \begin{bmatrix} I_{mm} & \mathbf{0}_{mn} \\ \mathbf{0}_{nm} & \mathbf{0}_{nn} \end{bmatrix},\ \Pi_3 = \begin{bmatrix} \mathbf{0}_{nn} & \mathbf{0}_{nn} \\ I_{nn} & \mathbf{0}_{nn} \end{bmatrix},$$
$$\Pi_4 = \begin{bmatrix} \mathbf{0}_{mm} & \mathbf{0}_{mn} \\ \mathbf{0}_{nm} & I_{nn} \end{bmatrix},\ \Pi_5 = \begin{bmatrix} \mathbf{0}_{mm} & \mathbf{0}_{mp} \\ \mathbf{0}_{nn} & \mathscr{N}(C) \end{bmatrix},$$

where $\mathscr{N}(C) \in \mathbb{R}^{n\times p}$ where p is the nullity [17] of matrix C. Now we are ready to state our result.

Lemma III.5. *(Constraints on $\overline{K}$ and determination of L). Consider dynamics* (7)*. Let* $\overline{K} = \begin{bmatrix} K_1 & K_2 \\ K_3 & \Theta \end{bmatrix} \in \mathbb{R}^{(m+n)\times 2n}$ *be a controller gain such that* $\overline{A}+\overline{B}\,\overline{K}$ *is Hurwitz. Then* $\overline{K} \in \mathscr{K}$ *and there exists a* $L \in \mathbb{R}^{n\times r}$ *given by* $L = -\Theta C^\top (CC^\top)^{-1}$ *if*

$$\overline{K}\,\Pi_1 + \Pi_2\,\overline{K}\,\Pi_3 = \mathbf{0},$$
$$\Pi_4\,\overline{K}\,\Pi_5 = \mathbf{0}.$$

Proof. As $\overline{K}\,\Pi_1 + \Pi_2\,\overline{K}\,\Pi_3 = \mathbf{0}$ is true so we have $\begin{bmatrix} K_1 & \mathbf{0} \\ K_3 & \mathbf{0} \end{bmatrix} + \begin{bmatrix} K_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{0}$. The only nonzero solution possible is $K_2 = -K_1$ and $K_3 = \mathbf{0}$. Thus $\overline{K} \in \mathscr{K}$.
The true statement $\Pi_4\,\overline{K}\,\Pi_5 = \mathbf{0}$ gives $\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Theta\mathscr{N}(C) \end{bmatrix} = \mathbf{0}$, i.e., $\Theta\mathscr{N}(C) = \mathbf{0}$. Therefore Θ belongs to $\mathscr{R}(C^\top)$ [17]. This ensures that there must exist an unique $L = -\Theta C^\top (CC^\top)^{-1} \in \mathbb{R}^{n\times r}$ such that $\Theta = -LC$ holds true. □

Equipped with Theorem III.4 and Lemma III.5 we now state equivalent reformulation of (9) for given $Z_0 = z_0 z_0^\top$.

$$\begin{aligned} \min_{\overline{K}, \overline{P}\succ 0} \quad & J = \mathrm{tr}(\overline{P} Z_0) \\ \text{s.t.} \quad & \overline{A}_c^\top \overline{P} + \overline{P}\,\overline{A}_c + \overline{Q} + \overline{K}^\top \overline{R}\,\overline{K} = \mathbf{0}, \\ & \overline{A}_c = \overline{A} + \overline{B}\,\overline{K}, \\ & \overline{K}\,\Pi_1 + \Pi_2\,\overline{K}\,\Pi_3 = \mathbf{0}, \\ & \Pi_4\,\overline{K}\,\Pi_5 = \mathbf{0}. \end{aligned} \tag{11}$$

In (11) when z_0 is a random variable with zero mean and covariance Z_0, then $J = \mathrm{tr}(\overline{P} Z_0)$ is the expected value of the objective function keeping the constraints same. In such cases, we take $Z_0 = I$. Once we compute $\overline{K}$ we obtain K, Θ and $L = -\Theta C^\top (CC^\top)^{-1}$ from Lemma III.5. We next propose an iterative solution procedure to solve (11).

C. Solution Procedure and analysis

To solve (11) for $\overline{K}$ of a specific structure we use the augmented Lagrangian method (ALM) [13], [18] with suitable modifications. ALM starts with a structured or unstructured initialization of $\overline{K}$ denoted by $\overline{K}^{-1}$ and minimizes series of unstructured problems. Finally these minimizers converge to the structured minimizer of (11) [13]. We define the augmented Lagrangian as follows,

$$\mathscr{L}_c(\overline{K},V,U) = J + J_{uv} + J_{cc}, \tag{12}$$

where

$$\begin{aligned} J &= \operatorname{tr}(\overline{P}Z_0),\\ J_{uv} &= \operatorname{tr}\big(V^\top(\overline{K}\,\Pi_1 + \Pi_2\,\overline{K}\,\Pi_3)\big) + \operatorname{tr}\big(U^\top(\Pi_4\,\overline{K}\,\Pi_5)\big),\\ J_{cc} &= \frac{c_1}{2}\|\,\overline{K}\,\Pi_1 + \Pi_2\,\overline{K}\,\Pi_3\|^2 + \frac{c_2}{2}\|\Pi_4\,\overline{K}\,\Pi_5\|^2. \end{aligned}$$

Here $c_1 > 0$, $c_2 > 0$ are the scalar penalty weights on the constraints. U, V in J_{uv} are the Lagrangian multipliers [18] of the constraints. J_{cc} is the quadratic penalty function which ensures fast convergence to the structured solution. Details regarding ALM can be read in [13], [18]. We now summarize our modified ALM in Algorithm 1.

Algorithm 1 Augmented Lagrangian method

Input: $\overline{A}$, $\overline{B}$, $\overline{K}^{-1}$, U^0, V^0, c_1, c_2, γ_1, γ_2, $\overline{Q}$, $\overline{R}$, Z_0, ε_1
Output: $\overline{K}$

1: Set $i \leftarrow 0$
2: **while** $\|\overline{K}^{i-1}\,\Pi_1 + \Pi_2\,\overline{K}^{i-1}\,\Pi_3\| + \|\Pi_4\,\overline{K}^{i-1}\,\Pi_5\| > \varepsilon_1$ **do**
3:
$$\overline{K}^i = \arg\min_{\overline{K}}\ \mathscr{L}(\overline{K}, U^i, V^i) \quad \triangleright \text{ see Algorithm 2}$$
4: $\quad V^{i+1} \leftarrow V^i + c_1^i(\overline{K}^i\,\Pi_1 + \Pi_2\,\overline{K}^i\,\Pi_3)$
5: $\quad U^{i+1} \leftarrow U^i + c_2^i(\Pi_4\,\overline{K}^i\,\Pi_5)$
6: $\quad c_1^{i+1} \leftarrow \gamma_1 c_1^i$
7: $\quad c_2^{i+1} \leftarrow \gamma_2 c_2^i$
8: $\quad i \leftarrow i+1$
9: **end while**
10: $\overline{K} \leftarrow \overline{K}^i$
11: **return** $\overline{K}$

From optimal $\overline{K}$ one can compute optimal K, L using Lemma III.5. Generally $V^0 = \mathbf{0}$, $U^0 = \mathbf{0}$, $c_1^0 \in [5,\ 10]$, $c_2^0 \in [5,\ 10]$, $\gamma_1 \in [3,\ 10]$, $\gamma_2 \in [3,\ 10]$, $\varepsilon_1 \in [10^{-6},\ 10^{-3}]$ give good results as observed from our numerical simulations. $\overline{K}^{-1}$ is unstructured stabilizing controller gain for the system $(\overline{A},\ \overline{B})$ obtained from any known methods from control theory [1], [2], [19].

To solve the minimization problem for the i^{th} iteration in Step 3 of Algorithm 1 to compute $\overline{K}^i$ we use gradient-descent method [18], [20]. First, we derive an expression for the gradient of the augmented Lagrangian function defined in (12).

Lemma III.6. *(Gradient of $\mathscr{L}$ with respect to $\overline{K}$). Consider $\mathscr{L}$ defined in* (12). *The gradient of $\mathscr{L}$ with respect to $\overline{K}$ is*

$$\nabla_{\overline{K}}\mathscr{L} = \nabla_{\overline{K}}J + \nabla_{\overline{K}}J_{uv} + \nabla_{\overline{K}}J_{cc}$$

where

$$\begin{aligned} \nabla_{\overline{K}}J_{uv} &= \Pi_4^\top\,U\,\Pi_5^\top + V\,\Pi_1^\top + \Pi_2^\top\,V\,\Pi_3^\top,\\ \nabla_{\overline{K}}J_{cc} &= c_1\big(\overline{K}\,\Pi_1\,\Pi_1^\top + \Pi_2\,\overline{K}\,\Pi_3\Pi_1^\top + \Pi_2^\top\,\overline{K}\,\Pi_1\,\Pi_3^\top\\ &\qquad + \Pi_2^\top\,\Pi_2\,\overline{K}\,\Pi_3\,\Pi_3^\top\big) + c_2(\Pi_4^\top\,\Pi_4\overline{K}\,\Pi_5\Pi_5^\top),\\ \nabla_{\overline{K}}J &= 2(\overline{R}\,\overline{K} + \overline{B}^\top\,\overline{P})\overline{W}, \end{aligned}$$

with $\overline{P} = \overline{P}^\top \succeq 0$ and $\overline{W} = \overline{W}^\top \in \mathbb{R}^{2n\times 2n}$ solutions of

$$(\overline{A} + \overline{B}\,\overline{K})^\top\overline{P} + \overline{P}\,(\overline{A} + \overline{B}\,\overline{K}) + \overline{Q} + \overline{K}^\top\,\overline{R}\,\overline{K} = \mathbf{0}, \tag{13a}$$

$$(\overline{A} + \overline{B}\,\overline{K})\overline{W} + \overline{W}\,(\overline{A} + \overline{B}\,\overline{K})^\top + Z_0 = \mathbf{0}. \tag{13b}$$

Proof. To derive our results, we will make use of known matrix properties [1]. For matrices X, Y, $\operatorname{tr}(X\,Y) = \operatorname{tr}(Y\,X)$, $\operatorname{tr}(X) = \operatorname{tr}(X^\top)$ provided the matrices are compatible for multiplication, and $\frac{\partial\,\operatorname{tr}(Y\,X)}{\partial X} = Y^\top$. Evaluation of $\nabla_{\overline{K}}J_{uv}$ and $\nabla_{\overline{K}}J_{cc}$ are straightforward using matrix operations. For $\nabla_{\overline{K}}J$ we follow approach in [21], differentiating (13a) with respect to $\overline{K}$. $\overline{P}_{\overline{K}}$ is the derivative of $\overline{P}$ with respect to $\overline{K}$ and $d\overline{K}$ is the differential of $\overline{K}$. We have

$$\begin{aligned} &(\overline{A} + \overline{B}\,\overline{K})^\top\overline{P}_{\overline{K}}\,d\overline{K} + \overline{P}_{\overline{K}}\,d\overline{K}\,(\overline{A} + \overline{B}\,\overline{K})\\ &+ (\overline{B}\,d\overline{K})^\top\overline{P} + \overline{P}\,(\overline{B}\,d\overline{K}) + d\overline{K}^\top\,\overline{R}\,\overline{K} + \overline{K}^\top\,\overline{R}\,d\overline{K} = \mathbf{0} \end{aligned} \tag{14}$$

Pre-multiplying (14) by $\overline{W}$, post-multiplying (13b) by $\overline{P}_{\overline{K}}d\overline{K}$, taking trace and doing matrix manipulations we have

$$\operatorname{tr}[2\overline{W}(\overline{K}^\top\,\overline{R} + \overline{P}\,\overline{B})] = \operatorname{tr}(Z_0\overline{P}_{\overline{K}}\,d\overline{K}).$$

With differential $dJ = \operatorname{tr}(\nabla_{\overline{K}}^\top J\,d\overline{K}) = \operatorname{tr}(Z_0\overline{P}_{\overline{K}}\,d\overline{K})$ we get the required result from comparison. $\square$

Typically $\varepsilon_2 \in [10^{-4},\ 10^{-2}]$ for fast convergence. Now we state the gradient-descent algorithm to minimize the augmented Lagrangian $\mathscr{L}$ in the i^{th} iteration in Step 3 of Algorithm 1.

Algorithm 2 Gradient descent algorithm

Input: $\overline{A}$, $\overline{B}$, $\overline{K}^{i-1}$, U^i, V^i, c_1^i, c_2^i, $\overline{Q}$, $\overline{R}$, Z_0, ε_2, α, β
Output: $\overline{K}^i$

1: Set $j \leftarrow 0$, $\overline{K}^j \leftarrow \overline{K}^{i-1}$
2: **while** $\|\nabla_{\overline{K}}\mathscr{L}(\overline{K}^j, U^i, V^i)\| > \varepsilon_2$ **do**
3: $\quad$ Solve (13a) and (13b) to obtain $\overline{P}^j, \overline{W}^j$
4: $\quad$ Compute $\nabla_{\overline{K}}\mathscr{L}(\overline{K}^j, U^i, V^i)$ using Lemma III.6
5: $\quad$ Compute step size s^j using *Armijo rule* $\triangleright$ See Algorithm 3
6: $\quad \overline{K}^{j+1} \leftarrow \overline{K}^j - s^j\nabla_{\overline{K}}\mathscr{L}(\overline{K}^j, U^i, V^i)$
7: $\quad j \leftarrow j+1$
8: **end while**
9: $\overline{K}^i \leftarrow \overline{K}^j$
10: **return** $\overline{K}^i$

The algorithm for *Armijo rule* [18] to determine the step size s^j is given next.

Algorithm 3 Armijo rule [18]

Input: α, β, $\overline{K}^j$, U^i, V^i
Output: s^j
1: Set $s^j \leftarrow 1$
2: **while** $\mathscr{L}(\overline{K}^j - s^j \nabla_{\overline{K}} \mathscr{L}(\overline{K}^j)) \geq \mathscr{L}(\overline{K}^j) - \alpha s^j \|\nabla_{\overline{K}} \mathscr{L}(\overline{K}^j, U^i, V^i)\|^2$ **do**
3: $\quad s^j = \beta s^j$,
4: **end while**
5: **return** s^j

Typically $\alpha = 0.3$ and $\beta = 0.5$. Next we analyze some properties of Algorithm 1 and the solution of problems (9), (11).

D. Analysis

In this subsection, we prove salient properties of problems (9), (11). We first show that a controller $\overline{K}$ belonging to the set $\mathscr{K}$ can never be computed from the solution of some algebraic Riccati equation (ARE) [22] for the cases when $\overline{Q} \succeq 0$ and $\overline{R} \succ 0$. These results shows the importance of a method similar to that presented in the paper in obtaining the optimal controller-observer gain pair when an observer cost is present in the formulation.

Theorem III.7. *(Non-determination of structured $\overline{K}$ from ARE). Consider the dynamics* (7)*, performance index* (8) *with $\overline{Q} \succeq 0$ and $\overline{R} \succ 0$ such that system $(A, \sqrt{\overline{Q}})$ is detectable. Let the optimal stabilizing controller gain $\overline{K}_O$ computed using $\overline{K}_O = -\overline{R}^{-1} \overline{B}^\top \overline{S}$ where $\overline{S} \succeq 0$ is the solution of the ARE*

$$\Psi := \overline{A}^\top \overline{S} + \overline{S}\overline{A} + \overline{Q} - \overline{S}\overline{B}\overline{R}^{-1} \overline{B}^\top \overline{S} = \mathbf{0}.$$

Now consider another stabilizing gain $\overline{K}$ for the system (A,B). If $\overline{K} \in \mathscr{K}$ then $\overline{K} \neq \overline{K}_O$.

Proof. Before stating the proof first we determine the structure of $\overline{S}$. Let $\overline{S} = \begin{bmatrix} S_1 & S_2 \\ S_2^\top & S_3 \end{bmatrix}$ and as Ψ is symmetric we have $\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^\top & \Psi_3 \end{bmatrix}$. Note that S_1, S_2, S_3 are of same size. Substituting $\overline{S}$ and using (7), (8) we have,

$$\Psi_1 := A^\top S_1 + S_1 A + Q_1 - S_1 B R_1^{-1} B^\top S_1 - S_2 R_2^{-1} S_2^\top = \mathbf{0},$$
$$\Psi_2 := A^\top S_2 + S_2 A - S_1 B R_1^{-1} B^\top S_2 - S_2 R_2^{-1} S_3^\top = \mathbf{0},$$
$$\Psi_3 := A^\top S_3 + S_3 A + Q_2 - S_2 B R_1^{-1} B^\top S_2 - S_3 R_2^{-1} S_3^\top = \mathbf{0}.$$

Now $(\overline{A}, \overline{B})$, (A, B), (A, I) are stabilizable by assumption. By virtue of $(\overline{A}, \sqrt{\overline{Q}})$ being detectable we have $(A, \sqrt{Q_1})$ and $(A, \sqrt{Q_2})$ to be detectable. When $S_2 = \mathbf{0}$, we have Ψ_2 is fulfilled and Ψ_1 and Ψ_3 are AREs with S_1 and S_3 as unknowns respectively. By [22, Corollary 13.8] there exists unique $S_1 = S_1^* \succeq 0$ and $S_3 = S_3^* \succeq 0$ as solutions to AREs Ψ_1 and Ψ_3. Therefore $\overline{S}^* = \begin{bmatrix} S_1^* & \mathbf{0} \\ \mathbf{0} & S_3^* \end{bmatrix} \succeq 0$ is the unique solution of Ψ due to [22, Corollary 13.8]. Gain $\overline{K}_O = -\overline{R}^{-1} \overline{B}^\top \overline{S}^* = \begin{bmatrix} -R_1^{-1} B^\top S_1^* & \mathbf{0} \\ \mathbf{0} & -R_2^{-1} S_3^* \end{bmatrix}$. Now if $\overline{K} \in \mathscr{K}$ is true then clearly from the structure of $\overline{K}_O$ we get $\overline{K} \neq \overline{K}_O$. $\square$

Theorem III.7 proves that any stabilizing $\overline{K} \in \mathscr{K}$ cannot be obtained by solving an ARE. The problem (5) (i.e., $Q_2 = \mathbf{0}_{nn}$, and $R_2 = \mathbf{0}_{nn}$) can be treated as the limiting case in Theorem III.7. However, a rigorous analysis is required to establish the applicability of Theorem III.7 in such a scenario and left as future work. We now derive a lower bound for the optimal objective function value of (11).

Theorem III.8. *(Lower bound for J). Consider the dynamics* (7)*, performance index* (8) *and reformulated controller-observer design problem* (11) *with $\overline{R} \succ 0$. Let $Z_0 = z_0 z_0^\top$ and J^* be the optimal objective function value of* (11)*. Then* $\text{tr}(\overline{S}Z_0) < J^*$ *where $\overline{S} \succeq 0$ is the solution of the ARE Ψ defined in Theorem III.7.*

Proof. From ARE theory [1], [22] the unique global minimum value the of J defined in (8) is $\text{tr}(\overline{S}Z_0)$. The global minimum optimal controller gain is $\overline{K}_O = -\overline{R}^{-1} \overline{B}^\top \overline{S}$. The feasible set of (11) is the subset of set $\mathscr{K}$. From Theorem III.7 we know that $\overline{K}_O \notin \mathscr{K}$ so we must have $\text{tr}(\overline{S}Z_0) < J^*$. $\square$

Thus, from Theorem III.8 we observe that the performance index J in (8) is lower bounded by the full state feedback LQR cost of the extended dynamics (7) computed using the ARE. However such an full state feedback optimal controller cannot be practically implemented.

Remark III.9. *(Convergence of Algorithm 1). The convergence of Algorithm 1 cannot be proved theoretical due to the non-convex nature of the problem* (11)*. However, the empirical studies [13], [23], suggest that AML works well when the value of penalty weights c_1, c_2 is sufficiently large as the term associated with them locally convexifies the objective function.*

IV. Example: Aircraft System

Consider the linearized model of lateral dynamics of an aircraft system in vertical plane [12]. The nominal system matrices are

$$A = \begin{bmatrix} -0.746 & 0.006 & -1 & 0.0369 \\ -12.9 & -0.746 & 0.387 & 0 \\ 4.31 & 0.024 & -0.174 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.0012 & 0.0092 \\ 6.05 & 0.952 \\ -0.416 & -1.76 \\ 6.05 & 0.952 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (15)$$

The model state vector $x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T$ consists of the side-slip angle (in *degrees*), the roll rate (in *degrees/s*), the lace speed (in *degrees/s*), and the roll angle θ(in *degrees*), respectively. The control inputs $u = [u_1 \;\; u_2]^T$ are the angle of elevator (in tenths of a *degree*) and the steering angle (in *degree*), respectively. The structure of C implies that we measure only three states. We consider performance index (8) with the weight matrices as $Q_1 = C^T C$, $Q_2 = \zeta_1 Q_1$, $R_1 = I_{22}$ and $R_2 = \zeta_2 I_{44}$. The following parameters are used in the implementation of the algorithm: $c_1^0 = 8$, $c_2^0 = 9$, $\gamma_1 = 5$, $\gamma_2 =$

7, $\varepsilon_1 = 10^{-6}$, $\alpha = 0.3$ $\beta = 0.5$, $\varepsilon_2 = 10^{-3}$. For representation purpose, we take $\overline{K}$ to be

$$\overline{K} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}.$$

When $\overline{K} = \overline{K}^*$ we have $K_2^* = -K_1^*$, $K_3^* = \mathbf{0}$ and we compute $K^* = K_1^*$, $L^* = -K_4^* C^\top (C C^\top)^{-1}$. We use $Z_0 = I_{88}$ while implementing Algorithm 1 to get a solution independent of the initial state, which is often not completely known. Note that $Z_0 = I_{88}$ implies the initial state of z, i.e., $z(0) = z_0$ is a random vector with zero mean and unit variance. However, for validation purposes, we take the initial state to be $x(0) = [25,\ -35,\ 80,\ 35]^\top$ and $\hat{x}(0) = \mathbf{0}_{41}$ giving us $z_0 = [25,\ -35,\ 80,\ 35\ ,25,\ -35,\ 80,\ 35]^\top$ wherever required. For comparison purposes we compute the optimal full-state feedback controller $K_R = -R_1^{-1} B^\top P_R$ where P_R is the solution of the ARE

$$A^\top P_R + P_R A + Q_1 - P_R B R_1^{-1} B P_R = \mathbf{0}.$$

For this, we evaluate (4) numerically for z_0 and using the full state feedback optimal controller $u = K_R x$ to get $f_c^R = 6686.5$ which is the global optimal value of f_c. We have used MATLAB R2021a [24] for simulations. To test the utility of the proposed algorithm, it's evaluated for various scenarios, and the results are listed below.

A. $\zeta_1 = 0,\ \zeta_2 = 0$

Considering the observer cost zero, we first report the results solving the optimization problem (5) using the proposed algorithm for various initial gain settings. We also evaluate the solution given by the proposed algorithm in case of full-state feedback i.e. $e(0) = 0$ and $C = I$.

1) Case 1: Pole Placement: For this scenario, we start with an initial gain $\overline{K}^{-1} \in \mathscr{K}$ where controller and observer gains are selected by placing the poles to $[-10\ -9\ -10\ -12]$, and $[-8\ -7\ -8\ -5]$, respectively. The initial value of J for $\overline{K}^{-1}$ is 189.85. Applying Algorithm 1 and computing optimal $\overline{K}^*$ the cost $J = 6.52$ yielding an improvement of 96%. Further, using K^*, L^* we numerically evaluate f_c in (4) for given z_0 to get $f_c = 6765.68$. Thus, using K^*, L^* computed using Algorithm 1 with incomplete state measurements leads to $\approx 1.18\%$ degradation compared to f_c^R justifying utility of our proposed theory.

2) Case 2: $K^0 = K_R$: For this scenario, we start with an initial gain $\overline{K}^{-1} \in \mathscr{K}$ where controller gain is selected as a LQR gain to system (A,B) with given $Q1$ and R_1. Meanwhile, the observer gains are selected by placing the observer poles to $[-8\ -7\ -3\ -5]$, respectively. The initial value of J for $\overline{K}^{-1}$ is 7.32. Applying Algorithm 1 and computing optimal $\overline{K}^*$ the cost $J = 6.52$ yielding an improvement of 10%.

3) Case 3: Full-state feedback: For this scenario, we start with an initial gain $\overline{K}^{-1} \in \mathscr{K}$ where controller gain is selected as a LQR gain to system (A,B) with given $Q1$ and R_1. Meanwhile, the observer gains are selected by placing the observer poles to (i) $[-8\ -7\ -3\ -5]$ and (ii) $[-8\ -7\ -13\ -15]$, respectively. We also have $x(0) = \hat{x}(0)$ and $C = I$. As expected, the Algorithm 1 terminated in the first iteration suggesting that the optimal controller observer pair for this scenario is $K^* = K_R$ with any stabilising observer gain L. We perform another test with an initial gain $\overline{K}^{-1} \in \mathscr{K}$ where controller and observer gains are selected by placing the poles to $[-10\ -9\ -10\ -12]$, and $[-8\ -7\ -8\ -5]$, respectively. The initial value of J for this $\overline{K}^{-1}$ is 94.73. In this test case, the Algorithm 1 terminated as soon as $K^* = K_R$ with a stabilising observer gain is achieved yielding the optimal cost 6.62 which is also similar to earlier test cases (i) and (ii). This verifies the rationality of the proposed algorithm.

B. $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$

When $\zeta \neq 0$ and $C \neq I$, the observer objective $f_o = \int_0^\infty [e^\top Q_2 e + u_e^\top R_2 u_e]\,\mathrm{d}t$ is also given importance in the optimization process. Consider a scenario when $\zeta_1 = \zeta_2 = 1$ for the example under consideration. We take $\overline{K}^{-1}$ as per Theorem III.7 where it has a block-diagonal structure. After application of Algorithm 1, we get $\overline{K}^*$ presented in (17) and is of the desired structure. Furthermore the average initial cost considering $Z_0 = I_{88}$ in Theorem III.8 i.e., $\mathrm{tr}(\overline{S} I_{88}) = 11.99$ is less than $J^* = 17.52$ obtained from Algorithm 1 and validates Theorem III.8.

$$\begin{aligned} K_1^0 &= \begin{bmatrix} -0.1153 & -0.5211 & 0.0363 & -0.9346 \\ -1.7056 & 0.2942 & 1.6640 & -0.4193 \end{bmatrix} \\ K_2^0 &= \mathbf{0}_{24}, \quad K_3^0 = \mathbf{0}_{44} \\ K_4^0 &= \begin{bmatrix} -3.2589 & 0.7674 & 0.3931 & 0.4290 \\ 0.7674 & -0.3894 & -0.2976 & -0.2287 \\ 0.3931 & -0.2976 & -1.1635 & -0.1219 \\ 0.4290 & -0.2287 & -0.1219 & -0.8468 \end{bmatrix} \end{aligned} \tag{16}$$

$$\begin{aligned} K_1^* &= \begin{bmatrix} 0.1585 & -0.3742 & -0.1036 & -0.7449 \\ -0.9881 & 0.1386 & 1.1257 & -0.4743 \end{bmatrix} \\ K_2^* &= -K_1^*, \quad K_3^* = \mathbf{0}_{44} \\ K_4^* &= \begin{bmatrix} 0.0000 & 1.4437 & 0.6503 & -0.1699 \\ 0.0000 & -0.9490 & -0.3801 & -0.0242 \\ 0.0000 & -0.4606 & -1.8823 & 0.1298 \\ 0.0000 & -0.2299 & 0.0456 & -1.3765 \end{bmatrix} \\ L^* &= \begin{bmatrix} -1.4437 & -0.6503 & 0.1699 \\ 0.9490 & 0.3801 & 0.0242 \\ 0.4607 & 1.8823 & -0.1298 \\ 0.2299 & -0.0456 & 1.3765 \end{bmatrix} \end{aligned} \tag{17}$$

Finally, to test how varying ζ_1 and ζ_2 affect the observer poles i.e. closed loop poles of the system (A,C), we apply the algorithm 1 with $\overline{K}^{-1}$ as given in (16) and the system under consideration. The results are presented in terms of a heat-map in Fig. 1. The trend shows that for a constant ζ_1, increasing ζ_2 shifts the poles towards the right or the imaginary line. At the same time, one can observe an opposite trend for a constant ζ_2, increasing ζ_1.

V. Conclusions

This paper presents a novel framework for designing an optimal observer-based controller for LTI systems. We refor-

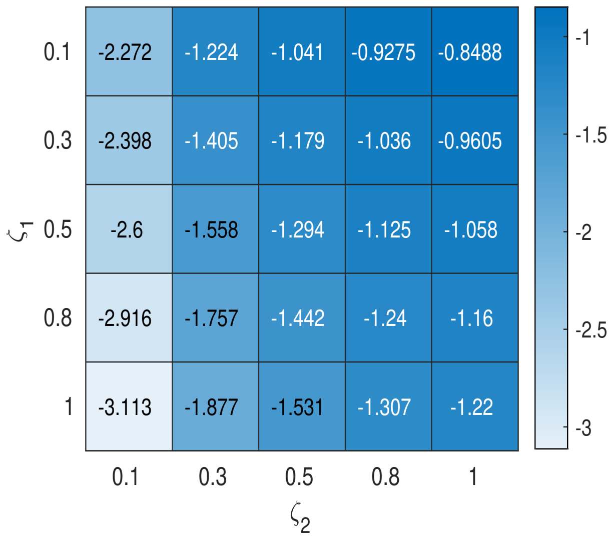

Fig. 1: Observer poles plot over varying ζ_1 and ζ_2. The values shown in each block represents the maximum value of the real part of poles of system (A, C) with observer gain L^*.

mulated the optimal observer-based controller using extended state dynamics and presented an augmented Lagrangian-based strategy to compute an optimal solution. We analytically show that the controller of the extended state dynamics has a specific structure to embed the controller and observer gains of the original system. We also prove that the specifically structured controller of the extended state dynamics cannot be optimally obtained by solving an algebraic Riccati equation. Finally, we justify our proposed theory with an example. Our future work involves studying the effect of increasing weightage on the observer objective, the optimization landscape of (5), adding input constraints in the formulation, computational aspects of Algorithm 1, and application to large-size practical examples.

References

[1] F. L. Lewis, D. Vrabie, and V. L. Syrmos, *Optimal control*. Hoboken, NJ, USA: John Wiley & Sons, 2012.

[2] H. Kwakernaak and R. Sivan, *Linear optimal control systems*. New York, NY, USA: Wiley-interscience, 1969, vol. 1072.

[3] V. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback—a survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[4] A. Ilka and N. Murgovski, "Novel results on output-feedback lqr design," *IEEE Transactions on Automatic Control*, 2022.

[5] A. Ilka, N. Murgovski, and J. Sjöberg, "An iterative newton's method for output-feedback lqr design for large-scale systems with guaranteed convergence," in *European Control Conference (ECC)*. Naples, Italy: IEEE, 2019, pp. 4849–4854.

[6] P. C. Müller, "Design of optimal state-observers and its application to maglev vehicle suspension control," *IFAC Proceedings Volumes*, vol. 10, no. 6, pp. 175–182, 1977.

[7] M. Newmann, "Optimal and sub-optimal control using an observer when some of the state variables are not measurable," *International Journal of Control*, vol. 9, no. 3, pp. 281–290, 1969.

[8] D. Rom and P. Sarachik, "The design of optimal compensators for linear constant systems with inaccessible states," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 509–512, 1973.

[9] H. Maeda and H. Hino, "Design of optimal observers for linear time–invariant systems," *International Journal of Control*, vol. 19, no. 5, pp. 993–1004, 1974.

[10] J. Duan, W. Cao, Y. Zheng, and L. Zhao, "On the optimization landscape of dynamic output feedback linear quadratic control," *IEEE Transactions on Automatic Control*, pp. 1–16, 2023.

[11] S. Ghali, A. Benallegue, and S. Elloumi, "A decentralized observer-based optimal control for interconnected systems using the block pulse functions," *Transactions of the Institute of Measurement and Control*, vol. 42, no. 15, pp. 3063–3075, 2020.

[12] I. S. Aousgi, S. Elloumi, and N. Benhadj Braiek, "Design of an observer based optimal control via block pulse functions," in *Mediterranean Conference on Control and Automation*, Palermo, Italy, 2014, pp. 569–573.

[13] F. Lin, M. Fardad, and M. R. Jovanovic, "Augmented lagrangian approach to design of structured optimal state feedback gains," *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2923–2929, 2011.

[14] W. J. Rugh, *Linear system theory*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.

[15] H. K. Khalil, *Nonlinear systems*. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[16] T. Abdel-Moneim, "Optimal compensators with pole constraints," *IEEE Transactions on Automatic Control*, vol. 25, no. 3, pp. 596–598, 1980.

[17] R. A. Horn and C. R. Johnson, *Matrix analysis*. New York, NY, USA: Cambridge university press, 2012.

[18] D. Bertsekas, *Nonlinear Programming*. Belmont, MA, USA: Athena Scientific, 1999.

[19] K. Ogata *et al.*, *Modern control engineering*. Upper Saddle River, NJ, USA: Prentice hall, Inc., 2010, vol. 5.

[20] B. D. Anderson, J. B. Moore, and B. Molinari, "Linear optimal control," *IEEE Transactions on Systems, Man, and Cybernetics*, no. 4, pp. 559–559, 1972.

[21] T. Rautert and E. W. Sachs, "Computational design of optimal output feedback controllers," *SIAM Journal on Optimization*, vol. 7, no. 3, pp. 837–852, 1997.

[22] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Englewood Cliffs, NJ, USA: Prentice hall, 1996.

[23] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[24] MATLAB, *version 9.10.0 (R2021a)*. Natick, MA: The MathWorks Inc., 2021.