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Small gain conditions for stability of infinite networks of time-delay systems and applications

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Abstract— **We prove a new theorem on sufficient conditions for ℓ_∞-global asymptotic stability of infinite networks composed of interconnected nonlinear time-delay systems. The sufficient conditions are formulated in terms of ISS Lyapunov-Razumikhin functions and suitable small-gain conditions. We demonstrate the applicability of our small-gain conditions to decentralized stabilization of infinite networks of nonlinear control systems with time delays.**

I. Introduction

Large-scale networks composed of many coupled subsystems, each of which has the state affecting the dynamics of its neighbours, have many applications in industry and engineering: vehicle platooning [3], [27], [39] and power networks [30], [29], multi-agent control systems [28], neural systems [5], computer networks [33], shared (wired or wireless) communication networks [18], networked force-reflecting telerobotic systems [41], [40], [9], economic systems and logistics [12], [10], chemical engineering [2], etc.

Stability analysis of large-scale networks composed of interconnected nonlinear subsystems is a difficult and challenging problem and one of its central methods is the theory of input-to-state stability (ISS) [43], [44] and small-gain theorems [21], [23], [14], [22], [15]. It becomes especially challenging in the case when the number of subsystems is not limited, i.e., some nodes or subsystems can arrive, some others can depart and one does not have any a priori estimate of their maximal possible number. For this case, the notions of string ISS or scalable ISS were introduced and some suitable small gain theorems were proved [39], [3], [4], [27]. It is also possible that the number of nodes is very large, as an example one can consider such complex neural systems as brain [5]. Therefore, it is natural to raise the problems of stability and control for *infinite* networks, which are composed of *infinite* set of nodes [8], [13], [24], [25], [26], [35]. Paper [8] deals with infinite networks of interconnected linear control systems, and recent works [13], [24], [25], [26], [35] deal with input-to-state stability (and decentralized stabilization [13], [35]) of infinite networks composed of interconnected nonlinear subsystems.

At the same time, one of substantial classes of dynamical systems is time-delay systems, and it is natural that the control theory as well as the ISS theory and stability analysis of large-scale networks of time-delay systems has received a lot of attention [17], [45], [16], [38], [37], [47], [19], [7], [11], [1], [28], [20], [3], [42], [6].

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A general small-gain theorem for *finite* large-scale networks of systems with time delays was proved in [11]. As regards the case of *infinite* networks of time-delay systems, general trajectory-based small-gain theorems were proved in recent works [31], [32]. However, to our best knowledge, there are no works devoted to Lyapunov-based small-gain theorems for infinite networks of time-delay systems as well as to their decentralized and distributed control at this moment. (Let us note that one needs Lyapunov-based small-gain theorems when solving the problems of decentralized/distributed control, see, e.g. [36], [13]).

The goal of this paper is to prove an analog of the small-gain Theorem 1 from [13] in the case when the infinite network is composed not of just a countably infinite set of systems of *ordinary differential equations (ODE)* as in [13] but of a countably infinite set of *time-delay systems*. Similarly to [13], we demonstrate how our small-gain theorem can be applied to decentralized control of infinite networks of control systems with time delays.

II. Preliminaries

Throughout the paper, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$ and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ denotes the quadratic norm of $\xi \in \mathbb{R}^N$. All vectors from $\mathbb{R}^N$ are treated as columns, i.e., $\mathbb{R}^N \cong \mathbb{R}^{N\times 1}$. For a, b from $\mathbb{R}$, by $]a, b[$ we denote the open interval $]a, b[:= \{s \in \mathbb{R} \mid a < s < b\}$; accordingly, we denote by $]a, b]$, $[a, b[$, the intervals defined by $]a, b] := \{s \in \mathbb{R} \mid a < s \le b\}$, $[a, b[:= \{s \in \mathbb{R} \mid a \le s < b\}$; intervals $]-\infty, b[$, $]a, +\infty[$, $]-\infty, b]$, and $[a, +\infty[$ are defined similarly.

If $\mathcal{A}$ is a finite set then $|\mathcal{A}|$ denotes the number of its elements (thus, $|\mathcal{A}| = 0$ for $\mathcal{A} = \emptyset$ and $|\mathcal{A}| \in \mathbb{N}$ otherwise).

We define ℓ_∞ as the normed vector space (actually, Banach space) of sequences of real numbers of the form $Z = \{z_i\}_{i\in\mathbb{N}}$ such that $\sup\limits_{i\in\mathbb{N}} |z_i| < +\infty$, and the norm of this space is defined by $||Z||_{\ell_\infty} := \sup\limits_{i\in\mathbb{N}} |z_i|$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$, and $\mathcal{K}_\infty$ is the set of all the unbounded $\mathcal{K}$-functions. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if for each fixed $t \ge 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \ge 0$, we have $\beta(s, t) \to 0$ as $t \to +\infty$ and $t \mapsto \beta(s, t)$ is decreasing.

Let $\mathcal{B}_i$, $i = 1, \ldots, N$ (with some $N \in \mathbb{N}$) be some Banach spaces. For convenience and brevity, we sometimes write $\prod_{i\in\{1,\ldots,N\}} \mathcal{B}_i$ instead of $\mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_N$.

Definition 1: Let $\mathcal{B}_i$, $i = 1, \ldots, N$ (with some $N \in \mathbb{N}$) be some Banach spaces and $n \in \mathbb{N}$ be some natural number.

As in [6], we say that a a map $F : \prod_{i\in\{1,\ldots,N\}} \mathcal{B}_i \to \mathbb{R}^n$ is *Lipschitz on bounded sets* if, for any $r > 0$, there exists $L(r) > 0$ such that

$$|F(\xi_1,\ldots,\xi_N) - F(\eta_1,\ldots,\eta_N)| \le L(r)\sum_{i=1}^{N} |\xi_i - \eta_i|. \quad (1)$$

As usually, any such $L(r) > 0$ from (1) is referred to as a *Lipschitz constant* for this map F and for this radius $r > 0$.

III. Problem Formulation and Main Definitions

In this paper, we consider infinite networks with time-delays of the following form

$$\dot{X}_i(t) = F_i(X_i^t, \{X_j^t\}_{j\in J(i)}), \qquad i \in \mathbb{N}, \qquad (2)$$

where $X_i^t(\tau) = X_i(t+\tau)$, $\tau \in [-\theta, 0]$, and $X_i \in \mathbb{R}^{N_i}$ is the state vector of the i-th subsystem.

In general, F_i are functions of $C([-\theta,0];\mathbb{R}^{N_i}) \times \prod_{j\in J(i)} C([-\theta,0];\mathbb{R}^{N_j})$ to $\mathbb{R}^{N_i}$ and, to provide the existence and the uniqueness of the solution to the corresponding Cauchy Problem we assume that they are Lipschitz continuous on bounded sets in the sense of Definition 1, see below.

Throughout the paper, we assume that *the entire state vector $X = \{X_i\}_{i=1}^{\infty}$ of system (2) is always an element of ℓ_∞* and system (2) has the following properties.

Standing Assumption (A1): The set $J(i) \subset \mathbb{N}$ of the neighbors of the i-th subsystem is always *finite*, and, by definition $i \notin J(i)$ for all $i \in \mathbb{N}$, i.e. i-th node is not a neighbor of itself. *Throughout this paper, we assume that* $\sup_{i\in\mathbb{N}} |J(i)| < \infty$ *and* $\sup_{i\in\mathbb{N}} N_i < \infty$. For brevity, we also denote $I(i) := \{i\} \cup J(i)$

Standing Assumption (A2): For each $i \in \mathbb{N}$, the corresponding vector field F_i of the ith subsystem (2) is Lipschitz on bounded sets in the sense of Definition 1 and, furthermore, for each fixed $i \in \mathbb{N}$ and each fixed $r > 0$ a suitable Lipschitz constant $L_i(r) > 0$ from (1) can be chosen such that we have $\sup_{i\in\mathbb{N}} L_i(r) < \infty$ for every fixed $r > 0$.

Standing Assumption (A3): For each $i \in \mathbb{N}$, the corresponding vector field F_i satisfies $F_i(0,0) = 0$.

Remark 1: Let us note that Assumptions (A2),(A3) also imply that, similarly to [13], the dynamics of (2) is "locally ℓ_∞- bounded" in the following sense: for each fixed $R > 0$, we have

$$\sup_{i\in\mathbb{N}} \sup_{\substack{||\xi_i||_{C([-\theta,0];\mathbb{R}^{N_i})} \le R \\ ||\xi_j||_{C([-\theta,0];\mathbb{R}^{N_j})} \le R,\ j\in J(i)}} |F_i(\xi_i, \{\xi_j\}_{j\in J(i)})| < \infty. \qquad (3)$$

For systems (2), we are interested in sufficient conditions for their global asymptotic stability.

Given any $\xi = \{\xi_i\}_{i=1}^{\infty}$ with $\xi_i \in C([-\theta,0];\mathbb{R}^{N_i})$ and with $\sup_{i\in\mathbb{N}} \parallel \xi_i \parallel_{C([-\theta,0];\mathbb{R}^{N_i})} < \infty$, let $t \mapsto X(t,0,\xi) = \{X_i(t,0,\xi)\}_{i=1}^{\infty}$ denote the maximal solution to (2) with the initial condition $X(0,0,\xi) = \xi$ such that $\sup_{i\in\mathbb{N}} |X_i(t,0,\xi)| < \infty$. This means that $t \mapsto X(t,0,\xi)$ is defined on some maximal interval $[0, t_{\max}[$ of its existence, satisfies (2) for all $t \in [0, t_{\max}[$ and $X(t,0,\xi) \in \ell_\infty$ for all $t \in [0, t_{\max}[$. Motivated by [38], [37], [12], [6] as well as by [13], [24], [34], [25], [26], [35], we give the following main definition.

Definition 2: System (2) is said to be ℓ_∞-globally asymptotically stable or ℓ_∞-GAS, if and only if there exists $\beta \in \mathcal{KL}$ such that for each $\xi = \{\xi_i\}_{i=1}^{\infty}$ with $\xi_i \in C([-\theta,0];\mathbb{R}^{N_i})$ and with $\sup_{i\in\mathbb{N}} \parallel \xi_i \parallel_{C([-\theta,0];\mathbb{R}^{N_i})} < \infty$, we have

$$||X(t,0,\xi)||_{\ell_\infty} \le \beta(\sup_{i\in\mathbb{N}} \parallel \xi_i \parallel_{C([-\theta,0];\mathbb{R}^{N_i})}, t). \qquad (4)$$

IV. Main Results

The current version of our main small gain theorem is as follows.

Theorem 1: Suppose (A1)-(A3) hold true and there exist positive definite functions $V_i(X_i)$ in $C^1(\mathbb{R}^{N_i}; [0,+\infty[)$, $i \in \mathbb{N}$ (i.e., with $V_i(0) = 0$ and with $V_i(X_i) > 0$, whenever $X_i \ne 0 \in \mathbb{R}^{N_i}$) that will be called ISS Lyapunov functions such that the following conditions hold

(i) There exists $\rho(\cdot) \in \mathcal{K}_\infty$ such that $V_i(X_i) \ge \rho(|X_i|)$ uniformly for all $X_i \in \mathbb{R}^{N_i}$, $i \in \mathbb{N}$ (i.e., $V_i(\cdot)$ are uniformly radially unbounded)

(ii) For each $R > 0$ we have:

$$\begin{aligned} &\sup_{i\in\mathbb{N}} \max_{|X_i|\le R} V_i(X_i) < +\infty, \\ &\sup_{i\in\mathbb{N}} \max_{|X_i|\le R} \left|\frac{\partial V_i(X_i)}{\partial X_i}\right| < +\infty \end{aligned} \qquad (5)$$

(iii) There exists $\alpha(\cdot) \in \mathcal{K}$, and there exists $\gamma(\cdot) \in \mathcal{K}$ such that $\gamma(r) < r$ for all $r > 0$ and such that, for each $\{\xi_j\}_{j\in I(i)} \in \prod_{j\in I(i)} C([-\theta,0];\mathbb{R}^{N_j})$, we have

$$\begin{aligned} &V_i(\xi_i(0)) \ge \max_{j\in I(i)} \left\{\gamma(\max_{\tau\in[-\theta,0]} V_j(\xi_j(\tau)))\right\} \Rightarrow \\ &\nabla V_i(\xi_i(0)) F_i(\xi_i, \{\xi_j\}_{j\in J(i)}) \le -\alpha(V_i(\xi_i(0))). \end{aligned} \qquad (6)$$

Then the following three statements hold true:

(I) Given any $\xi = \{\xi_i\}_{i=1}^{\infty}$ with $\xi_i \in C([-\theta,0];\mathbb{R}^{N_i})$ and with $\sup_{i\in\mathbb{N}} \parallel \xi_i \parallel_{C([-\theta,0];\mathbb{R}^{N_i})} < \infty$, the corresponding solution $t \mapsto X(t,0,\xi)$ to (2) with the initial condition $X(0,0,\xi) = \xi$ is well defined on the entire $[0,+\infty[$, it is unique and satisfies the condition $\sup_{t\in[-\theta,+\infty[} \sup_{i\in\mathbb{N}} |\xi_i(t)| < \infty$.

(II) Define the Lyapunov function for system (2) by

$$V(X) := \sup_{i\in\mathbb{N}} \{V_i(X_i)\} \quad \text{for all } X = \{X_i\}_{i\in\mathbb{N}} \in \ell_\infty. \qquad (7)$$

Then, for every solution $t \mapsto X(t) = X(t,0,\xi)$ defined on the entire $[0,+\infty[$ (see Item (I) above) by any initial $\xi = \{\xi_i\}_{i=1}^{\infty}$ with $\xi_i \in C([-\theta,0];\mathbb{R}^{N_i})$ and with $\sup_{i\in\mathbb{N}} \parallel \xi_i \parallel_{C([-\theta,0];\mathbb{R}^{N_i})} < \infty$, the corresponding function $t \mapsto V(X(t))$ is locally absolutely continuous on $[t_0,+\infty[$ and there exist $\hat{\alpha}(\cdot) \in \mathcal{K}$, and $\hat{\gamma}(\cdot) \in \mathcal{K}$ such that $\hat{\alpha}(r) \le \alpha(r)$ and $r > \hat{\gamma}(r) \ge \gamma(r)$ for all $r > 0$ and such that

$$\begin{aligned} &V(X(t)) \ge \hat{\gamma}(\max_{\tau\in[-\theta,0]} V(X(t+\tau))) \Rightarrow \\ &\tfrac{d[V(X(t))]}{dt} \le -\hat{\alpha}(V(X(t))) \quad \text{a.e. on } [0,+\infty[. \end{aligned} \qquad (8)$$

(III) System (2) is ℓ_∞-GAS in the sense of Definition 2.

Remark 2: Based on the notation and terminology proposed in [45], [12], [6], we see that it is indeed possible to say that $V_i(\cdot)$ are ISS Lyapunov functions (or even ISS Lyapunov-Razumikhin functions) for the corresponding subsystems of (2) and to say that Theorem 1 is a small-gain theorem, or a Lyapunov-based small-gain theorem (or even a Lyapunov-Razumikhin-based small-gain theorem).

V. PROOF OF THEOREM 1

First of all, we note that (ii) implies the existence of $\hat{\rho}(\cdot) \in \mathcal{K}_\infty$ such that $V_i(X_i) \le \hat{\rho}(|X_i|)$ uniformly for all $X_i \in \mathbb{R}^{N_i}$, $i \in \mathbb{N}$.

Take any $\xi = \{\xi_i\}_{i=1}^{\infty}$ with $\xi_i \in C([-\theta, 0]; \mathbb{R}^{N_i})$ and with $\sup_{i\in\mathbb{N}} \| \xi_i \|_{C([-\theta,0];\mathbb{R}^{N_i})} < \infty$, Without loss of generality, we assume that $\xi \ne 0$, i.e., there is $i \in \mathbb{N}$ such that $\xi_i \ne 0 \in C([-\theta, 0]; \mathbb{R}^{N_i})$ (otherwise the solution is unique and is identically equal to zero, and the statement is trivial). Then, we define

$$\begin{array}{l} V^\star := V(\xi(0)) > 0, \\ V^0 := \hat{\rho}(\sup_{i\in\mathbb{N}} \| \xi_i \|_{C([-\theta,0];\mathbb{R}^{N_i})}) > 0. \end{array} \tag{9}$$

As in the Proof of Theorem 1 from [13], we split our proof of Theorem 1 into the following steps.

Step 1. In this Step 1 we prove the existence of $s^\star > 0$ such that the corresponding solution $t \mapsto X(t, 0, \xi)$ to (2) is well-defined on $[0, s^\star]$. First, define the following constants

$$\begin{aligned} M := \sup_{i\in\mathbb{N}} \Big\{ 1 + & \sup_{\substack{|X_i| \le \rho^{-1}(2V^0) \\ ||\xi_i||_{C([-\theta,0];\mathbb{R}^{N_i})} \le \rho^{-1}(2V^0), \\ ||\xi_j||_{C([-\theta,0];\mathbb{R}^{N_j})} \le \rho^{-1}(2V^0) \\ j \in J(i)}} \left| \frac{\partial V_i(X_i)}{\partial X_i} \right| |F_i(\xi_i, \{\xi_j\}_{j\in J(i)})| \\ & + \sup_{\substack{||\xi_i||_{C([-\theta,0];\mathbb{R}^{N_i})} \le \rho^{-1}(2V^0), \\ ||\xi_j||_{C([-\theta,0];\mathbb{R}^{N_j})} \le \rho^{-1}(2V^0) \\ j \in J(i)}} |F_i(\xi_i, \{\xi_j\}_{j\in J(i)})| \Big\}; \\ s^\star := & \tfrac{V^0}{4M+1}; \end{aligned} \tag{10}$$

and then define the Picard's iterations $\{\{X_i(q,\cdot)\}_{i\in\mathbb{N}}\}_{q=0}^{\infty}$ on $[0, s^\star]$ by

$$\begin{array}{l} X_i(0,t) := \xi_i(0), \qquad t \in [0, s^\star], \\ X_i(q+1,t) = \xi_i(0) + \int_0^t F_i(X_i^s(q,\cdot), \{X_j^s(q,\cdot)\}_{j\in J(i)}) ds, \\ t \in [0, s^\star], \qquad i \in \mathbb{N}, \qquad q \in \mathbb{N}. \end{array} \tag{11}$$

Using (7),(9),(10),(11), we by induction on $q = 1, 2, \ldots$, that $\frac{1}{2}V^\star \le V(X(q,t)) \le \frac{3}{2}V^\star$ for all $t \in [0, s^\star]$, $q \in \mathbb{N}$, where $X(q,t) = \{X_i(q,\cdot)\}_{i\in\mathbb{N}}$. Then, using (A2), and arguing as for usual Picard's iterations for Lipszhitz continuous dynamics in finite-dimensional spaces, we find $X(\cdot) = \{X_i(\cdot)\}_{i\in\mathbb{N}}$ such that $\| X_i(q,\cdot) - X_i(\cdot) \|_{C([0,s^\star];\mathbb{R}^{N_i})} \to 0$ as $q \to +\infty$. Combining this with (11), we obtain that $X(\cdot) = \{X_i(\cdot)\}_{i\in\mathbb{N}}$ is the desired solution $t \mapsto X(t, 0, \xi)$ to (2) on $[0, s^\star]$ with $X(0) = \xi$. Its uniqueness can be proved by using (A2) and the Gronwall-Bellman lemma. By the above construction, this solution (along with $X(q,t)$) satisfies the inequality

$$\frac{1}{2}V^\star \le V(X(t)) \le \frac{3}{2}V^\star \quad \text{for all} \;\; t \in [0, s^\star]. \tag{12}$$

Step 2. In this Step 2, we prove (8) on $[0, s^\star]$, then extend it to $[0, +\infty[$, and obtain (I)-(III).

Define $\hat{\alpha}(\cdot) := \frac{1}{2}\alpha(\cdot) \in \mathcal{K}$, and fix any $\hat{\gamma}(\cdot) \in \mathcal{K}$ such that $\hat{\alpha}(r) < \alpha(r)$ and $r > \hat{\gamma}(r) > \gamma(r)$ for all $r > 0$. Define

$$\begin{array}{c} \varepsilon := \frac{1}{2} \min \big\{ \min\limits_{\frac{1}{2}V^\star \le V \le \frac{3}{2}V^\star} (\alpha(V) - \hat{\alpha}(V)), \\ \min\limits_{\frac{1}{2}V^\star \le V \le \frac{3}{2}V^\star} |V - \hat{\gamma}(V)| \big\}. \end{array} \tag{13}$$

From (A2) and (5), we obtain the existence of $L = L(V^0, V^\star) > 0$ such that

$$\begin{array}{c} \forall i \in \mathbb{N} \qquad \forall t' \in [0, s^\star] \qquad \forall t'' \in [0, s^\star] \\ |V_i(X_i(t')) - V_i(X_i(t''))| \le L(V^0, V^\star)|t' - t''|, \end{array} \tag{14}$$

and then

$$\begin{array}{c} \forall t' \in [0, s^\star] \qquad \forall t'' \in [0, s^\star] \\ |V(X(t')) - V(X(t''))| \le L(V^0, V^\star)|t' - t''|. \end{array} \tag{15}$$

Hence $t \mapsto V(X(t))$ is absolutely continuous and differentiable almost everywhere on $[0, s^\star]$.

Then, there is $\tau \in]0, \frac{s^\star}{2}]$ such that

$$\begin{array}{c} \forall t' \in [0, s^\star] \qquad \forall t'' \in [0, s^\star] \forall i \in \mathbb{N} \\ |V_i(X_i(t')) - V_i(X_i(t''))| \le \frac{\varepsilon}{4}, \end{array} \tag{16}$$

For every $t \in [0, s^\star]$ and every $\delta \in]0, \varepsilon]$, by $\Upsilon(\delta, t) \subset \mathbb{N}$ denote the following set of indices

$$\Upsilon(\delta, t) := \{j \in \mathbb{N} \mid V_j(X_j(t)) \ge V(X(t)) - \delta\}. \tag{17}$$

Then, as in [13] (proof of Theorem 1, Step 2) we easily obtain the following Lemma.

Lemma 1: For each $i \in \mathbb{N}$, each $t \in [0, s^\star - \tau]$ and each $h \in [0, \tau]$ we have:

$$\begin{array}{c} V_i(X_i^*(t+h)) \ge V(X(t)) - \frac{3\varepsilon}{4} \Rightarrow \\ \dot{V}_i(X_i^*(t+h)) \le -\alpha\big(V_i(X_i^*(t+h))\big) \end{array}$$

and

$$V_i(X_i^*(t+h)) \le \max\{V_i(X_i^*(t)), V(X(t)) - \frac{3\varepsilon}{4}\}.$$

and

$$\forall i \in \Upsilon(\varepsilon/2, t) \qquad \dot{V}_i(X_i(t+h)) \le -\alpha\big(V_i(X_i(t+h))\big).$$

Next we note that $\Upsilon(\delta_1, t) \subset \Upsilon(\delta_2, t)$ for all $0 < \delta_1 < \delta_2 \le \varepsilon$ and for all $t \in [0, s^\star]$ and all $\delta \in]0, \frac{\varepsilon}{4}]$ we have

$$V(X(t)) = \sup_{i\in\mathbb{N}} V_i(X_i(t)) = \sup_{i\in\Upsilon(\delta,t)} V_i(X_i^*(t)) \tag{18}$$

and

$$\forall i \in \Upsilon(\delta, t) \qquad \dot{V}_i(X_i(t+h)) \le -\alpha\big(V_i(X_i(t+h))\big). \tag{19}$$

Then arguing as in Step 2 from Proof of Theorem 1 from [13], and taking into account (iii) along with the inequalities

$$
\begin{aligned}
& V(X(t)) \geq \hat{\gamma}(\max_{\tau\in[-\theta,0]} V(X(t+\tau))) \Rightarrow \\
& \forall i \in \mathbb{N} \qquad\qquad V(X(t)) \geq \hat{\gamma}(\max_{\tau\in[-\theta,0]} V_i(X(t+\tau))) \\
& \geq \gamma(\max_{\tau\in[-\theta,0]} V_i(X(t+\tau))),
\end{aligned}
$$

we obtain (8) almost everywhere on $[0, s^\star]$. Then we consider Picard's iterations (11) on $[s^\star, 2s^\star]$, and extend our solution to $[s^\star, 2s^\star]$, $\ldots$, $[ks^\star, (k+1)s^\star]$, $\ldots$, $[t_0, +\infty[$, and eventually obtain (8) almost everywhere on $[t_0, +\infty[$. On every step, the length $s^\star > 0$ does not decrease because $V(X(t))$ does not increase and $s^\star > 0$ was defined by (10). Hence the Zeno effect is not possible. Finally, we note that (II) implies (III) (function $\beta \in \mathcal{KL}$ from (4) from Definition 2 can be constructed by using our function $\gamma(\cdot) < \mathrm{id}$ and by following standard arguments), and this completes our proof of Theorem 1.

VI. Applications

Motivated by many papers devoted to backstepping designs for time-delay control systems in pure-feedback form, see, for instance, [47], [46], let us consider the following problem.

As in related paper [13], we consider a countably infinite network of switched control systems in the following form

$$
\begin{cases}
\dot{x}_{i,k}(t) = x_{i,k+1}(t) + \Delta_{i,1}(x_{i,k}(t), \mathcal{X}_{i,k}(t), x_{i,k}(t-\theta), \\
\mathcal{X}_{i,k}(t-\theta)), \qquad\qquad k = 1, \ldots, n-1, \\
\dot{x}_{i,n}(t) = u_i + \Delta_n(X_{i,n}(t), \mathcal{X}_{i,n}(t), X_{i,n}(t-\theta), \mathcal{X}_{i,n}(t-\theta)) \\
\qquad\qquad i \in \mathbb{N}
\end{cases}
\tag{20}
$$

with controls $u_i \in \mathbb{R}^1$, $i \in \mathbb{N}$, with states $X_{i,n} := [x_{i,1}, ..., x_{i,n}]^\top \in \mathbb{R}^n$ of each ith subsystem and with the entire state vector $X = \{X_{i,n}\}_{i\in\mathbb{N}} \in \ell_\infty$, where vectors $X_{i,k}$ $\mathcal{X}_{i,k}$, are defined by

$$
X_{i,k} := [x_{i,1}, ..., x_{i,k}]^\top, \quad \mathcal{X}_{i,k} = \{X_{\varkappa,k}\}_{\varkappa\in J(i)}, \quad k = 1, ..., n \tag{21}
$$

for all $i \in \mathbb{N}$, and where each $J(i) \subset \mathbb{N}$ and denotes the set of the "neighbors" affecting i-th agent (node) of (20), and, as above, we assume that $i \notin J(i)$.

Assume that (20) satisfies the following conditions:

(C1) Every $J(i) \subset \mathbb{N}$ is *finite* for each $i \in \mathbb{N}$ and $N := \sup_{i\in\mathbb{N}} |J(i)| < +\infty$;

(C2) All $\Delta_{i,k}(\cdot)$ are functions of class C^1, $k = \overline{1,n}$, $i \in \mathbb{N}$, and $\Delta_{i,k}(0,0) = 0$ for all $k = \overline{1,n}$;

(C3) All $\Delta_{i,k}(\cdot,\cdot)$ and their partial derivatives are uniformly locally bounded, i.e., for each $R>0$ we have:

$$
\sup_{i\in\mathbb{N}} \max_{1\leq k\leq n} \max_{\substack{|X_{i,k}|_\infty \leq R, \\ |\chi_{i,k}|_\infty \leq R}} |\Delta_{i,k}(X_{i,k}, \chi_{i,k})| < \infty; \tag{22}
$$

$$
\sup_{i\in\mathbb{N},\, 1\leq j\leq k\leq n} \max_{|X_{i,k}|_\infty \leq R,\, |\chi_{i,k}|_\infty \leq R} \Big\{ \max_{\varkappa\in J(i)\cup\{i\}} \Big| \frac{\partial \Delta_{i,k}(X_{i,k}, \chi_{i,k})}{\partial x_{\varkappa,j}} \Big| \Big\} < \infty. \tag{23}
$$

Our corollary of Theorem 1 is as follows.

Theorem 2: Suppose that system (20) satisfies (C1)-(C3). Then, there is a *decentralized* feedback $u_i = u_i(X_{i,n})$ of class C^1 with $u_i(0) = 0$, $i \in \mathbb{N}$ which renders the closed-loop system (20) with this feedback $u_i = u_i(X_{i,n})$, $i \in \mathbb{N}$, ℓ_∞-GAS in the sense of Definition 2.

The proof of Theorem 2 is obtained by the following backstpping argument.

The auxiliary statement, which corresponds to the *Base Case* of the corresponding backstepping design for (20), deals with the following auxiliary infinite network of one-dimensional systems *of ODE*

$$
\dot{x}_{i,1} = x_{i,2} + \Delta_{i,1}(x_{i,1}, \mathcal{X}_{i,1}, \xi_{i,1}, \Xi_{i,1}), \quad i \in \mathbb{N} \tag{24}
$$

with *virtual controls* $x_{i,2} \in \mathbb{R}^1$, $i \in \mathbb{N}$, with states $x_{i,1} \in \mathbb{R}^1$ of each ith subsystem, with the entire state vector $\{x_{i,1}\}_{i\in\mathbb{N}} \in \ell_\infty$, and with *virtual external disturbance vectors* $(\mathcal{X}_{i,1}, \xi_{i,1}, \Xi_{i,1})$ in $\mathbb{R}^{|J(i)|} \times \mathbb{R}^1 \times \mathbb{R}^{|J(i)|}$. Here $\xi_{i,1} \in \mathbb{R}^1$, and $\Xi_{i,1} := \{\xi_{j,1}\}_{j\in J(i)}$.

Thus, (24) is obtained from (20) by taking each first equation with $x_{i,2} \in \mathbb{R}^1$ treated as a virtual control and by replacing $(\mathcal{X}_{i,1}(t), x_{i,1}(t-\theta), \mathcal{X}_{i,1}(t-\theta))$ with $(\mathcal{X}_{i,1}, \xi_{i,1}, \Xi_{i,1})$. In other words, motivated by [45], we treat each $(\mathcal{X}_{i,1}(t), x_{i,1}(t-\theta), \mathcal{X}_{i,1}(t-\theta))$ in (20) as the *virtual disturbance input* of each $x_{i,1}$-subsystem of (20) in the Base Case of our backstepping gain assignment. We also define the following "virtual Lyapunov functions"

$$
W_i(x_{i,1}) := \frac{x_{i,1}^2}{2}, \quad U_i(\xi_{i,1}) := \frac{\xi_{i,1}^2}{2}, \qquad i \in \mathbb{N}. \tag{25}
$$

Then, for auxiliary infinite network (24), the following lemma holds.

Lemma 2: Suppose that system (20) satisfies (C1)-(C3). Then, for any $\gamma \in]0,1[$, there is a *decentralized* feedback $x_{i,2} = \alpha_{i,1}(x_{i,1})$ of class C^∞ with $\alpha_{i,1}(0) = 0$, $i \in \mathbb{N}$ such that the following properties hold:

(P1) All $\alpha_{i,1}(x_{i,1})$ and their partial derivatives are uniformly locally bounded, i.e., for every $R > 0$ they are bounded on the set $|x_{i,1}| \leq R$ uniformly w.r.t. $i \in \mathbb{N}$.

(P2) Each ith subsystem of (24) with its virtual feedback $x_{i,2} = \alpha_{i,1}(x_{i,1})$ satisfies the following ISS Lyapunov inequality

$$
\begin{aligned}
& W_i(x_{i,1}) \geq \gamma \max\Big\{ \max_{j\in J(i)} W_i(x_{i,1}), \max_{j\in I(i)} U_i(\xi_{i,1}) \Big\} \Rightarrow \\
& \frac{dW_i(x_{i,1}, \mathcal{X}_{i,1}, \xi_{i,k}, \Xi_{i,1})}{dt} \Big|_{\substack{(24), \\ x_{i,2} = \alpha_{i,1}(x_{i,1})}} \leq -W_i(x_{i,1}).
\end{aligned}
\tag{26}
$$

Proof of Lemma 2 is a special case (corresponding to $p_1 = 1$) of Step 2 (Base Case) Proof of Theorem 2 from [13], see Eqs. (43)-(50) with $p_1 = 1$.

Let us emphasize that, if $n = 1$, then Theorem 2 is already proved: properties (P1)-(P2) of Lemma 2 imply that the network of interconnected $x_{i,1}$-subsystems of (20) with the feedbacks $x_{i,2} = \alpha_{i,1}(x_{i,1})$ satisfies (A1)-(A3) and conditions (i)-(iii) of our main Theorem 1.

For our *Inductive Step,* we take the standard backstepping transformation $z_i = x_{i,1}$, $w_i = x_{i,2} - \alpha_{i,1}(x_{i,1})$, and $\eta_i = \xi_{i,2} - \alpha_{i,1}(\xi_{i,1})$, where $\xi_{i,2}$ appears after formal replacement

$x_{i,2}(t-\theta) \mapsto \xi_{i,2}$ (similarly to the above formal replacement $x_{i,1}(t-\theta) \mapsto \xi_{i,1}$). Then we denote $\mathcal{Z}_i := \{z_j\}_{j\in J(i)}$, $\mathcal{W}_i := \{w_j\}_{j\in J(i)}$, $\mathcal{Y}_i := \{\eta_j\}_{j\in J(i)}$, and replace each $(x_{i,1}, x_{i,2})$-subsystem of (20) with the corresponding auxiliary infinite network of two-dimensional systems *of ODE*

$$\begin{aligned} \dot{z}_i &= w_i + \alpha_{i,1}(z_i) + \Delta_{i,1}(z_i, \mathcal{Z}_i, \xi_{i,1}, \Xi_{i,1}), \\ \dot{w}_i &= x_{i,3} + \hat{\Delta}_i(z_i, w_i, \mathcal{Z}_i, \mathcal{W}_i, \xi_{i,1}, \eta_i, \Xi_{i,1}, \mathcal{Y}_i); \\ & \qquad i \in \mathbb{N} \end{aligned} \tag{27}$$

with *virtual controls* $x_{i,3} \in \mathbb{R}^1$, $i \in \mathbb{N}$, with states $[z_i, w_i] \in \mathbb{R}^2$ of each ith subsystem, with the entire state vectors $\{[z_i, w_i]\}_{i\in\mathbb{N}} \in \ell_\infty$, and with *virtual external disturbance vector* $(w_i, \mathcal{Z}_i, \mathcal{W}_i, \xi_{i,1}, \eta_i, \Xi_{i,1}, \mathcal{Y}_i)$ (here, $\hat{\Delta}_i$ is defined by $\Delta_{i,2}$ and by our backstepping transformation $z_i = x_{i,1}$, $w_i = x_{i,2} - \alpha_{i,1}(x_{i,1})$, $\xi_{i,1} = \xi_{i,1}$ and $\eta_i = \xi_{i,2} - \alpha_{i,1}(\xi_{i,1})$,).

Then we apply Theorem 3 from [36] (in the special case, when there is no swtiching signals $\sigma(\cdot)$ and there is no uncertainties θ) to each ith $[z_i, w_i]$-subsystem of our system (27) (note that (C1)-(C3) from [36] and (14) from [36] are again satisfied for each ith $[z_i, w_i]$-subsystem of our system (27)). Then we eventually obtain the existence of a decentralized feedback $x_{i,3} = \alpha_{i,2}(z_i, w_i)$ which (after the inverse transformation $[z_i, w_i] \mapsto [x_{i,1}, x_{i,2}]$) provides that the network of interconnected $[x_{i,1}, x_{i,2}]$-subsystems of (20) with this feedback satisfies (A1)-(A3) and conditions (i)-(iii) of our main Theorem 1 (the uniform boundness for the obtained feedback follows from the construction in the proof of Theorem 3 from [36] and from our current condition (C3) with (22),(23)). If $n = 2$, then Theorem 2 is proved; otherwise, we follow the standard backstepping transformation of states for $n = 2, 3, \ldots$ and eventually prove Theorem 2 for any n.

VII. Conclusion

We proved a new small gain theorem for infinite networks of time-delay systems in terms of ISS Lyapunov-Razumikhin functions, and we demonstrated how to apply this result to decentralized stabilization of infinite networks of hierarchically interconnected time-delay nonlinear systems in strict-feedback form.

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