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Robust Output Synchronization of Discrete-Time Linear-Time-Invariant Multi-Agent Systems Using Phase Tool

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***Abstract*— In this paper, the output synchronization in large-scale discrete-time networks is examined by utilizing the novel phase tool, where the agent dynamics are allowed to be significantly heterogeneous. The synchronization synthesis problem is formulated and thoroughly investigated, with the goal of characterizing the allowable heterogeneity among the agents to ensure synchronization under a uniform controller. The solvability condition is provided in terms of the phases of the residue matrices of the agents at the persistent modes. When the condition is satisfied, a design procedure is given, producing a low-gain synchronizing controller. Numerical examples are given to illustrate the results.**

I. Introduction

In the last decades, the consensus and synchronization problems have been popular in the field of coordination and control of multi-agent systems, and have broad applications in diverse domains, from robotics and autonomous vehicles to distributed computing and social networks [3], [8], [19].

Output synchronization, in its essence, pertains to the coordinated behaviour of multiple interconnected agents to achieve a common output or goal. In the applications such as a group of autonomous drones tasked with forming and maintaining a particular geometric shape in the sky, ensuring that all agents or subsystems work harmoniously to produce a synchronized output is essential to optimize performance and efficiency.

In the simplest case of the synchronization problem, where all agents are identical integrators, synchronization can be achieved with a uniform static network controller, as shown in [11], [20], [22]. A more general scenario is when the agents are general Linear Time-Invariant (LTI) systems but remain homogeneous [9], [15], [16], [26]. Most research efforts in this domain commonly exploit the intuitive approach of utilizing a uniform controller across all agents.

Recently, attention has been paid to the heterogeneous agents with complicated dynamics [1], [2], [14], [21]. A challenging issue is to characterize the allowable diversity among agents to ensure that they can converge to a common trajectory. One way is to treat the agents as nominal models with perturbations, which are described by gain, passivity, gap metric, IQC, and so forth [5], [12], [13]. When the controllers are uniform, the extent of the uncertainties, i.e., the diversity among the agents allowable to ensure problem solvability according to robust control, is rather small. Mathematically, allowing for different controllers makes the synchronization problem easy to solve. Nevertheless, it is crucial to acknowledge that this comes with a substantial increase in both design and implementation costs. Conversely, imposing the condition of a uniform controller significantly enhances the complexity of the research problem.

The recently developed phase tool provides a new perspective [6], [7], [17], [18], [24]. This tool has demonstrated its utility in accommodating large diversities, especially for systems with various gains. In this paper, we will study the synchronization problem by exploiting the phase theory for multi-input multi-output (MIMO) LTI systems. We will demonstrate how the notion of phase provides distinct advantages in addressing the diverse heterogeneity within these networks, leading to a collection of novel results and deeper insights.

The rest of paper is organized as follows. Necessary background and preliminaries are provided in Section II, which are particularly related to graph theory, sectorial matrix and matrix phases. The problem formulation of synchronization is given in Section III. Main result is introduced in Section IV. Section V presents the simulation results. The paper is concluded in Section VI. Due to page limit, we omit all the proofs in this paper.

Notation used in this paper is mostly standard. Let $\mathbb{R}$ and $\mathbb{C}$ be the set of real and complex numbers, respectively. For a matrix $A \in \mathbb{C}^{m\times m}$, A^* denotes its complex conjugate transpose. The sets of eigenvalues and their angles of A are denoted by $\lambda(A)$ and $\angle\lambda(A)$. The Kronecker product of two matrices A and B is denoted by $A\otimes B$. The identity matrix is denoted by I. For a vector $x \in \mathbb{C}^m$, x^* denotes its complex conjugate transpose. We use $\mathbf{1}$ to denote the vectors with all entries equal to 1. Denote by $\mathcal{R}^{m\times m}$ the set of $m \times m$ real rational transfer matrices and let $\mathcal{RH}_\infty^{m\times m} \subset \mathcal{R}^{m\times m}$ contain all its proper stable elements. In this paper, we will adopt the z-transform in discrete time. Therefore, $\mathcal{RH}_\infty^{m\times m}$ is the set of real rational transfer matrices with poles in the open unit disk.

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II. Preliminary

A. Graph theory

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a set of vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ and a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. For the edge (v_i, v_j), the nodes v_i and v_j are called the tail and head respectively. A sequence of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ with $(v_{j-1}, v_j) \in \mathcal{E}$ for $j = \{2, \ldots, k\}$ is called a directed path from node v_1 to node v_k. Here, we assume that the directed graph does not have self loops. A directed graph $\mathcal{G}$ is strongly connected if every vertex is reachable from every other vertex. A directed graph has a spanning tree if there exists at least one node called root with directed paths to all other nodes. A weighted graph is a graph in which the weight is assigned to each edge. A weighted directed graph is balanced if for each node, the total coming weights are equal to the total leaving weights.

Define the (in-degree) Laplacian matrix as

$$L_{ij} = \begin{cases} -a_{ij} & i \neq j, \\ \sum_{j \neq i} a_{ij} & i = j. \end{cases}$$

where a_{ji} is a positive real number (representing the weight) if $(v_i, v_j) \in \mathcal{E}$ and $a_{ji} = 0$ otherwise. A Laplacian matrix always has a zero eigenvalue with an associated right eigenvector $\mathbf{1}$. A necessary and sufficient condition for $\mathbf{1}$ being also the corresponding left eigenvector is that the graph is balanced [20]. Furthermore, the zero eigenvalue is simple if and only if the graph has a spanning tree [22].

B. Matrix phases

Given a matrix $A \in \mathbb{C}^{m \times m}$, the numerical range of A is defined to be

$$W(A) = \{x^* A x : x \in \mathbb{C}^m, \|x\|_2 = 1\}.$$

This is a convex and compact subset of the complex plane [10, Section 1.2] and contains the spectrum of A. The matrix A is said to be semi-sectorial if the origin is not in the interior of $W(A)$. A semi-sectorial matrix is said to be quasi-sectorial if the origin is not on the smooth boundary of $W(A)$. Furthermore, it is said to be sectorial if the origin is not contained in $W(A)$.

For a nonzero semi-sectorial matrix A, its numerical range $W(A)$ is contained in a closed half plane. Define the (largest and smallest) phases of A by

$$\overline{\phi}(A) = \sup_{x \neq 0, x^* A x \neq 0} \angle x^* A x,$$
$$\underline{\phi}(A) = \inf_{x \neq 0, x^* A x \neq 0} \angle x^* A x.$$

so that $[\underline{\phi}(A), \overline{\phi}(A)] \subset [\theta(A) - \pi/2, \theta(A) + \pi/2]$ with $\theta(A) = \theta_0(A) + 2k\pi, k \in \mathbb{Z}$ for some $\theta_0(A) \in [-\pi, \pi)$. One can see that values of the phases are determined mod 2π. When $\theta(A) = \theta_0(A)$, the phases are said to take the principal values.

The phases defined above have many nice properties. We first give the relation between the phases of a semi-sectorial matrix and its compression.

Lemma 1 ([23]): Let $A \in \mathbb{C}^{m \times m}$ be a nonzero semi-sectorial matrix with phases in $[\theta(A) - \pi/2, \theta(A) + \pi/2]$ and $\tilde{A} \in \mathbb{C}^{(m-k) \times (m-k)}$ be a nonzero compression of A. Then $\tilde{A}$ is semi-sectorial and

$$\underline{\phi}(A) \leq \underline{\phi}(\tilde{A}) \leq \overline{\phi}(\tilde{A}) \leq \overline{\phi}(A).$$

Another interesting property is about the matrix product.

Lemma 2: Let $A, B \in \mathbb{C}^{m \times m}$ be semi-sectorial and sectorial. Then the number of nonzero eigenvalues of AB is equal to the rank of A, and the inequality

$$\underline{\phi}(A) + \underline{\phi}(B) \leq \angle \lambda_i(AB) \leq \overline{\phi}(A) + \overline{\phi}(B) \qquad (1)$$

is satisfied if $\angle \lambda_i(AB)$ take values in $(\theta(A) + \theta(B) - \pi, \theta(A) + \theta(B) + \pi)$.

C. Matrix essential phases

In various applications, it is common to encounter a matrix that is not semi-sectorial but can be transformed into a semi-sectorial matrix through diagonal similarity transformation. For a matrix $A \in \mathbb{C}^{m \times m}$, its (largest and smallest) essential phases are defined by

$$\overline{\phi}_{\text{ess}}(A) = \inf_{D \in \mathcal{D}} \overline{\phi}(D^{-1} A D) \text{ and } \underline{\phi}_{\text{ess}}(A) = \sup_{D \in \mathcal{D}} \underline{\phi}(D^{-1} A D),$$

where $\mathcal{D}$ is the set of positive definite diagonal matrices. Here the infimum and supremum are taken over all diagonal positive definite matrices such that $D^{-1} A D$ is semi-sectorial and $\overline{\phi}(D^{-1} A D)$ and $\underline{\phi}(D^{-1} A D)$ take their principal values. The essential phases lack a general analytic representation for arbitrary matrices. However, for specific matrix classes like Laplacian matrices, an analytic representation can be obtained. For a real matrix A, if it can be made semi-sectorial through diagonal similarity transformation, then either $D^{-1} A D$ or $-D^{-1} A D$ is accretive, i.e., having positive semi-definite Hermitian part. It follows that $-\underline{\phi}_{\text{ess}}(A) = \overline{\phi}_{\text{ess}}(A)$. Hereinafter, we denote $\overline{\phi}_{\text{ess}}(A)$ by $\phi_{\text{ess}}(A)$ for notational simplicity.

Here we study the essential phases of Laplacians of the graphs that have a spanning tree. This is the least requirement on the graph connectedness in our application. The Laplacian matrix in this case is reducible, and can be reduced to the Frobenius normal form.

Lemma 3 ([4]): If the graph has a spanning tree, then under a proper permutation the Laplacian matrix can be written as a block lower diagonal matrix

$$L = \begin{bmatrix} L_{11} & 0 & \ldots & 0 \\ L_{21} & L_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{k1} & L_{k2} & \ldots & L_{kk} \end{bmatrix}, \qquad (2)$$

where L_{11} is an irreducible Laplacian matrix or a zero matrix with dimension one and $L_{ii}, i = 2, \ldots, k$ is irreducible with at least one row having positive row sum.

The essential phases of $L_{ii}, i = 1, \ldots, k$, take effect in the synchronization context. A square matrix M is called an M-matrix if it can be written as $sI - A$, where A is nonnegative and $s \geq \rho(A)$. The matrix L_{11} is a special form of M-matrix, which is a Laplacian of a strongly connected subgraph formed by all the roots. One can also observe that $L_{ii}, i = 2, \ldots, k$, are nonsingular diagonal-dominant M-matrices. An upper bound of essential phases of L_{ii} is provided in the following lemma.

Lemma 4: For a Laplacian matrix L in the form (2), there holds

$$\phi_{\text{ess}}(L_{ii}) \leq \overline{\phi}(D_{0i}^{-1} L_{ii} D_{0i}) \leq \frac{\pi}{2}, i = 1, \ldots, k,$$

where $D_{0i} = \text{diag}(\sqrt{x_{i1}/y_{i1}}, \ldots, \sqrt{x_{in}/y_{in}})$, x_i and y_i are right and left eigenvector corresponding to the smallest real eigenvalue of L_{ii}.

Note that the analytic formula for the essential phases of L_{11} can be obtained, i.e., $\phi_{\text{ess}}(L_{11}) = \overline{\phi}(D_{01}^{-1} L_{11} D_{01})$. The essential phases of L_{ii} in general do not have a closed-form expression. Nevertheless, a numerical solution can be obtained. One may refer to [25] for more details.

III. Problem formulation

Consider a complex discrete-time dynamic network. The agents are heterogeneous systems described by

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u_i(k), \\ y_i(k) &= C_i x_i(k),\ x_i(0) \neq 0, \end{aligned}$$

for $i = 1, 2, \ldots, n$, where $x_i(k) \in \mathbb{R}^{p_i}$, $u_i(k) \in \mathbb{R}^m$ and $y_i(k) \in \mathbb{R}^m$ represent the state, control input and output of each agent i, $A_i \in \mathbb{R}^{p_i \times p_i}, B_i \in \mathbb{R}^{p_i \times m}, C_i \in \mathbb{R}^{m \times p_i}$ represent the state, input and output matrices respectively. Let (A_i, B_i) be controllable and (C_i, A_i) be observable for all i.

Assume that agents are all semi-stable in the sense that all eigenvalues of A_i are contained in the closed unit disk. Let A_i share the common eigenvalues on the unit disk, denoted by $e^{j\Omega} = \{e^{j0}, e^{\pm j\omega_1}, \ldots, e^{\pm j\omega_q}, e^{j\pi}\}$. With these internal modes on the unit circle, the agents are all able to generate same common persistent outputs autonomously. The stable modes, i.e., the eigenvalues inside the unit disk of the agents can be completely different.

The transfer function of each agent is

$$P_i(z) = C_i(zI - A_i)^{-1} B_i.$$

Denote the z-transform of input u_i and output y_i by $\hat{u}_i$ and $\hat{y}_i$, then

$$\hat{y}_i(z) = P_i(z)\hat{u}_i(z) + K_i(z)x_i(0),$$

where $K_i(z) = zC_i(zI - A_i)^{-1}$.

The partial fractional expansion of $P_i(z)$ is in the form

$$\begin{aligned} P_i(z) =& \frac{N_{0i}}{z-1} + \frac{N_{\pi i}}{z+1} + \frac{N_{1i}}{z - e^{j\omega_1}} + \frac{\bar{N}_{1i}}{z - e^{-j\omega_1}} \\ &+ \cdots + \frac{N_{qi}}{z - e^{j\omega_q}} + \frac{\bar{N}_{qi}}{z - e^{-j\omega_q}} + P_i^s(z), \end{aligned} \tag{3}$$

where $0 < \omega_1 < \cdots < \omega_q < \pi$ are the frequencies of persistent modes, $N_{0i}, N_{\pi i} \in \mathbb{R}^{m \times m}$ are the residues of $P_i(z)$ at the pole 1 and -1, $N_{li} \in \mathbb{C}^{m \times m}$ for $l = 1, \ldots, q$, are the residues of $P_i(z)$ at the pole $e^{j\omega_l}$, and $P_i^s(z)$ is stable and strictly proper.

A communication protocol utilizing relative output feedback is given by

$$u_i(z) = \sum_{(i,j)\in\mathcal{E}} a_{ij} C(z)(\hat{y}_j(z) - \hat{y}_i(z)), \tag{4}$$

where a_{ij} are the weight of edges and $C(z)$ is a uniform controller. The weights a_{ij} are given *a priori* while the controller $C(z)$ is to be designed. Assume that the digraph has a spanning tree. Denote the corresponding Laplacian matrix by L_0.

Since the agents are heterogeneous and their state dimensions may differ, it is not possible to expect the state synchronization among the agents. Therefore, the aim is to design the controller $C(z)$ such that the output synchronization is reached in the network. The multi-agent system is said to reach output synchronization if $\lim_{k\to\infty}(y_i(k) - y_j(k)) = 0, \forall i, j \in \{1, 2, \ldots, n\}$ and all initial conditions.

By denoting $x(0), \hat{u}(z)$ and $\hat{y}(z)$ as the concatenated vectors of $\begin{bmatrix} x_1(0)' & \cdots & x_n(0)' \end{bmatrix}'$, $\begin{bmatrix} \hat{u}_1(z)' & \cdots & \hat{u}_n(z)' \end{bmatrix}'$ and $\begin{bmatrix} \hat{y}_1(z)' & \cdots & \hat{y}_n(z)' \end{bmatrix}'$, respectively, the network dynamics can be written as

$$\begin{aligned} \hat{y}(z) &= P(z)\hat{u}(z) + K(z)x(0), \\ \hat{u}(z) &= -L_0 \otimes C(z)\hat{y}(z), \end{aligned}$$

where

$$\begin{aligned} P(z) &= \text{diag}\{P_1(z), \ldots, P_n(z)\}, \\ K(z) &= \text{diag}\{K_1(z), \ldots, K_n(z)\}. \end{aligned}$$

It can be obtained that

$$\hat{y}(z) = (I + P(z)(L_0 \otimes C(z)))^{-1} K(z)x(0). \tag{5}$$

The synchronization framework is shown in Fig 1. Introduce the variable

$$\hat{e}(z) = (J \otimes I_m)\hat{y}(z), \tag{6}$$

where $J = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$. Here $\hat{e}(z)$ is referred to as the disagreement vector. The matrix J has a simple eigenvalue 0 with a corresponding right eigenvector $\mathbf{1}_n$. It can be seen that $\lim_{k\to\infty} e(k) = 0$ if and only if y reaches synchronization. In other words, the synchronization problem is converted to a feedback stability problem. Hence, the techniques dealing with stability problem can be naturally applied.

Let Q be an isometry whose columns form the basis of the orthogonal complement of $\text{span}\{\mathbf{1_n}\}$. Denote $U = \begin{bmatrix} Q & \frac{1}{\sqrt{n}}\mathbf{1}_n \end{bmatrix} \otimes I_m$. In view of (5) and (6), we have

$$\begin{aligned} \hat{e}(z) &= JUU'(I + P(z)(L_0 \otimes C(z)))^{-1} UU'K(z)x(0) \\ &= Q \begin{bmatrix} S(z) & 0 \end{bmatrix} U'K(z)x(0) \\ &= (Q \otimes I_m)S(z)(Q' \otimes I_m)K(z)x(0), \end{aligned} \tag{7}$$

where

$$S(z) = (I_{nm-m} + (Q \otimes I_m)' P(z)(L_0 \otimes C(z))(Q \otimes I_m))^{-1}.$$

Here $\hat{e}$ can be treated as tracking error of the reference signal $K(z)x(0)$. Thus, $\lim_{k\to\infty} e(k) = 0$ is equivalent to that $S(z)$

is stable and the internal model of $K(z)$ is contained in the loop transfer matrices. The latter is automatically satisfied since the internal model of $K(z)$ is contained in the agent dynamics.

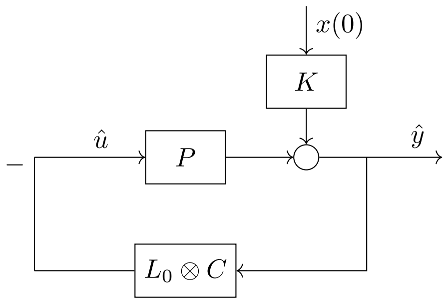

Fig. 1. Block diagram of synchronization.

IV. Main Result

In this section, we proceed to investigate the synchronization synthesis problem with the attempt to design a uniform controller such that the synchronization is enforced. There are two major issues that need to be considered in the study. The first is the synchronizability of the network, i.e., whether there exists a uniforn controllers such that the heterogeneous agents can achieve synchronization. It will be delineated that the phase serves as a good characterization of the diversity of the agents. The second is to provide a construction method of the controller for the synchronizable multi-agent system.

We introduce the following definition.

Definition 1: The discrete-time multi-agent system (3) with a fixed graph is said to be synchronizable under protocol (4) if there exists a stable controller $C(z)$ such that the output synchronization can be enforced, i.e.,

$$\lim_{k\to\infty} |y_i(k) - y_j(k)| = 0, \forall i, j \in \{1, 2, \ldots, n\},$$

regardless of the initial conditions of all the agents.

By properly labelling the agents, the Laplacian matrix can be written in the form (2). Divide the agents into k group $P_{1*}, \ldots, P_{k*}$ according to (2). The size of P_{i*} is compatible with the size of L_{ii}. The agents in each group are strongly connected. In particular, the first group contains all the roots of the graph. It hence serves as the steering group as they have directed path to all other nodes while receive no information from other groups. When L_{11} is a zero matrix with dimension one, the multi-agent system only has one leader. The agents in i-th group for $i = 2, \ldots, k$ can be treated as followers. Let $r_0 = 0$ and $r_i = \sum_{j=1}^{i} \text{size}(L_{jj}), i = 1, \ldots, k$. Let $N_{li*} = \text{diag}\{N_{l(r_{i-1}+1)}, \ldots, N_{lr_i}\}, l = 0, 1, \ldots, q, \pi$, and $i = 1, \ldots, k$. We have the following main result.

Theorem 1: The synchronization problem is solvable if there exist nonsingular matrices $K_0, K_\pi \in \mathbb{R}^{m\times m}$ and $K_1, \ldots, K_q \in \mathbb{C}^{m\times m}$ such that

$$\overline{\phi}(e^{-j\omega_l} N_{li*}(I_{r_i - r_{i-1}} \otimes K_l)) < \frac{\pi}{2} - \phi_{\text{ess}}(L_{ii}),$$
$$\underline{\phi}(e^{-j\omega_l} N_{li*}(I_{r_i - r_{i-1}} \otimes K_l)) > -\frac{\pi}{2} + \phi_{\text{ess}}(L_{ii}),$$

for $l = 0, 1, \ldots, q, \pi$, and $i = 1, \ldots, k$.

When the conditions in Theorem 1 are satisfied, then one synchronizing controller is given by $C(z) = \epsilon H(z)$ for all $\epsilon \in (0, \epsilon^*)$, where $H(z) \in \mathcal{RH}_\infty^{m\times m}$ satisfies $H(e^{j\omega_l}) = K_l$ and $\epsilon^* > 0$ can be estimated from given data.

Next we provide a design procedure of $H(z) \in \mathcal{RH}_\infty^{m\times m}$ such that

$$H(e^{j\omega_l}) = K_l, l = 0, 1, \ldots, q, \pi,$$
$$H(e^{-j\omega_l}) = \bar{K}_l, l = 1, \ldots, q.$$

Let $f(z) = \frac{z-0.5}{z+0.5}$. Denote $z_0 = 1, z_1 = -1$ and $z_{2i} = e^{j\omega_i}, z_{2i+1} = e^{-j\omega_i}$ for $i = 1, \ldots, q$. With the aid of Lagrange polynomial, an $H(z)$ can be given by

$$H(z) = K_0 \prod_{j=1}^{2q+1} \frac{f - f(z_j)}{f(z_0) - f(z_j)} + K_\pi \prod_{\substack{j=0,\\ j\neq 1}}^{2q+1} \frac{f - f(z_j)}{f(z_1) - f(z_j)} + \sum_{t=1}^{q} (K_t \prod_{\substack{j=0,\\ j\neq 2t}}^{2q+1} \frac{f - f(z_j)}{f(z_{2t}) - f(z_j)} + \bar{K}_t \prod_{\substack{j=0,\\ j\neq 2t+1}}^{2q+1} \frac{f - f(z_j)}{f(z_{2t+1}) - f(z_j)}).$$

One can see that conditions in Theorem 1 only require the information of residue matrices at the semi-stable modes. The stable part of each agent does not appear in the theorem, showing that the proposed controller design technique can tolerate large heterogeneity among the agents. The method is robust against the perturbations. Furthermore, the synchronizability condition only depends on the phase information. The gain of each agent can be arbitrarily large. Therefore, the synchronization problem is likely solvable if the agents have vastly different sizes but similar shapes.

The design of synchronizing controllers also suggests the use of low gain controller, indicating that the coordination among the agents does not need strong action. Instead it is more critical to have the right directions of the action.

The consensus problem is a special case of synchronization problem, where all the agents share only one common pole, i.e.,

$$P_i(z) = \frac{N_{0i}}{z-1} + P_i^s(z).$$

Corollary 1: The multi-agent system is consensusable if there exists nonsingular $K \in \mathbb{R}^{m\times m}$ such that

$$\overline{\phi}(N_{0i*}(I_{r_i - r_{i-1}} \otimes K)) < \frac{\pi}{2} - \phi_{\text{ess}}(L_{ii})$$

for $i = 1, \ldots, k$. The controller is given by $C(z) = \epsilon K$ for all $\epsilon \in (0, \epsilon^*)$, where $\epsilon^* > 0$ can be estimated from given data.

The very early studies assume that the agents are simply all identical integrators $P_i(z) = \frac{1}{z-1}$. While the integrator in continuous-time is passive, the integrator in discrete-time is not passive resulting from the sampling process. The controller $C(z) = K$ with a sufficient small positive K solves the problem, which is consistent with the result in the literature.

V. Simulation

We use a numerical example with five agents to illustrate our theoretical result on the uniform synchronizing controller design. Consider a group of agents $P_1, \ldots, P_5$ with network shown in Fig. 2, where the agent dynamics are given in (11). The Laplacian matrix of the network is

$$L_0 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 4 & 0 & -1 \\ -2 & 0 & -1 & 3 & 0 \\ 0 & -2 & 0 & -4 & 6 \end{bmatrix}.$$

Here

$$L_{11} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, L_{22} = \begin{bmatrix} 4 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -4 & 6 \end{bmatrix}.$$

It follows that $\overline{\phi}_{\text{ess}}(L_{11}) = 0$, $\overline{\phi}_{\text{ess}}(L_{22}) = 0.2707$. The agents can be divided into two strongly connected groups accordingly.

Solving the LMIs

$$e^{-j(\pi/4)} N_{1i} K_1 + e^{j(\pi/4)} (N_{1i} K_1)^* > 0, i = 1, 2,$$
$$e^{-j(\pi/4-0.2707)} N_{1i} K_1 + e^{j(\pi/4-0.2707)} (N_{1i} K_1)^* > 0, i = 3, 4, 5,$$
$$e^{-j(\pi/4+0.2707)} N_{li} K_1 + e^{j(\pi/4+0.2707)} (N_{1i} K_1)^* > 0, i = 3, 4, 5,$$

yields

$$K_1 = \begin{bmatrix} 11.96 + 14.11i & 5.22 + 10.00i \\ 8.97 + 15.13i & 2.82 + 7.24i \end{bmatrix}.$$

One synchronizing controller is given by

$$C(z) = 0.014 \times \begin{bmatrix} \frac{36.05z-21.03}{z+0.5} & \frac{22.29z-16.53}{z+0.5} \\ \frac{34.8z-24.48}{z+0.5} & \frac{15.18z-12.45}{z+0.5} \end{bmatrix},$$

which can indeed enforce synchronization as confirmed in Fig 3 and Fig 4.

VI. Conclusion

In this paper, we examine the output synchronization in large-scale discrete-time heterogeneous networks by utilizing the developed phase tool. Matrix essential phases are introduced to reduce the conservatism of matrix phases in the application. The synchronization synthesis problem is formulated and investigated. We provide a sufficient condition, answering the solvability question that under what condition there exists a uniform controller such that a group of agents will reach synchronization. If the condition is satisfied, a design procedure is given, which produces a low gain synchronizing controller. Numerical examples are given to demonstrate the effectiveness of phase tool.

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$$
\begin{aligned}
P_1(z) &= \frac{\begin{bmatrix} -0.22+4.06i & 0.88-5.42i \\ 5.03+2.75i & -4.83-3.43i \end{bmatrix}}{z-e^{j\pi/4}} + \frac{\begin{bmatrix} -0.22-4.06i & 0.88+5.42i \\ 5.03-2.75i & -4.83+3.43i \end{bmatrix}}{z-e^{-j\pi/4}} + \frac{\begin{bmatrix} -3.5z+2.605 & 4.1z-2.6 \\ -11z+7.8 & 12.3z-7.7 \end{bmatrix}}{z^2-1.5z+0.7}, \\
P_2(z) &= \frac{\begin{bmatrix} -2.78+5.86i & 4.28-8.43i \\ 3.70+5.21i & -3.31-7.20i \end{bmatrix}}{z-e^{j\pi/4}} + \frac{\begin{bmatrix} -2.78-5.86i & 4.28+8.43i \\ 3.70-5.21i & -3.31+7.20i \end{bmatrix}}{z-e^{-j\pi/4}} + \frac{\begin{bmatrix} -1.1z+0.8 & 1.2z-0.7 \\ -11z+8.227 & 13.2z-8.7 \end{bmatrix}}{z^2-1.545z+0.625}, \\
P_3(z) &= \frac{\begin{bmatrix} -6.09+12.01i & 8.08-17.84i \\ 6.76+3.94i & -5.38-6.81i \end{bmatrix}}{z-e^{j\pi/4}} + \frac{\begin{bmatrix} -6.09-12.01i & 8.08+17.84i \\ 6.76-3.94i & -5.38+6.81i \end{bmatrix}}{z-e^{-j\pi/4}} + \frac{\begin{bmatrix} -1.1z+1.4 & 3.3z-3.1 \\ -15z+10.9 & 16.9z-10.6 \end{bmatrix}}{z^2-1.2z+0.7}, \\
P_4(z) &= \frac{\begin{bmatrix} -0.98+8.41i & 1.28-11.83i \\ 2.20-18.29i & 1.81+17.88i \end{bmatrix}}{z-e^{j\pi/4}} + \frac{\begin{bmatrix} -0.98-8.41i & 1.28+11.83i \\ 2.20+18.29i & 1.81-17.88i \end{bmatrix}}{z-e^{-j\pi/4}} + \frac{\begin{bmatrix} -6z+5 & 9.2z-7 \\ 12.9z-10.8 & -19.8z+15 \end{bmatrix}}{z^2-1.2z+0.6}, \\
P_5(z) &= \frac{\begin{bmatrix} -2.95+4.39i & 4.03-8.01i \\ 1.53-17.06i & 2.57+16.00i \end{bmatrix}}{z-e^{j\pi/4}} + \frac{\begin{bmatrix} -2.95-4.39i & 4.03+8.01i \\ 1.53+17.06i & 2.57-16.00i \end{bmatrix}}{z-e^{-j\pi/4}} + \frac{\begin{bmatrix} 0.97z-0.27 & 0.5z-1.1 \\ 12.9z-10.7 & -19.4z+14.5 \end{bmatrix}}{z^2-1.1z+0.5}.
\end{aligned}
\tag{11}
$$

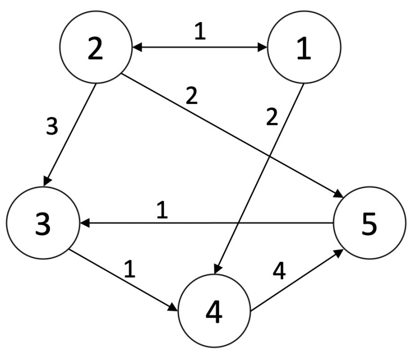

Fig. 2. A directed graph.

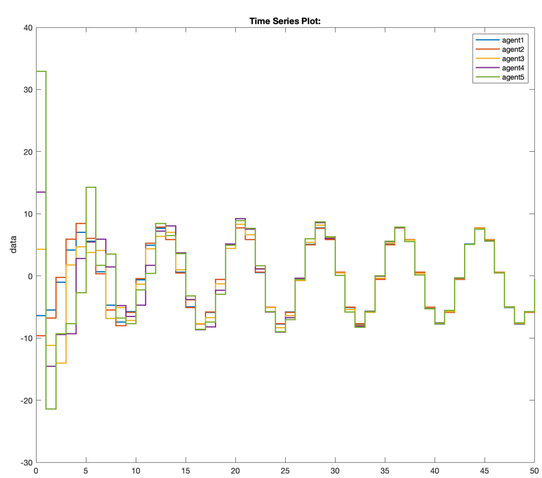

Fig. 3. Trajectories of first output.

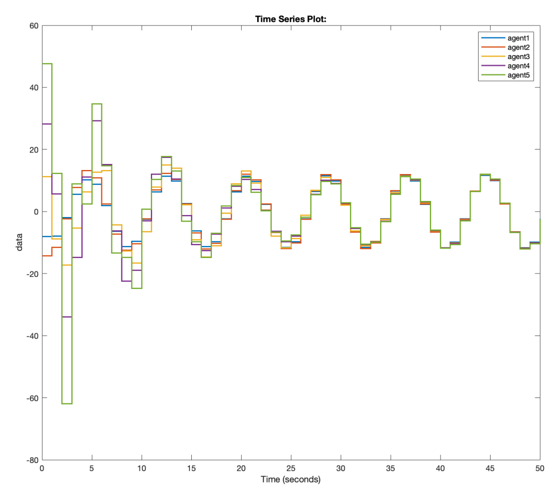

Fig. 4. Trajectories of second output.