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# An Adaptive Nonlinear $\mathcal{H}_\infty$ Control with Exact Parameter Estimation for Mechanical Systems*

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***Abstract*— While exact parameter estimation is desired in numerous applications, many adaptive controllers require the fulfillment of the persistency of excitation (PE) condition to achieve that objective. Relaxing the PE condition poses a challenging theoretical problem and many research works have been devoted to addressing this issue. In this paper, we propose a novel adaptive robust nonlinear $\mathcal{H}_\infty$ optimal controller for trajectory tracking of mechanical systems subjected to unknown parameters. The proposed method includes an extra term, based on the dynamic regressor extension and mixing (DREM) approach, into the adaptive law which enables exact parameter estimation even without fulfilling the PE condition. The effectiveness of the proposed adaptive controller is corroborated with numerical experiments involving a CRS-A465 robot manipulator and comparison analyses with a classic adaptive nonlinear $\mathcal{H}_\infty$ controller.**

## I. INTRODUCTION

When considering idealized situations, it is possible to establish a precise mathematical representation of a mechanical system using classical methods such as the Euler-Lagrange (EL) equations of motion. Nevertheless, practical applications often introduce uncertainties in the model parameters, posing challenges in achieving trajectory tracking when implementing control strategies. To handle these challenges, two of the most common control strategies employed are the robust control approach, which focuses on maintaining stability and performance in the presence of uncertainties, and the adaptive control approach which aims to modify the controller based on evolving system dynamics.

Regarding robust control, the $\mathcal{H}_2$ [1] and $\mathcal{H}_\infty$ [2] control strategies have gathered considerable attention. These methods were originally formulated within the frequency domain to handle single-input-single-output (SISO) systems that are represented by transfer functions [3]. In this domain, the $\mathcal{H}_2$ controller seeks to minimize the energy of the system impulse response from the disturbance to the output [4], while the $\mathcal{H}_\infty$ controller minimizes the maximum gain given by the closed-loop system to a disturbance signal [5]. In the context of multiple-input-multiple-output (MIMO) systems in state-space representation, the $\mathcal{H}_\infty$ control was first introduced in [6] and has later paved the way for the development of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers based on Linear Matrix Inequalities (LMIs) [7], [8]. As for nonlinear systems, the $\mathcal{H}_\infty$ control strategy has been first introduced in [9], with the problem being formulated in the $\mathcal{L}_2$ space using dissipative systems theory [10]. Concerning the development of nonlinear $\mathcal{H}_2$ controllers for mechanical systems, [1] stands out as a pioneering work. This work has formulated the nonlinear $\mathcal{H}_2$ control problem for fully actuated mechanical systems represented by EL equations via game theory, offering an analytical solution to the resulting Hamilton-Jacobi (HJ) equation. Building on this foundation, subsequent studies, like [11], have extended the nonlinear $\mathcal{H}_\infty$ control strategy to the same class of systems. Additionally, in [12], the development of the nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control strategy for mechanical systems with actuation redundancy has been achieved.

Regarding adaptive controllers, the EL equations present some structural characteristics, notably, the capacity of representing the model as a product of a matrix, referred to as the “regressor”, and a vector of constant parameters, which can be helpful when designing these controllers. Leveraging this structural feature and based on [1], in [13], an adaptive nonlinear $\mathcal{H}_\infty$ controller has been formulated for fully actuated mechanical systems. This adaptive controller aims to estimate uncertain parameters and external disturbances while performing trajectory tracking with guaranteed stability. Nonetheless, it fails to achieve parameter estimation convergence due to the restrictive requirement of satisfying the persistency of excitation (PE) condition [14]. To ensure parameter convergence, parameter drift must be prevented by guaranteeing that the regressor vector is persistently excited. In [15], it has been demonstrated that the necessary and sufficient condition to guarantee the convergence of the parameters is that the regressor matrix satisfies the PE condition along the reference trajectories, enabling a priori verification.

Recently, there has been a noteworthy effort on relaxing the requirement for PE, exemplified by the consideration of interval excitation (IE) [16]. In this study, an approach known as Composite Learning Robot Control (CLRC) has been designed to achieve fast and accurate parameter estimation under the less stringent IE condi-

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tion. Within this context, one particularly technique has garnered attention. The Dynamic Regressor Extension and Mixing (DREM) method unfolds in two stages [17]. First, it involves the generation of new regression forms through the application of dynamic operators to the original regression dataset. Subsequently, these newly generated data forms are combined to construct the desired final regression structure, enabling the application of the conventional parameter estimation techniques. In [18], a composite scheme, combining he standard gradient adaptive law with a DREM-based additional term, has been proposed and applied to a robotic manipulator. It has been shown that the new parameter convergence condition is more relaxed than the PE one, but stronger than the initial and interval excitation assumptions, and can also be partially verified online.

Accordingly, this paper proposes a novel adaptive nonlinear $\mathcal{H}_\infty$ controller with exact parameter estimation. The proposed approach introduces an additional term based on the DREM method to estimate unknown parameters. The controller is capable of ensuring exponential trajectory tracking and, in cases where PE conditions are not ensured, the achievement of a less restrictive condition leads to exact parameter estimation.

## II. Preliminaries

Consider the problem of estimation of constant parameters of the p-dimensional linear regression

$$y(t) = \boldsymbol{m}'(t)\boldsymbol{\theta}, \tag{1}$$

with $y : \mathbb{R}_+ \to \mathbb{R}$ and $\boldsymbol{m} : \mathbb{R}_+ \to \mathbb{R}^p$ being known, bounded functions of time, and $\theta \in \mathbb{R}^p$ being the vector of unknown parameters. The classic gradient estimator $\dot{\hat{\boldsymbol{\theta}}}(t) = -\boldsymbol{\Psi}\boldsymbol{m}(t)\left(\boldsymbol{m}'(t)\hat{\boldsymbol{\theta}}(t) - y(t)\right)$, with a given constant positive definite adaptation gain matrix $\boldsymbol{\Psi} \in \mathbb{R}^{p\times p}$, yields the parameter error equation

$$\dot{\tilde{\boldsymbol{\theta}}}(t) = -\boldsymbol{\Psi}\boldsymbol{m}(t)\boldsymbol{m}'(t)\tilde{\boldsymbol{\theta}}(t), \tag{2}$$

where $\tilde{\boldsymbol{\theta}} \triangleq \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is the vector of parameter estimation error. It is well-known that the zero equilibrium point of the time-varying linear system (2) is uniformly globally exponentially stable iff the regressor vector $\boldsymbol{m}(t)$ satisfies the PE condition [19]

$$\int_t^{t+T} \boldsymbol{m}(\tau)\boldsymbol{m}(\tau)'d\tau \geq \delta \boldsymbol{I}, \tag{3}$$

for $T$, $\delta > 0$ and for all $t \geq 0$. Nevertheless, if $\boldsymbol{m}(t)$ does not satisfy (3), as is common in many practical scenarios, very little can be inferred about the convergence of the parameter estimation error.

Due to the difficulty of meeting the PE condition, the DREM method becomes an alternative [17]. The DREM generates $p$ new, one-dimensional, regression models to independently estimate each of the parameters under conditions on the regressor $\boldsymbol{m}(t)$ that differ from the PE condition (3). To employ the DREM, initially, $p-1$ linear stable $\mathcal{L}_\infty$ operators $H_i : \mathcal{L}_\infty \to \mathcal{L}_\infty$, for $i \in \{1, 2, p-1\}$, are introduced. These operators can be, for example, exponentially stable linear filters of the form $\dot{y}_{f_i}(t) = -b_i y_{f_i}(t) + a_i y(t)$, where $a_i \neq 0$ and $b_i > 0$. Hence, $p-1$ filtered outputs can be obtained

$$y_{f_i}(t) = \boldsymbol{m}'_{f_i}\boldsymbol{\theta}. \tag{4}$$

Pilling up the original linear regression (1) with the $p-1$ filtered regressors (4), we can construct the augmented regressor system

$$\boldsymbol{y}_f(t) = \boldsymbol{Y}_f(t)\boldsymbol{\theta}, \tag{5}$$

where $\boldsymbol{y}_f = [y \;\; y_{f_1} \;\; \ldots \;\; y_{f_{p-1}}]'$, and $\boldsymbol{Y}_f = [\boldsymbol{m}' \;\; \boldsymbol{m}'_{f_1} \;\; \ldots \;\; \boldsymbol{m}'_{f_{p-1}}]'$. Using the fact that $\text{adj}(\boldsymbol{Y}_f)\boldsymbol{Y}_f = \boldsymbol{Y}_f\text{adj}(\boldsymbol{Y}_f) = \phi\boldsymbol{I}$, in which $\phi \triangleq \det(\boldsymbol{Y}_f)$ and $\text{adj}(\cdot)$ is the adjugate matrix of $(\cdot)$, we multiply both sides of (5) by the adjugate matrix $\text{adj}(\boldsymbol{Y}_f)$ of $\boldsymbol{Y}_f \in \mathbb{R}^{p\times p}$ to obtain $p$ decoupled equations of the form $y_{e_j} = \phi\theta_j, \;\; \forall j \in \{1, 2, ..., p\}$, with $\boldsymbol{y}_e = \text{adj}(\boldsymbol{Y}_f)\boldsymbol{y}_f$, where $(\cdot)_j$is the $j$-th element of the vector $(\cdot)$. Thus, $p$-decoupled estimators of the form $\dot{\hat{\theta}}_j = -\gamma_j\phi\left(\phi\hat{\theta}_j - y_e\right)$ are obtained with $\gamma_j > 0$, which leads to the following parameter error dynamics:

$$\dot{\tilde{\theta}}_j = -\gamma_j\phi^2\tilde{\theta}_j, \tag{6}$$

where $\tilde{\theta}_j$ is the $j$-th element of the error vector of unknown parameters. Then, from (6), one can infer that

$$\phi \notin \mathcal{L}_2 \implies \lim_{t\to\infty}\tilde{\theta}_j = 0, \;\; \forall j \in \{1, 2, ..., p\}. \tag{7}$$

As demonstrated in [20], it is always possible to find operators $H_i$ such that if the PE condition (3) is satisfied, then so is $\phi \notin \mathcal{L}_2$, leading to the validity of equation (7). It can be concluded that the DREM condition, (7), is easier to be satisfied than the PE condition, (3).

## III. Mathematical Modeling

The mathematical model of a $n$-degrees of freedom mechanical system can be expressed by the EL equation in the canonical form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}(\dot{\boldsymbol{q}}, \boldsymbol{q})\dot{\boldsymbol{q}}(t) + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_l, \tag{8}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$ is the vector of generalized coordinates, $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $\boldsymbol{C}(\dot{\boldsymbol{q}}, \boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the coriolis and centripetal force matrix, $\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^n$ is the vector of gravitational forces, $\boldsymbol{\tau} \in \mathbb{R}^n$ is the vector of generalized control inputs, and $\boldsymbol{\tau}_l \in \mathbb{R}^n$ represents the external disturbances. Equation (8) possesses the following properties [21].

*Property 1:* The inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is symmetric and positive definite for every $\boldsymbol{q}(t) \in \mathbb{R}^n$.

*Property 2:* There exist constants $\lambda_m$, $\lambda_M \in \mathbb{R}$, such that the inequality $\lambda_m||\tilde{\boldsymbol{x}}||^2 \leq \tilde{\boldsymbol{x}}'\boldsymbol{M}\tilde{\boldsymbol{x}} \leq \lambda_M||\tilde{\boldsymbol{x}}||^2$ holds $\forall \boldsymbol{q}(t) \in \mathbb{R}^n, \tilde{\boldsymbol{x}} \in \mathbb{R}^n$, and $0 < \lambda_m \leq \lambda_M < \infty$.

*Property 3:* Matrix $\boldsymbol{N} = \dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}$ is skew-symmetric, provided that the coriolis and centripetal forces matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is derived using Christoffel symbols of the first kind.

*Property 4:* The EL equation (8) is linear parametrizable for a suitable selection of the vector of parameters $\boldsymbol{\theta} \in \mathbb{R}^p$, allowing to write

$$\boldsymbol{M}(\boldsymbol{q}, \boldsymbol{\theta})\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\theta})\dot{\boldsymbol{q}}(t) + \boldsymbol{G}(\boldsymbol{q}, \boldsymbol{\theta}) = \mathcal{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\theta}, \tag{9}$$

where $\mathcal{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is called the regressor matrix.

For control design purposes, we define the state vector $\tilde{\boldsymbol{x}} = \begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}' & \tilde{\boldsymbol{q}}'\end{bmatrix}'$, where $\tilde{\boldsymbol{q}} \triangleq \boldsymbol{q}(t) - \boldsymbol{q}_r(t)$, with $\boldsymbol{q}_r(t) \in \mathcal{C}^2$ being the desired reference, and consider the state-space transformation [1]

$$\boldsymbol{z} = \begin{bmatrix}\boldsymbol{z}_1\\ \boldsymbol{z}_2\end{bmatrix} = \boldsymbol{T}_0\tilde{\boldsymbol{x}} = \begin{bmatrix}\boldsymbol{T}_1\\ \boldsymbol{T}_2\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ \tilde{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{T}_{11} & \boldsymbol{T}_{12}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{q}}}\\ \tilde{\boldsymbol{q}}\end{bmatrix}, \tag{10}$$

where $\boldsymbol{T}_{11}, \boldsymbol{T}_{12} \in \mathbb{R}^{n\times n}$ are constant matrices to be determined. In what follows, $\boldsymbol{T}_{11}$ is assumed to be a diagonal matrix such that $\boldsymbol{T}_{11} = t_{11}\boldsymbol{I}$ for some $t_{11} > 0$. Taking into account *Property 4*, one can write

$$\dot{\boldsymbol{z}} = \begin{bmatrix}-\boldsymbol{M}^{-1}\boldsymbol{C} & \boldsymbol{0}\\ \boldsymbol{T}_{11}^{-1} & -\boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\end{bmatrix}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{T}_{11}\boldsymbol{M}^{-1}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right), \tag{11}$$

with $\boldsymbol{B} = [\boldsymbol{I}\ \ \boldsymbol{0}]'$ and $\boldsymbol{Y}\boldsymbol{\theta} = \boldsymbol{M}(\boldsymbol{q})\left(\ddot{\boldsymbol{q}}_r(t) - \boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\dot{\tilde{\boldsymbol{q}}}\right) + \boldsymbol{C}(\dot{\boldsymbol{q}}, \boldsymbol{q})\left(\dot{\boldsymbol{q}}_r(t) - \boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\tilde{\boldsymbol{q}}\right) + \boldsymbol{G}(\boldsymbol{q})$. Applying the state space transformation (10) into (11) results in the following dynamic equation:

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{f}(\tilde{\boldsymbol{x}}, t)\tilde{\boldsymbol{x}} + \boldsymbol{g}(\tilde{\boldsymbol{x}}, t)\boldsymbol{u} + \boldsymbol{g}(\tilde{\boldsymbol{x}}, t)\boldsymbol{d}, \tag{12}$$

where $\boldsymbol{f}(\tilde{\boldsymbol{x}}, t) \triangleq \boldsymbol{T}_0^{-1}\begin{bmatrix}-\boldsymbol{M}^{-1}\boldsymbol{C} & \boldsymbol{0}\\ \boldsymbol{T}_{11}^{-1} & -\boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\end{bmatrix}\boldsymbol{T}_0$, $\boldsymbol{g}(\tilde{\boldsymbol{x}}, t) \triangleq \boldsymbol{T}_0^{-1}\boldsymbol{B}\boldsymbol{M}^{-1}$, $\boldsymbol{d} \triangleq \boldsymbol{T}_{11}\boldsymbol{\tau}_l$, $\boldsymbol{u} \triangleq \boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau}\right)$.

## IV. Adaptive Nonlinear $\mathcal{H}_\infty$ Control Design

In this section, we develop an adaptive robust nonlinear $\mathcal{H}_\infty$ optimal control law for the system (12), with the objective of achieving trajectory tracking while exact estimating the vector of parameters $\boldsymbol{\theta}$ and maintaining robustness against external disturbances. This control approach is derived by combining the gradient adaptive algorithm and the DREM method.

Before proposing the adaptive nonlinear $\mathcal{H}_\infty$ controller, we recall the following result from [13].

*Corollary 1 (Adapted from [13]):* Consider the nonlinear system described by (12). Let $\boldsymbol{R} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Q} \in \mathbb{R}^{2n\times 2n}$ be given symmetric positive definite matrices. Suppose $\rho^2\boldsymbol{I} > \boldsymbol{R}$ and both $\boldsymbol{T}_0$ and $\boldsymbol{K}$, with $\boldsymbol{K} = \boldsymbol{K}' > 0$ and $\boldsymbol{K} \in \mathbb{R}^{n\times n}$, satisfy the algebraic Riccati equation

$$\begin{bmatrix}\boldsymbol{0} & \boldsymbol{K}\\ \boldsymbol{K} & \boldsymbol{0}\end{bmatrix} + \boldsymbol{Q} - \boldsymbol{T}_0'\boldsymbol{B}\left(\boldsymbol{R}^{-1} - \frac{1}{\rho^2}\boldsymbol{I}\right)\boldsymbol{B}'\boldsymbol{T}_0 = 0. \tag{13}$$

Then, the adaptive robust nonlinear $\mathcal{H}_\infty$ control law

$$\dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{K}_\theta^{-1}\boldsymbol{Y}'\boldsymbol{T}_{11}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}}, \tag{14}$$

$$\boldsymbol{\tau} = \boldsymbol{Y}\hat{\boldsymbol{\theta}} + \boldsymbol{T}_{11}^{-1}\boldsymbol{u}^*, \tag{15}$$

$$\boldsymbol{\tau}_l = \boldsymbol{T}_{11}^{-1}\boldsymbol{d}^*, \tag{16}$$

with $\boldsymbol{u}^* = -\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}}$, $\boldsymbol{d}^* = \frac{1}{\rho^2}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}}$, and the symmetric and positive definite matrix $\boldsymbol{K}_\theta \in \mathbb{R}^{p\times p}$, is a solution to the nonlinear $\mathcal{H}_\infty$ optimal control problem

$$\min_{\boldsymbol{u}\in\mathcal{L}_2}\max_{\boldsymbol{d}\in\mathcal{L}_2}\int_{t_0}^{t}\left(||\tilde{\boldsymbol{x}}||_{\boldsymbol{Q}}^2 + ||\boldsymbol{u}||_{\boldsymbol{R}}^2 - \rho^2||\boldsymbol{d}||^2\right)d\tau \le 0, \tag{17}$$

in which $\rho \in \mathbb{R}$ is the provided $\mathcal{H}_\infty$ attenuation level.

Aiming at exact parameter estimation guarantees, next, we redesign the adaptive robust nonlinear $\mathcal{H}_\infty$ optimal control law (14)-(16) using the DREM approach.

First, we construct the augmented regressor matrix $\boldsymbol{Y}_f$ to obtain a relationship similar to (5). To do so, we define the filtered regressor $\boldsymbol{Y}_{\phi_1} \in \mathbb{R}^{n\times p}$,

$$\dot{\boldsymbol{Y}}_{\phi_1} = -\lambda_{\phi_1}\boldsymbol{Y}_{\phi_1} + \lambda_{\phi_1}\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}),\quad \boldsymbol{Y}_{\phi_1}(t_0) = \boldsymbol{0}, \tag{18}$$

with $\lambda_{\phi_1} > 0$, and the filtered input $\boldsymbol{\tau}_{\phi_1} \in \mathbb{R}^n$,

$$\dot{\boldsymbol{\tau}}_{\phi_1} = -\lambda_{\phi_1}\boldsymbol{\tau}_{\phi_1} + \lambda_{\phi_1}\boldsymbol{\tau},\quad \boldsymbol{\tau}_{\phi_1}(t_0) = \boldsymbol{0}. \tag{19}$$

*Remark 1:* For implementation purposes, integration by parts can be applied to (18) to avoid the need for knowledge of $\ddot{\boldsymbol{q}}(t)$ [18].

According to the Swapping Lemma [14], (18) and (19) yields $\boldsymbol{\tau}_{\phi_1} = \boldsymbol{Y}_{\phi_1}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))\boldsymbol{\theta}$. Next, we assume that $n < p$ to construct a square augmented regressor matrix $\boldsymbol{Y}_f$, by following the steps:

(1) Define the first $n$ rows of $\boldsymbol{Y}_f$ as the first $n$ rows of $\boldsymbol{Y}_{\phi_1}$ such that $p - n$ extra rows are required.

(2) Define $\tilde{n}$ as the integer total number of matrices $\boldsymbol{Y}_{\phi_1}, \ldots, \boldsymbol{Y}_{\phi_{\tilde{n}}}$ required to form the square matrix $\boldsymbol{Y}_f$, so $\tilde{n} - 1$ extra matrices are necessary.

(3) If

$$\tilde{n} = \frac{p}{n}, \tag{20}$$

is an integer, then $\tilde{n}-1$ matrices $\boldsymbol{Y}_{\phi_2}, \ldots, \boldsymbol{Y}_{\phi_{\tilde{n}}} \in \mathbb{R}^{n\times p}$ are generated by applying $\tilde{n}-1$ exponentially stable linear filters $H_j : \mathcal{L}_\infty \to \mathcal{L}_\infty$ for $j = 2, \ldots, \tilde{n}$ to get

$$\boldsymbol{Y}_f = \begin{bmatrix}\boldsymbol{Y}_{\phi_1}' & \cdots & \boldsymbol{Y}_{\phi_{\tilde{n}}}'\end{bmatrix}',\quad \in \mathbb{R}^{p\times p} \tag{21}$$

where $\dot{\boldsymbol{Y}}_{\phi_j} = -b_j\boldsymbol{Y}_{\phi_j} + a_j\boldsymbol{Y}_{\phi_1}$, $\boldsymbol{Y}_{\phi_j}(t_0) = \boldsymbol{0}$, with $a_j \neq 0$ and $b_j > 0$ for $j = 2, \ldots, \tilde{n}$.

(4) In the case where (20) is not an integer, the DIV operator can be used to obtain an integer (e.g. $\frac{9}{4} = 2.25$, but $9\mathrm{DIV}4 = 2$). This implies $\tilde{n} = (p\mathrm{DIV}n) + 1$. In this case, the last term of matrix $\boldsymbol{Y}_{\phi_{\tilde{n}}}$ in (21) is of different dimension. The dimension of this matrix can be computed using the MOD operator which gives the remainder of an integer division (e.g. $9\mathrm{MOD}4 = 1$). Thus implying $\boldsymbol{Y}_{\phi_{\tilde{n}}} \in \mathbb{R}^{(p\mathrm{MOD}n)\times p}$.

Furthermore, similar reasoning is applied to the outputs as we apply the same exponentially stable linear filters to the filtered output such that

$$\boldsymbol{\tau}_f = \begin{bmatrix}\boldsymbol{\tau}_{\phi_1}' & \ldots & \boldsymbol{\tau}_{\phi_{\tilde{n}}}'\end{bmatrix}'. \tag{22}$$

Thus, we can write

$$\boldsymbol{\tau}_f = \boldsymbol{Y}_f\boldsymbol{\theta}. \tag{23}$$

Multiplying (23) on the left-hand by $\mathrm{adj}(\boldsymbol{Y}_f)$ yields

$$\boldsymbol{\tau}_e = \mathrm{adj}(\boldsymbol{Y}_f)\boldsymbol{Y}_f\boldsymbol{\theta} = \phi\boldsymbol{\theta}, \tag{24}$$

where $\phi = \det(\boldsymbol{Y}_f)$ and $\boldsymbol{\tau}_e = \mathrm{adj}(\boldsymbol{Y}_f)\boldsymbol{\tau}_f$. It is worth highlighting that (24) is designed in a way that yields $p$ decoupled equations.

Now consider the following modification to the adaptive law (14):

$$\dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{K}_\theta^{-1}\left(\boldsymbol{Y}'\boldsymbol{T}_{11}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}} + \boldsymbol{f}_\theta\right). \tag{25}$$

Note that (25) is a composite estimation law that depends on both the system state, $\tilde{\boldsymbol{x}}$, and the DREM component vector, $\boldsymbol{f}_\theta$, whose elements are given by

$$f_{\theta_i} = \gamma_i\phi\left(\phi\hat{\theta}_i - \tau_{e_i}\right), \tag{26}$$

where $\gamma_i > 0$ is the $i$-th component of the positive definite diagonal matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{p\times p}$, with $\boldsymbol{\Gamma} \triangleq \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_p)$.

Given that the unknown parameters are constant, the closed-loop dynamics of the parameter error can be described by

$$\dot{\tilde{\boldsymbol{\theta}}} = -\boldsymbol{K}_\theta^{-1}\left(\boldsymbol{Y}'\boldsymbol{T}_{11}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}} + \bar{\boldsymbol{f}}_\theta\right), \tag{27}$$

with

$$\bar{\boldsymbol{f}}_\theta = \boldsymbol{\Gamma}\phi^2\tilde{\boldsymbol{\theta}}. \tag{28}$$

To derive (28), we have substituted (24) into (26). It is noteworthy that $\boldsymbol{f}_\theta = \bar{\boldsymbol{f}}_\theta$; however, $\boldsymbol{f}_\theta$ is implementable, whereas $\bar{\boldsymbol{f}}_\theta$ is not. The latter is used only for stability analysis purposes.

The main contribution of this work is summarized in the subsequently presented Theorem 1. To facilitate the proof of this theorem, the following factorizations are considered:

$$\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12}\\ * & \boldsymbol{Q}_{22}\end{bmatrix} = \begin{bmatrix}\boldsymbol{Q}_1'\boldsymbol{Q}_1 & \boldsymbol{Q}_{12}\\ * & \boldsymbol{Q}_2'\boldsymbol{Q}_2\end{bmatrix}, \tag{29}$$

and

$$\boldsymbol{R}_1'\boldsymbol{R}_1 = \left(\boldsymbol{R}^{-1} - \frac{1}{\rho^2}\boldsymbol{I}\right)^{-1}. \tag{30}$$

Then, taking into account (29) and (30), we compute

$$\boldsymbol{T}_{11} = \boldsymbol{R}_1'\boldsymbol{Q}_1, \qquad \boldsymbol{T}_{12} = \boldsymbol{R}_1'\boldsymbol{Q}_2, \tag{31}$$

and

$$\boldsymbol{K} = \frac{1}{2}\left(\boldsymbol{Q}_1'\boldsymbol{Q}_2 + \boldsymbol{Q}_2'\boldsymbol{Q}_1\right) - \frac{1}{2}\left(\boldsymbol{Q}_{12}' + \boldsymbol{Q}_{12}\right) > 0. \tag{32}$$

Considering the factorizations (29) and (30), one can show via direct calculation that (31) and (32) compose a solution of the Riccati equation (13).

*Theorem 1:* Let $\boldsymbol{R} = r^2\boldsymbol{I} > 0$ and

$$\boldsymbol{Q} = \begin{bmatrix} a_1^2\boldsymbol{I}_{n\times n} & \boldsymbol{0}\\ * & a_2^2\boldsymbol{I}_{n\times n}\end{bmatrix} > 0, \tag{33}$$

with given parameters $r, a \in \mathbb{R}_+$. Consider a desired disturbance attenuation level, $\rho$, such that $\rho^2\boldsymbol{I} > \boldsymbol{R}$. Then, the adaptive robust nonlinear $\mathcal{H}_\infty$ optimal control law

$$\dot{\hat{\boldsymbol{\theta}}} = -\frac{1}{a_1}\boldsymbol{Y}'\begin{bmatrix}a_1\boldsymbol{I} & a_2\boldsymbol{I}\end{bmatrix}\tilde{\boldsymbol{x}} - \frac{1}{t_{11}^2}\boldsymbol{f}_\theta, \tag{34}$$

$$\boldsymbol{\tau} = -\frac{1}{a_1 r^2}\begin{bmatrix}a_1\boldsymbol{I} & a_2\boldsymbol{I}\end{bmatrix}\tilde{\boldsymbol{x}} + \boldsymbol{Y}\hat{\boldsymbol{\theta}}, \tag{35}$$

$$\boldsymbol{\tau}_l = \frac{1}{a_1\rho^2}\begin{bmatrix}a_1\boldsymbol{I} & a_2\boldsymbol{I}\end{bmatrix}\tilde{\boldsymbol{x}}, \tag{36}$$

with $t_{11} = \frac{r\rho a_1}{\sqrt{\rho^2 - r^2}}$, is a solution to the nonlinear $\mathcal{H}_\infty$ optimal control problem (17). Moreover, since

$$\boldsymbol{\Phi} \triangleq \boldsymbol{Q} + \begin{bmatrix}a_1^2\boldsymbol{I} & a_1a_2\boldsymbol{I}\\ * & a_2^2\boldsymbol{I}\end{bmatrix} > 0, \tag{37}$$

and if

$$\phi^2 \geq \frac{1}{2}\left(\frac{\lambda_{min}(\boldsymbol{\Phi})}{\lambda_{min}(\boldsymbol{\Gamma})}\right), \tag{38}$$

with $\phi = \det(\boldsymbol{Y}_f)$, and

$$\kappa = \left(\frac{\lambda_{min}(\boldsymbol{\Phi})}{2\lambda_2}\right), \tag{39}$$

then, exponential stability is guaranteed for the closed-loop system (8) with the control law (34)-(35) for the worst-case disturbance (36), ensuring that both $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{\theta}}$ converges to zero with decay rate $\kappa$.

*Proof:* From $\boldsymbol{R} = r^2\boldsymbol{I}$, we have that (30) becomes

$$\boldsymbol{R}_1'\boldsymbol{R}_1 = \frac{r^2\rho^2}{\rho^2 - r^2}\boldsymbol{I} \implies \boldsymbol{R}_1 = \frac{r\rho}{\sqrt{\rho^2 - r^2}}\boldsymbol{I}. \tag{40}$$

From (31), (33) and (40), we have that

$$\boldsymbol{T}_{11} = t_{11}\boldsymbol{I} = \left(\frac{r\rho a_1}{\sqrt{\rho^2 - r^2}}\right)\boldsymbol{I}, \quad \boldsymbol{T}_{12} = \left(\frac{r\rho a_2}{\sqrt{\rho^2 - r^2}}\right)\boldsymbol{I}. \tag{41}$$

Also, from the definitions of $\boldsymbol{B}$ and $\boldsymbol{T}_0$, we obtain

$$\boldsymbol{B}'\boldsymbol{T}_0 = \frac{\rho r}{\sqrt{\rho^2 - r^2}}\begin{bmatrix}a_1\boldsymbol{I} & a_2\boldsymbol{I}\end{bmatrix}. \tag{42}$$

Replacing (41) and (42) into (15), (16) and (25), and choosing $\boldsymbol{K}_\theta = t_{11}^2\boldsymbol{I}$, render the adaptive robust nonlinear $\mathcal{H}_\infty$ control law (34)-(36). Next, for the sake of convenience, let's rewrite (12) in terms of $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_l$

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}} = {}& \boldsymbol{T}_0^{-1}\begin{bmatrix}-\boldsymbol{M}^{-1}\boldsymbol{C} & \boldsymbol{0}\\ \boldsymbol{T}_{11}^{-1} & -\boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\end{bmatrix}\boldsymbol{T}_0\tilde{\boldsymbol{x}}\\ &+ \boldsymbol{T}_0^{-1}\boldsymbol{B}\boldsymbol{M}^{-1}\boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right),\end{aligned} \tag{43}$$

and consider the candidate Lyapunov function

$$V(\mathcal{X}, t) = V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}, t) + V_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}}), \tag{44}$$

with the augmented state vector $\boldsymbol{\mathcal{X}} = [\tilde{\boldsymbol{x}}'\ \tilde{\boldsymbol{\theta}}']'$, such that

$$V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}, t) \triangleq \frac{1}{2}\tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\begin{bmatrix}\boldsymbol{M}(\boldsymbol{q}) & \boldsymbol{0}\\ * & \boldsymbol{K}\end{bmatrix}\boldsymbol{T}_0\tilde{\boldsymbol{x}}, \tag{45}$$

$$V_{\tilde{\boldsymbol{\theta}}}(\tilde{\boldsymbol{\theta}}) \triangleq \frac{1}{2}\tilde{\boldsymbol{\theta}}'\boldsymbol{K}_\theta\tilde{\boldsymbol{\theta}}, \tag{46}$$

which satisfies

$$\lambda_1||\boldsymbol{\mathcal{X}}||^2 \leq V(\boldsymbol{\mathcal{X}}, t) \leq \lambda_2||\boldsymbol{\mathcal{X}}||^2, \tag{47}$$

where $\lambda_1 = \frac{1}{2}\min\{\lambda_h, \lambda_{min}(\boldsymbol{K}_\theta)\}$, with $\lambda_h \triangleq \lambda_{min}(\boldsymbol{H})$ in which $\boldsymbol{H} \triangleq \boldsymbol{T}_0'\mathrm{blkdiag}(\boldsymbol{M}, \boldsymbol{K})\boldsymbol{T}_0$. Also, $\lambda_2 = \frac{1}{2}\max\{\lambda_H, \lambda_{max}(\boldsymbol{K}_\theta)\}$, $\lambda_H = \lambda_{max}(\boldsymbol{H})$, where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ stand for the smallest and highest eigenvalue of $(\cdot)$, respectively. Then, the time derivative of (44) is

$$\dot{V}(\mathcal{X}, t) = \frac{\partial V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}, t)}{\partial t} + \left(\frac{\partial V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}, t)}{\partial \tilde{\boldsymbol{x}}}\right)'\dot{\tilde{\boldsymbol{x}}} + \tilde{\boldsymbol{\theta}}'\boldsymbol{K}_\theta\dot{\tilde{\boldsymbol{\theta}}}. \tag{48}$$

Taking the partial derivative of (45) with respect to $\tilde{\boldsymbol{x}}$ and multiplying by (43) yields

$$\begin{aligned}\frac{\partial V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}, t)}{\partial \tilde{\boldsymbol{x}}}'\dot{\tilde{\boldsymbol{x}}} = {}& \tilde{\boldsymbol{x}}\boldsymbol{T}_0'\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix}\boldsymbol{T}_0\dot{\tilde{\boldsymbol{x}}} + \frac{1}{2}\sum_{i=1}^{2n}\tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\begin{bmatrix}\frac{\partial \boldsymbol{M}}{\partial \tilde{\boldsymbol{x}}_i}\dot{\tilde{\boldsymbol{x}}}_i & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{T}_0\tilde{\boldsymbol{x}}\\ = {}& \tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\begin{bmatrix}-\boldsymbol{C} & \boldsymbol{0}\\ \boldsymbol{K}\boldsymbol{T}_{11}^{-1} & -\boldsymbol{K}\boldsymbol{T}_{11}^{-1}\boldsymbol{T}_{12}\end{bmatrix}\boldsymbol{T}_0\tilde{\boldsymbol{x}}\\ &+ \frac{1}{2}\sum_{i=1}^{2n}\tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\begin{bmatrix}\frac{\partial \boldsymbol{M}}{\partial \tilde{\boldsymbol{x}}_i}\dot{\tilde{\boldsymbol{x}}}_i & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{T}_0\tilde{\boldsymbol{x}}\\ &+ \tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\boldsymbol{B}\boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right).\end{aligned} \tag{49}$$

Using *Property 3* renders

$$\begin{aligned}\frac{\partial V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}, t)}{\partial \tilde{\boldsymbol{x}}}'\dot{\tilde{\boldsymbol{x}}} = {}& \tilde{\boldsymbol{x}}'\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{K} & \boldsymbol{0}\end{bmatrix}\tilde{\boldsymbol{x}} + \frac{1}{2}\tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\begin{bmatrix}\sum_{i=1}^{2n}\frac{\partial \boldsymbol{M}}{\partial \tilde{\boldsymbol{x}}_i}\dot{\tilde{\boldsymbol{x}}}_i - \dot{\boldsymbol{M}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{T}_0\tilde{\boldsymbol{x}}\\ &+ \tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\boldsymbol{B}\boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right).\end{aligned} \tag{50}$$

In the following, the partial derivative of (45) with respect to time is computed as

$$\frac{\partial V_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}},t)}{\partial t} = \frac{1}{2}\tilde{\boldsymbol{x}}'\boldsymbol{T}_0' \begin{bmatrix} \frac{\partial M}{\partial t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{T}_0\tilde{\boldsymbol{x}}. \tag{51}$$

Since $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}(\tilde{\boldsymbol{q}} + \boldsymbol{q}_r(t)) = \boldsymbol{M}(\tilde{\boldsymbol{x}}, t)$ , then $\dot{\boldsymbol{M}}(\tilde{\boldsymbol{x}}, t) = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial \tilde{\boldsymbol{x}}}\dot{\tilde{\boldsymbol{x}}}$ . Considering this and substituting (27) and (50)-(51) into (48) yields

$$\begin{aligned}\dot{V}(\mathcal{X}, t) = \tilde{\boldsymbol{x}}' &\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{K} & \mathbf{0} \end{bmatrix} \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\boldsymbol{B}\boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right) \\ &- \tilde{\boldsymbol{\theta}}'\boldsymbol{Y}'\boldsymbol{T}_{11}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{\theta}}'\bar{\boldsymbol{f}}_\theta. \end{aligned} \tag{52}$$

Note that we can write (52) as

$$\begin{aligned}\dot{V}(\mathcal{X}, t) = \frac{1}{2}\tilde{\boldsymbol{x}}' &\begin{bmatrix} \mathbf{0} & \boldsymbol{K} \\ \boldsymbol{K} & \mathbf{0} \end{bmatrix} \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\boldsymbol{B}\boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right) \\ &- \tilde{\boldsymbol{\theta}}'\boldsymbol{Y}'\boldsymbol{T}_{11}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{\theta}}'\bar{\boldsymbol{f}}_\theta. \end{aligned} \tag{53}$$

Using (13) and (30), yields

$$\begin{aligned}\dot{V}(\mathcal{X}, t) &= \frac{1}{2}\tilde{\boldsymbol{x}}'\left(-\boldsymbol{Q} + \boldsymbol{T}_0'\boldsymbol{B}\left(\boldsymbol{R}_1'\boldsymbol{R}_1\right)^{-1}\boldsymbol{B}'\boldsymbol{T}_0\right)\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{\theta}}'\bar{\boldsymbol{f}}_\theta \\ &+ \tilde{\boldsymbol{x}}'\boldsymbol{T}_0'\boldsymbol{B}\boldsymbol{T}_{11}\left(-\boldsymbol{Y}\boldsymbol{\theta} + \boldsymbol{\tau} + \boldsymbol{\tau}_l\right) - \tilde{\boldsymbol{\theta}}'\boldsymbol{Y}'\boldsymbol{T}_{11}\boldsymbol{B}'\boldsymbol{T}_0\tilde{\boldsymbol{x}}. \end{aligned} \tag{54}$$

Substituting (34)-(36), (41), and (42) into (54) yields

$$\dot{V}(\mathcal{X}, t) = -\frac{1}{2}\tilde{\boldsymbol{x}}'\left(\boldsymbol{Q} + \begin{bmatrix} a_1^2\boldsymbol{I} & a_1a_2\boldsymbol{I} \\ * & a_1^2\boldsymbol{I} \end{bmatrix}\right)\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{\theta}}'\bar{\boldsymbol{f}}_\theta. \tag{55}$$

Replacing (28) into (55) and given inequalities (37)-(38),

$$\begin{aligned}\dot{V}(\mathcal{X}, t) &= -\frac{1}{2}\tilde{\boldsymbol{x}}'\boldsymbol{\Phi}\tilde{\boldsymbol{x}} - \phi^2\tilde{\boldsymbol{\theta}}'\boldsymbol{\Gamma}\tilde{\boldsymbol{\theta}}, \\ &\leq -\frac{1}{2}\sum_{i=1}^{2n}\lambda_{min}\left(\boldsymbol{\Phi}\right)(\tilde{x}_i)^2 - \sum_{j=1}^{p}\lambda_{min}\left(\boldsymbol{\Gamma}\right)\phi^2(\tilde{\theta}_j)^2, \\ &\leq -\sum_{i=1}^{2n}\left(\frac{\lambda_{min}\left(\boldsymbol{\Phi}\right)}{2}\right)(\tilde{x}_i)^2 - \sum_{j=1}^{p}\left(\frac{\lambda_{min}\left(\boldsymbol{\Phi}\right)}{2}\right)(\tilde{\theta}_j)^2, \end{aligned} \tag{56}$$

which implies $\dot{V}(\mathcal{X}, t) \leq -\sum_{k=1}^{2n+p}\left(\frac{\lambda_{min}(\boldsymbol{\Phi})}{2}\right)(\mathcal{X}_k)^2 = -\left(\frac{\lambda_{min}(\boldsymbol{\Phi})}{2}\right)||\boldsymbol{\mathcal{X}}||_2^2$, where $\mathcal{X}_k$ is the $k$-th element of the augmented state vector $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{2n+p}$. Using (47), we obtain

$$\begin{aligned}\dot{V}(\mathcal{X}, t) &\leq -\left(\frac{\lambda_{min}\left(\boldsymbol{\Phi}\right)}{2}\right)\frac{1}{\lambda_2}\left(\lambda_2||\boldsymbol{\mathcal{X}}||^2\right), \\ &= -\left(\frac{\lambda_{min}\left(\boldsymbol{\Phi}\right)}{2\lambda_2}\right)V(\boldsymbol{\mathcal{X}}, t) \leq -\kappa V(\boldsymbol{\mathcal{X}}, t), \end{aligned} \tag{57}$$

with $\kappa$ given in (39). According to the Comparison Lemma [22], the solution of (58) is given by $V(\boldsymbol{\mathcal{X}}, t) \leq e^{-\kappa t}V(\boldsymbol{\mathcal{X}}(0), t)$, which completes the proof of the exponential stability of the system (12) in closed-loop with the control law (34)-(36) and, consequently, for the system (8). ∎

## V. Results

In this section, numerical experimental results are presented to corroborate the efficacy of the proposed adaptive nonlinear $\mathcal{H}_\infty$ control strategy. The numerical experiment is conducted considering a simplified version of the CRS-A465 robot model [23]. In this simplified model, we consider only joints 2, 3, and 5, which have been renamed as 1, 2, and 3, respectively, while the other degrees of freedom remain fixed, following the approach established in [18]. This system is described by the EL equation

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}(\dot{\boldsymbol{q}}, \boldsymbol{q})\dot{\boldsymbol{q}}(t) + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{58}$$

with the elements $\boldsymbol{M}_{ij}$ and $\boldsymbol{G}_i$ corresponding to the $i$-th row and $j$-th columns of the inertia and the gravity vector computed as $\boldsymbol{M}_{11} = \theta_1 + 2\theta_2 s_2 + 2c_3\theta_3 + 2\theta_4 s_{23}$, $\boldsymbol{M}_{12} = \boldsymbol{M}_{21} = \theta_2 s_2 + 2\theta_3 c_3 + \theta_4 s_{23} + \theta_5$, $\boldsymbol{M}_{13} = \boldsymbol{M}_{31} = \theta_3 c_3 + \theta_4 s_{23} + \theta_6$, $\boldsymbol{M}_{22} = 2\theta_3 c_3 + \theta_5$, $\boldsymbol{M}_{23} = \boldsymbol{M}_{32} = \theta_3 c_3 + \theta_6$, $\boldsymbol{M}_{33} = \theta_6$, $\boldsymbol{G}_1 = \theta_7 c_1 + \theta_8 s_{12} + \theta_9 s_{123}$, $\boldsymbol{G}_2 = \theta_8 s_{12} + \theta_9 s_{123}$, and $\boldsymbol{G}_3 = \theta_9 s_{123}$, and the coriolis and centrifugal forces matrix can be obtained from the inertia matrix by computing the Christoffel symbols of the first kind [21]. In addition, $s_2 = \sin(q_2)$, $s_3 = \sin(q_3)$, $s_{12} = \sin(q_1 + q_2)$, $s_{23} = \sin(q_2 + q_3)$, $s_{123} = \sin(q_1 + q_2 + q_3)$, $c_1 = \cos(q_1)$, $c_2 = \cos(q_2)$, $c_3 = \cos(q_3)$, $c_{23} = \cos(q_2 + q_3)$, $\dot{q}_{12} = \dot{q}_1 + \dot{q}_2$, $\dot{q}_{23} = \dot{q}_2 + \dot{q}_3$, and $\dot{q}_{123} = \dot{q}_1 + \dot{q}_2 + \dot{q}_3$. The parameters are $\theta_1 = 6.3922\,\mathrm{kg\ m^2}$, $\theta_2 = 1.4338\,\mathrm{kg\ m^2}$, $\theta_3 = 0.0706\,\mathrm{kg\ m^2}$, $\theta_4 = 0.0653\,\mathrm{kg\ m^2}$, $\theta_5 = 2.4552\,\mathrm{kg\ m^2}$, $\theta_6 = 0.2868\,\mathrm{kg\ m^2}$, $\theta_7 = 113.6538\,\mathrm{N\ m}$, $\theta_8 = 46.1168\,\mathrm{N\ m}$, $\theta_9 = 2.0993\,\mathrm{N\ m}$.

In the numerical experiment, the robot manipulator started at the initial conditions $\boldsymbol{q}(0) = [0\ 0\ 90]'$(deg) and $\dot{\boldsymbol{q}}(0)$ with $\hat{\boldsymbol{\theta}}(0) = \mathbf{0}$ and was requested to track the desired trajectory $\boldsymbol{q}_r(t) = [90\ -90\ 0]$(deg) , with $\dot{\boldsymbol{q}}_r(t) = \ddot{\boldsymbol{q}}_r(t) = \mathbf{0}$. The adaptive nonlinear $\mathcal{H}_\infty$ controller was implemented considering (34)-(35) and synthesized with the tuning parameters $\boldsymbol{Q} = 0.6\boldsymbol{I}$, $\boldsymbol{R} = (0.01)^2\boldsymbol{I}$, $\rho = 0.2$, with the DREM term gain $\Gamma = 10\boldsymbol{I}$. Additionally, $\lambda_\phi$ was set to 1, in the filtered regressor (18). The values $b_2 = 0.2$, $b_3 = 0.3$, and $a_2 = a_3 = 100$ were used to compute $Y_f$ in (21).The adaptive nonlinear $\mathcal{H}_\infty$ controller proposed in [13] was also designed for the robot manipulator and implemented considering (14)-(15) with the same tuning parameters. That controller is from now on called $\mathcal{H}_\infty$ (Chen et. all 1997), whereas ours is called $\mathcal{H}_\infty$ + DREM.

The results of the numerical experiment are presented in Figs. 1-3. Note that both adaptive nonlinear $\mathcal{H}_\infty$ controllers achieved the desired positions with nearly identical performance and with bounded control inputs. Nevertheless, the $\mathcal{H}_\infty$ + DREM controller achieved exact parameter estimation since condition (38) is satisfied, as shown in Fig. 3. It is noteworthy that, for the requested desired trajectory, the PE condition is not fulfilled [15]. Thus, the $\mathcal{H}_\infty$ (Chen et. all 1997) controller was not able to achieve exact parameter estimation.

## VI. Conclusions

This work proposed a novel adaptive nonlinear $\mathcal{H}_\infty$ controller based on the DREM approach for trajectory tracking with exact parameter estimation of mechanical systems. The proposed adaptive nonlinear $\mathcal{H}_\infty$ controller was designed for a simplified version of the CRS-A465 robot manipulator. Comparison analyses were performed with the adaptive nonlinear $\mathcal{H}_\infty$ controller proposed in [13]. The results demonstrated that both controllers achieved similar performance with respect to the tracking of the states; however, different from the adaptive control

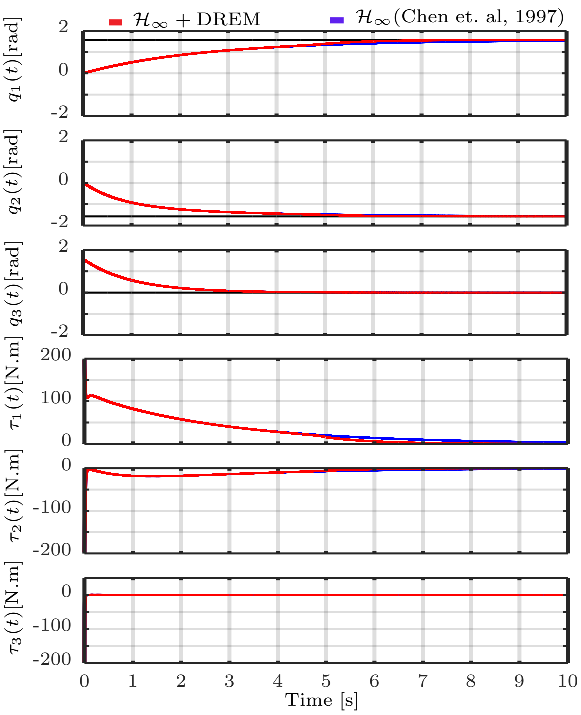


Fig. 1. Temporal evolution of the states and torques for both the $\mathcal{H}_\infty$ + DREM and the $\mathcal{H}_\infty$ (Chen et. all 1997) controllers.

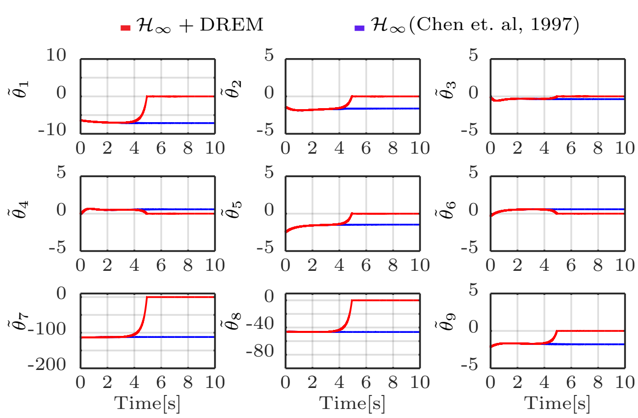


Fig. 2. Temporal evolution of the parameters estimation error.

strategy in [13], the controller here proposed can ensure trajectory tracking with exact parameter estimation even when the PE condition is not fulfilled. Future works includes extending the adaptive nonlinear $\mathcal{H}_\infty$ controller with exact parameter estimation to its counterpart in the Sobolev Spaces, namely the $\mathcal{W}_\infty$ controller [24].

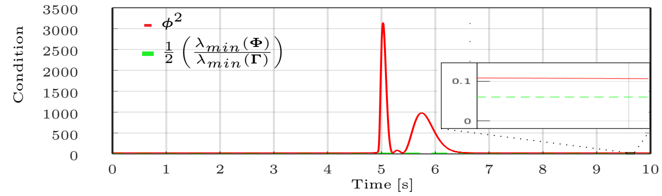


Fig. 3. Verification of the condition (38), which ensures exact parameter convergence.

## References


[1] R. Johansson, "Quadratic optimization of motion coordination and control," *IEEE TAC*, vol. 35, no. 11, pp. 1197–1208, 1990.

[2] A. J. Van Der Schaft, "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$ control," *IEEE TAC*, vol. 37, no. 6, pp. 770–784, 1992.

[3] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback control theory*. Courier Corporation, 2013.

[4] J. C. Geromel, J. Bernussou, and M. C. Oliveira, "$\mathcal{H}_2$-norm optimization with constrained dynamic output feedback controllers: decentralized and reliable control," *IEEE TAC*, vol. 44, no. 7, pp. 1449–1454, 1999.

[5] B. A. Francis and J. C. Doyle, "Linear control theory with an $\mathcal{H}_\infty$ optimality criterion," *SIAM J. on Control and Optimization*, vol. 25, no. 4, pp. 815–844, 1987.

[6] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE TAC*, vol. 34, no. 8, pp. 831–847, 1989.

[7] S. K. Nguang and P. Shi, "$\mathcal{H}_\infty$ fuzzy output feedback control design for nonlinear systems: an LMI approach," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 3, pp. 331–340, 2003.

[8] L.-S. Hu, J. Lam, Y.-Y. Cao, and H.-H. Shao, "A linear matrix inequality (LMI) approach to robust $\mathcal{H}_2$ sampled-data control for linear uncertain systems," *IEEE Trans. on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 33, no. 1, pp. 149–155, 2003.

[9] A. J. Van der Schaft, "On a state space approach to nonlinear $\mathcal{H}_\infty$ control," *Systems & Control Letters*, vol. 16, no. 1, pp. 1–8, 1991.

[10] J. C. Willems, "Dissipative dynamical systems," *European Journal of Control*, vol. 13, no. 2-3, pp. 134–151, 2007.

[11] B.-S. Chen, T.-S. Lee, and J.-H. Feng, "A nonlinear $\mathcal{H}_\infty$ control design in robotic systems under parameter perturbation and external disturbance," *IJC*, vol. 59, no. 2, pp. 439–461, 1994.

[12] A. A. Siqueira, M. H. Terra, and B. C. Maciel, "Nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control applied to manipulators via actuation redundancy," *CEP*, vol. 14, no. 4, pp. 327–335, 2006.

[13] B.-S. Chen, Y.-C. Chang, and T.-C. Lee, "Adaptive control in robotic systems with $\mathcal{H}_\infty$ tracking performance," *Automatica*, vol. 33, no. 2, pp. 227–234, 1997.

[14] P. A. Ioannou and J. Sun, *Robust adaptive control*. PTR Prentice-Hall Upper Saddle River, NJ, 1996, vol. 1.

[15] A. Loría, R. Kelly, and A. R. Teel, "Uniform parametric convergence in the adaptive control of mechanical systems," *European Journal of Control*, vol. 11, no. 2, pp. 87–100, 2005.

[16] Y. Pan and H. Yu, "Composite learning robot control with guaranteed parameter convergence," *Automatica*, vol. 89, pp. 398–406, 2018.

[17] S. Aranovskiy, A. Bobtsov, R. Ortega, and A. Pyrkin, "Performance enhancement of parameter estimators via dynamic regressor extension and mixing," *IEEE TAC*, vol. 62, no. 7, pp. 3546–3550, 2017.

[18] M. A. Arteaga, "On the exact parameter estimation for robot manipulators without persistence of excitation," *IEEE TAC*, vol. 69, no. 1, pp. 410–417, 2024.

[19] S. Sastry, M. Bodson, and J. F. Bartram, *Adaptive control: stability, convergence, and robustness*. Acoustical Society of America, 1990.

[20] M. Korotina, S. Aranovskiy, R. Ushirobira, and A. Vedyakov, "On parameter tuning and convergence properties of the drem procedure," in *IEEE ECC*, 2020, pp. 53–58.

[21] M. W. Spong, S. Hutchinson, and M. Vidyasagar, *Robot modeling and control*. John Wiley & Sons, 2020.

[22] H. Khalil, "Nonlinear systems, printice-hall," *Upper Saddle River, NJ*, vol. 3, 1996.

[23] M. A. Arteaga, A. Gutiérrez-Giles, and J. Pliego-Jiménez, *Local stability and ultimate boundedness in the control of robot manipulators*. Springer, 2022.

[24] D. N. Cardoso, S. Esteban, and G. V. Raffo, "A robust optimal control approach in the weighted sobolev space for underactuated mechanical systems," *Automatica*, vol. 125, p. 109474, 2021.