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# Robust Safety-Critical Control for Input-Delayed System with Delay Estimation

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***Abstract*— This paper proposes a Control Barrier Function (CBF)-based delay adaptive controller design to accomplish robust safety in the presence of unknown but bounded constant input delay. To this end, we first estimate the input delay by using a gradient descent method minimizing the discrepancy between the current state and the estimated state. Then, we establish the state prediction feedback with the estimated input delay, which is leveraged to attenuate the effect of the input delay. However, due to the error between the true delay and the estimated delay, there is a state prediction error that leads to violations of safety if we use the normal CBFs. To remedy this, we use ideas from Measurement Robust Control Barrier Functions (MRCBFs) that enforce the robust safety constraint against the state prediction error. Specifically, we bound the state prediction error in connection with the input delay estimation error and incorporate the worst case error bound into the safety constraint. The proposed method is verified in the simulations under the connected automated vehicles scenario.**

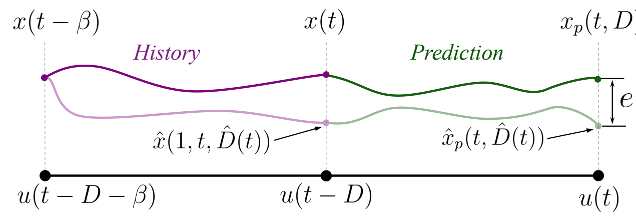


Fig. 1: Illustration of the state predictions on each side: history and prediction. $\hat{\boldsymbol{x}}(1,t,\hat{D}(t))$ is the estimated current state during the past interval from $-\beta$ to 0. It is used to estimate a constant input delay by minimizing the error from $\boldsymbol{x}(t)$. The estimated input delay is subsequently leveraged to predict the future state at the time, $t+D$. The predicted state, $\hat{\boldsymbol{x}}_p$ is finally used to compensate for the effect of the input delay and ensures robust safety.

## I. Introduction

With the increasing demand for automated system applications in urban settings, the importance of systems with safety-critical control has increased. The frameworks to ensure safety have been studied in the context of forward invariance through *Control Barrier Functions* (CBFs) [1], CBFs-based Quadratic Programs (CBFs-QP) [2], Robust adaptive CBFs (RaCBFs) [3], reachability-based approaches [4], and predictive safety control [5] in delay-free systems. However, time delays occur in many real implementations such as a connected automated truck system [6], trajectory tracking [7], and bilateral force teleoperation [8] with unsafe behaviors, which cannot be addressed efficiently with the delay-free model-based safety-critical controllers. In this vein, the safety in the systems subject to the delays presents unique challenges.

Several works have been recently proposed to ensure safety in time-delayed systems. To address the safety in systems with constant known input delay, CBFs have been employed in linear systems [9], and nonlinear systems [10], [11] integrated with the state predictor feedback proposed by [12]. Additionally, the effectiveness of the CBFs-based synthetic framework has been explored in dynamic environments [13]. Furthermore, Tunable Input-to-State-Safety CBFs (Tissf-CBFs) [14] have been leveraged to minimize safety violations in the presence of input disturbances, and Tissf-CBFs are combined with the state prediction feedback control to ensure safety in the presence of input delay and show the successful performance in the connected automated vehicles system [6]. Moreover, the safety with known time-varying input delay is also addressed in the state prediction feedback controller-based framework proposed by [15].

In the context of ensuring safety with unknown input delays, recent studies have introduced CBFs with Integral Quadratic Constraints (IQC) [16] as a tool to bound unmodeled dynamics subject to bounded unknown input delays to ensure safety. Furthermore, tube-based CBFs [17] extended from the IQC-based tube approach allow to design the robust safety-critical controller subject to the unknown input delays.

In the scope of estimating the delays, various delay adaptation methods have been proposed in linear [18], [19] and nonlinear systems [20], [21], [22]. Especially, the delay adaptive controller combined with the estimation of input delay based on the state predictor-based approach [22] successfully demonstrates the local input-to-state stability in a robot manipulator. However, adaptively estimating the unknown input delay and combining it with safety-critical control design has not been studied actively.

In this paper, we propose a robust safety-critical control framework with delay adaptation in a synthetic fashion. The main contributions of this work are twofold. Firstly, we propose a robust safety-critical control framework with the delay adaptation method [22], inspired by measurement-robust CBFs [23]. This leads to the achievable robust safety in the presence of the unknown constant input delay. Secondly, we evaluate the proposed safety-critical control framework

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in the numerical simulations under the connected automated vehicles scenario.

The paper is organized as follows. Section II describes the problem resolved in the paper and Section III introduces the concepts of CBF frameworks and the input delay estimation. The proposed method is provided in Section IV and presents the proof of the safety. Subsequently, we apply the proposed method to connected automated vehicles and provide the results of simulation in Section V. Finally, conclusion is presented in Section VI.

## II. Problem Formulation

Consider the following control affine system with input delay $D > 0$:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})(\boldsymbol{u}(t - D)), \tag{1}$$

where $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^n$ is the system state, $f : \mathcal{X} \to \mathbb{R}^n$ and $g : \mathcal{X} \to \mathbb{R}^n$ are locally Lipschitz continuous functions, $\boldsymbol{u} \in \mathcal{U} \subset \mathbb{R}^n$ represents a control input, and $D \in [\underline{D}, \overline{D}]$ is a bounded input delay.

There are several methods to ensure the safety of system (1) under the assumption that the input delay is known [13], [15]. However, the existing methods cannot easily ensure the system's safety subject to the unknown input delay. In this paper, we estimate the unknown constant input delay in an adaptive scheme and apply it to ensure that (1) is safe; the following problem will be solved:

*Problem 1:* Consider a control system (1) where $D$ is an unknown but bounded constant delay. Design a control input $\boldsymbol{u} \in \mathcal{U}$ that renders the system (1) safe with respect to a given set $\mathcal{S} \subset \mathcal{X}$.

The proposed solution to Problem 1 enforces robustness against an input delay into a safety constraint, inspired by the idea of Measurement Robust Control Barrier Function (MRCBF) [23], while combining the delay estimator [22] in the safe controller design. The proposed barrier function is called *Delay adaptive CBF* (DaCBF) and is presented in Section IV.

## III. Preliminaries

This section contains background information on control barrier functions and input delay estimation.

### A. Safety in Delay-Free System

Consider the following delay-free control affine system:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^n$ is the state and $\boldsymbol{u} \in \mathcal{U} \subset \mathbb{R}^m$ is the control input. We define a set

$$\mathcal{S} = \{\boldsymbol{x} \in \mathcal{X} \mid h(\boldsymbol{x}) \geq 0\}, \tag{3}$$

where $h : \mathcal{X} \to \mathbb{R}$ is a continuously differentiable function. The system (2) is *safe* w.r.t $\mathcal{S}$ if $\mathcal{S}$ is forward invariant.

*Theorem 1 ([1]):* The function, $h$ is a *Control Barrier Function* (CBF) for the system (2) if there exist a control input $\boldsymbol{u}$ and an extended class $\mathcal{K}_\infty^e$ function $\alpha$ such that:

$$\sup_{\boldsymbol{u} \in \mathcal{U}} [L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}] \geq -\alpha\big(h(\boldsymbol{x})\big) \quad \forall \boldsymbol{x} \in \mathcal{S}. \tag{4}$$

If $h$ is a CBF for system (2) then the system is safe w.r.t $\mathcal{S}$.

### B. Safety in Systems with Known Input Delay

Consider system (1), if the input delay is known, then safety is achieved for (1) based on the following:

*Theorem 2 ([13]):* The function $h$ is a CBF for the system (1) if there exist control input $\boldsymbol{u}(t)$ and an extended class $\mathcal{K}_\infty^e$ function $\alpha$ such that for all $\boldsymbol{x} \in \mathcal{S}$:

$$\sup_{\boldsymbol{u} \in \mathcal{U}} [L_f h(\boldsymbol{x}_p) + L_g h(\boldsymbol{x}_p)\boldsymbol{u}(t)] \geq -\alpha\big(h(\boldsymbol{x}_p)\big), \tag{5}$$

where $\boldsymbol{x}_p = \boldsymbol{x}(t + D)$ is the state prediction. If $h$ is a CBF for the system (1), then the system is safe w.r.t $\mathcal{S}$ for all $t \geq D$ if $\boldsymbol{x}(\tau) \in \mathcal{S}, \forall \tau \in [0, D]$.

### C. Safety in Erroneous State Measurement

Let a state-dependent measurement be:

$$\boldsymbol{z} = \boldsymbol{p}(\boldsymbol{x}), \tag{6}$$

where $\boldsymbol{p} : \mathbb{R}^n \to \mathbb{R}^k$ is locally Lipschitz continuous function, and $k$ is the dimension of measurement. We assume that there is the deterministic relationship between $\boldsymbol{z}$ and $\boldsymbol{x}$, and from (6), let a locally Lipschitz continuous function, $\boldsymbol{q}$ be:

$$\boldsymbol{q}(\boldsymbol{z}) = \boldsymbol{x}. \tag{7}$$

Since we have imperfections in (7) in many applications, we assume that the estimate of the state can be defined:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{q}}(\boldsymbol{z}) = \boldsymbol{x} + \boldsymbol{e}(\boldsymbol{x}) \tag{8}$$

where $\boldsymbol{e} : \mathbb{R}^n \to \mathbb{R}^n$ is an unknown uncertainty with upper bounds. Subsequently, the following pointwise set of the state is considered:

$$\mathcal{X}(\boldsymbol{z}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \exists\ \boldsymbol{e} \in \mathcal{E}(\boldsymbol{z}) \text{ s.t. } \hat{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{e}\}. \tag{9}$$

where $\mathcal{E} : \mathbb{R}^k \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued function with the power set, $\mathcal{P}$.

*Definition 1 ([23]):* The continuously differentiable function $h$ is a MRCBF for (2) on $\mathcal{S}$ with parameter function $(a, b) : \mathbb{R}^n \to \mathbb{R}_+^2$ if there exists an extended class $\mathcal{K}_\infty^e$ function $\alpha$ such that for all $\boldsymbol{z}, \hat{\boldsymbol{x}} \in \mathcal{S}$:

$$\sup_{\boldsymbol{u} \in \mathcal{U}} [L_f h(\hat{\boldsymbol{x}}) + L_g h(\hat{\boldsymbol{x}})\boldsymbol{u}(t) - (a(\boldsymbol{z}) + b(\boldsymbol{z})||\boldsymbol{u}(t)||)] \geq -\alpha\big(h(\hat{\boldsymbol{x}})\big). \tag{10}$$

*Theorem 3 ([23]):* Let $h$ be a MRCBF for the system (2). Assume the functions $L_f h, L_g h$, and $\alpha \circ h$ are Lipschitz continuous on $\mathcal{S}$ with Lipschitz coefficients $\mathfrak{L}_{L_f h}, \mathfrak{L}_{L_g h}$, and $\mathfrak{L}_{\alpha \circ h}$, respectively. Further assume there exists a locally Lipschitz function $\eta : \mathbb{R}^k \to \mathbb{R}_+$ such that $\max_{\boldsymbol{e} \in \mathcal{E}(\boldsymbol{z})} ||\boldsymbol{e}|| \leq \eta(\boldsymbol{z})$ for all $\boldsymbol{z} \in \boldsymbol{p}(\mathcal{S})$. Then, any locally Lipschitz continuous controller $\boldsymbol{u}$ satisfying:

$$L_f h(\hat{\boldsymbol{x}}) + L_g h(\hat{\boldsymbol{x}})\boldsymbol{u}(t) - (a(\boldsymbol{z}) + b(\boldsymbol{z})||\boldsymbol{u}(t)||) \geq -\alpha\big(h(\hat{\boldsymbol{x}})\big), \tag{11}$$

where $a(\boldsymbol{z}) = \eta(\boldsymbol{z})(\mathfrak{L}_{L_f h} + \mathfrak{L}_{\alpha \circ h})$ and $b(\boldsymbol{z}) = \eta(\boldsymbol{z})\mathfrak{L}_{L_g h}$, renders (2) safe w.r.t $\mathcal{S}$ for all $t \geq 0$.

In our case from *Problem 1*, we can measure all states, thus $\boldsymbol{p}$ is an identity matrix ($n = k$), and additionally $||\boldsymbol{e}||$ will be globally bounded in the proposed method.

### D. Input Delay Estimation

To estimate input delay, we predict the state of the system (1) within the interval from $t-\beta$ to $t$ where $\beta > 0$ such that for $\hat{D}(t)$ [22]:

$$\hat{\boldsymbol{x}}(\delta, t, \hat{D}(t)) = \boldsymbol{x}(t-\beta) + \beta \int_0^{\delta} f_0(\hat{\boldsymbol{x}}, \boldsymbol{u}_p) dy \qquad (12)$$
$$\text{with} \quad f_0(\hat{\boldsymbol{x}}, \boldsymbol{u}_p) = f(\hat{\boldsymbol{x}}) + g(\hat{\boldsymbol{x}})\boldsymbol{u}_p,$$

where $\boldsymbol{u}_p(\delta, t, \hat{D}(t)) = \boldsymbol{u}(t-\hat{D}(t)+\beta(\delta-1)), \quad \delta \in [0,1]$. By using the fact that $\hat{\boldsymbol{x}}(1,t,\hat{D}(t))$ should be equal to $\boldsymbol{x}(t)$ if $\hat{D}(t)$ converges to $D$, we define the following cost function to derive an adaptation law based on a gradient descent approach [22]:

$$\min_{\hat{D}} J = \frac{1}{2}\Big|\Big|\hat{\boldsymbol{x}}(1,t,\hat{D}(t)) - \boldsymbol{x}(t)\Big|\Big|^2. \qquad (13)$$

Taking the gradient of the cost function (13), the following equation is derived:

$$\frac{\partial J}{\partial \hat{D}(t)}(t, \hat{D}(t)) = \Big(\hat{\boldsymbol{x}}(1,t,\hat{D}(t)) - \boldsymbol{x}(t)\Big)^{\top} \frac{\partial \hat{\boldsymbol{x}}}{\partial \hat{D}}(1,t,\hat{D}(t)) \qquad (14)$$

with the estimator for the input delay proposed by [22] as follows:

$$\dot{\hat{D}}(t) = \gamma \text{Proj}_{[\underline{D},\overline{D}]}\Big\{\hat{D}(t), \rho_D(t)\Big\} \qquad (15)$$

with

$$\rho_D(t) = \frac{-\frac{\partial J}{\partial \hat{D}}(t, \hat{D}(t))}{1 + \Big|\Big|\frac{\partial \hat{\boldsymbol{x}}}{\partial \hat{D}}(1,t,\hat{D}(t))\Big|\Big|^2}, \qquad (16)$$

where $\gamma > 0$ is the adaptation rate of the estimator, and $\text{Proj}_{[a,b]}$ is the general project operator between the range $[a,b]$ to define a threshold of the estimated input delay. See [22] for the detailed assumptions and proofs for convergence.

## IV. Proposed Method

In this section, we show how safety guarantees can be ensured in the presence of the unknown constant input delay.

### A. State Prediction Error

The following presents a bound on the state prediction error, when the input delay is not known exactly.

*Theorem 4:* Let trajectory $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{R}^n$ be the solution of (1) from initial condition $\boldsymbol{x}(0) \in \mathbb{R}^n$ and $\boldsymbol{y} : \mathbb{R}_+ \to \mathbb{R}^n$ be the solution of

$$\dot{\boldsymbol{y}} = f(\boldsymbol{y}) + g(\boldsymbol{y})\boldsymbol{u}(t-\hat{D}), \qquad (17)$$

from initial condition $\boldsymbol{y}(0) = \boldsymbol{x}(0)$, where $\mathfrak{L}_f$ and $\mathfrak{L}_g$ are Lipschitz constants of $f$ and $g$ respectively and

$$||\boldsymbol{u}(t-D) - \boldsymbol{u}(t-\hat{D})|| \leq \epsilon_{\max}, \quad \epsilon_{\max} \in \mathbb{R}_+. \qquad (18)$$

The prediction error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$ for all $t \geq 0$ is bounded as:

$$||\boldsymbol{e}(t)|| \leq \epsilon_{\max} \int_0^t e^{a(t-\tau)} ||g(\boldsymbol{y}(\tau))|| d\tau, \qquad (19)$$

where $a = \mathfrak{L}_f + \mathfrak{L}_g(u_{\max} + \epsilon_{\max})$, and $||\boldsymbol{u}(t)|| \leq u_{\max}$ for all $t \geq 0$.

*Proof:* The dynamics of the prediction error is computed from (1) and (17) as:

$$\dot{\boldsymbol{e}} = f(\boldsymbol{x}) - f(\boldsymbol{y}) + (g(\boldsymbol{x}) - g(\boldsymbol{y}))\boldsymbol{u}(t-D) + g(\boldsymbol{y})\boldsymbol{\epsilon} \quad (20)$$

where $\boldsymbol{\epsilon} = \boldsymbol{u}(t-D) - \boldsymbol{u}(t-\hat{D})$. Since $f$ and $g$ are Lipschitz, the prediction error is bounded as

$$||\dot{\boldsymbol{e}}(t)|| \leq \mathfrak{L}_f||\boldsymbol{e}(t)|| + \mathfrak{L}_g(u_{\max}+\epsilon_{\max})||\boldsymbol{e}(t)|| + ||g(\boldsymbol{y}(t))||\epsilon_{\max}, \quad (21)$$

which is equivalent to

$$||\dot{\boldsymbol{e}}(t)|| \leq (\mathfrak{L}_f + \mathfrak{L}_g(u_{\max} + \epsilon_{\max}))||\boldsymbol{e}(t)|| + ||g(\boldsymbol{y}(t))||\epsilon_{\max}. \qquad (22)$$

By the comparison lemma, it is seen that $\boldsymbol{e}(t)$ can be bounded by the solution to a linear differential equation with zero initial condition, since $\boldsymbol{e}(0) = 0$, i.e.

$$||\boldsymbol{e}(t)|| \leq \underbrace{\epsilon_{\max} \int_0^t e^{a(t-\tau)} ||g(\boldsymbol{y}(\tau))|| d\tau}_{e_{\max}(t)} \qquad (23)$$

where $a = \mathfrak{L}_f + \mathfrak{L}_g(u_{\max} + \epsilon_{\max})$. ■

It should be noted that the following state estimation error should be considered $\bar{\boldsymbol{e}}(t) = \boldsymbol{x}(t+D) - \boldsymbol{y}(t+\hat{D})$; thus, the previous result is extended as follows.

*Corollary 1:* Let trajectory $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{R}^n$ be the solution of (1) from initial condition $\boldsymbol{x}(0) \in \mathbb{R}^n$ and $\boldsymbol{y} : \mathbb{R}_+ \to \mathbb{R}^n$ be the solution of (17) from initial condition $\boldsymbol{y}(0) = \boldsymbol{x}(0)$ and let the delay estimation error be bounded by $\tilde{D}_{\max}$. Then the state estimation error $\boldsymbol{x}(D) - \boldsymbol{y}(\hat{D})$ is bounded as:

$$||\boldsymbol{x}(D) - \boldsymbol{y}(\hat{D})|| \leq e_{\max}(\hat{D} + \tilde{D}_{\max}) + \Delta y_{\max}, \qquad (24)$$

where $e_{\max}(\hat{D} + \tilde{D}_{\max})$ is given in (23) and

$$\Delta y_{\max} = \max_{\tilde{D} \in [-\tilde{D}_{\max}, \tilde{D}_{\max}]} ||\boldsymbol{y}(\hat{D} + \tilde{D}) - \boldsymbol{y}(\hat{D})||. \qquad (25)$$

*Proof:* The prediction error is rewritten by adding and subtracting $\boldsymbol{y}(D)$

$$\boldsymbol{x}(D) - \boldsymbol{y}(\hat{D}) = \boldsymbol{x}(D) - \boldsymbol{y}(D) + \boldsymbol{y}(D) - \boldsymbol{y}(\hat{D}) \qquad (26)$$
$$= \bar{\boldsymbol{e}}(D) + \boldsymbol{y}(D) - \boldsymbol{y}(\hat{D}). \qquad (27)$$

From (19), it is seen that the bound on the prediction error is non-decreasing; hence,

$$||\boldsymbol{x}(D) - \boldsymbol{y}(\hat{D})|| \leq e_{\max}(\hat{D} + \tilde{D}_{\max}) + ||\boldsymbol{y}(D) - \boldsymbol{y}(\hat{D})||. \qquad (28)$$

Since $D$ is not known, $||\boldsymbol{y}(D) - \boldsymbol{y}(\hat{D})||$ is over-approximated by using the bound on the delay estimation error

$$||\boldsymbol{x}(D) - \boldsymbol{y}(\hat{D})|| \leq e_{\max}(\hat{D} + \tilde{D}_{\max}) + \Delta y_{\max} \qquad (29)$$

where

$$\Delta y_{\max} = \max_{\tilde{D} \in [-\tilde{D}_{\max}, \tilde{D}_{\max}]} ||\boldsymbol{y}(\hat{D} + \tilde{D}) - \boldsymbol{y}(\hat{D})||. \qquad (30)$$

■

### B. Robust Safety with Delay Adaptation

This section presents a control barrier function for systems with input delay similar to (5) in Theorem 2, where the state prediction has a bounded error.

Consider the system (1), and the following state prediction:

$$\dot{\boldsymbol{x}}_p = f(\boldsymbol{x}_p) + g(\boldsymbol{x}_p)\boldsymbol{u}(t), \tag{31}$$

where $\boldsymbol{x}_p = \boldsymbol{x}(t + D)$. Since the delay $D$ is not known exactly, it is only possible to compute the following:

$$\dot{\hat{\boldsymbol{x}}}_p = f(\hat{\boldsymbol{x}}_p) + g(\hat{\boldsymbol{x}}_p)\boldsymbol{u}(t), \tag{32}$$

where $\hat{\boldsymbol{x}}_p$ is the estimated state prediction with delay estimate $\hat{D}$ given by (15):

$$\hat{\boldsymbol{x}}_p = \boldsymbol{x}(t + \hat{D}(t)) = \Psi(\vartheta, \boldsymbol{x}(t), \boldsymbol{u}_t), \quad \vartheta \in [0, \hat{D}(t)]. \tag{33}$$

$\Psi(\vartheta, \boldsymbol{x}(t), \boldsymbol{u}_t)$ is the forward integration of the system defined as:

$$\begin{aligned}\Psi(\vartheta, \boldsymbol{x}(t), \boldsymbol{u}_t) = \boldsymbol{x}(t) + \int_0^{\vartheta} \Big( & f(\Psi(y, \boldsymbol{x}(t), \boldsymbol{u}_t)) \\ & + g(\Psi(y, \boldsymbol{x}(t), \boldsymbol{u}_t))\boldsymbol{u}_t(y - \hat{D}(t))\Big) dy, \end{aligned} \tag{34}$$

where $\boldsymbol{u}_t$ is the input history defined by $\boldsymbol{u}_t(s) = \boldsymbol{u}(t + s), \quad s \in \big[-\hat{D}(t), 0\big)$.

In the following, we provide a condition under which safety is guaranteed for (1) in the presence of the unknown input delay, despite an erroneous state prediction defined as:

$$\hat{\boldsymbol{x}}_p = \boldsymbol{x}_p + \boldsymbol{e}_p, \tag{35}$$

where a bound on $\boldsymbol{e}_p$ is provided in Corollary 1.

*Definition 2:* The continuously differentiable function $h$ is a *Delay adaptive Control Barrier Function* (DaCBF) for (1) on $\mathcal{S}$ if $||\boldsymbol{u}(t-D) - \boldsymbol{u}(t-\hat{D})|| \leq \epsilon_{\max}$, $||\boldsymbol{u}(t)|| \leq u_{\max}$ for all $t \geq 0$, and there exists function $\alpha \in \mathcal{K}_\infty^e$ such that for all $\hat{\boldsymbol{x}}_p \in \mathcal{S}$:

$$\sup_{\boldsymbol{u} \in \mathcal{U}} [L_f h(\hat{\boldsymbol{x}}_p) + L_g h(\hat{\boldsymbol{x}}_p)\boldsymbol{u}(t) - d(t)] \geq -\alpha\big(h(\hat{\boldsymbol{x}}_p)\big), \tag{36}$$

where

$$d(t) = (\mathfrak{L}_{L_f h} + \mathfrak{L}_{\alpha \circ h}) e_{p,\max} + \mathfrak{L}_{L_g h} e_{p,\max} ||\boldsymbol{u}(t)|| \tag{37}$$

with $e_{p,\max} = e_{\max}(\hat{D} + \tilde{D}_{\max}) + \Delta y_{\max}$ given in Corollary 1. Since we bound the error, $\boldsymbol{e}_p$ between $\hat{\boldsymbol{x}}_p$ and $\boldsymbol{x}_p$ according to *Corollary* 1, the following proposition is obtained:

*Theorem 5:* Let $h$ be a DaCBF, and the functions $L_f h, L_g h$, and $\alpha \circ h$ are Lipschitz continuous on $\mathcal{S}$ with Lipschitz coefficients $\mathfrak{L}_{L_f h}, \mathfrak{L}_{L_g h}$, and $\mathfrak{L}_{\alpha \circ h}$, respectively. Then, any locally Lipschitz continuous controller $\boldsymbol{u}$ satisfying:

$$L_f h(\hat{\boldsymbol{x}}_p) + L_g h(\hat{\boldsymbol{x}}_p)\boldsymbol{u}(t) - d(t) \geq -\alpha\big(h(\hat{\boldsymbol{x}}_p)\big), \tag{38}$$

where $d(t) = (\mathfrak{L}_{L_f h} + \mathfrak{L}_{\alpha \circ h}) e_{p,\max} + \mathfrak{L}_{L_g h} e_{p,\max} ||\boldsymbol{u}(t)||$, renders the system (1) safe with respect to $\mathcal{S}$ such that $\forall t \geq D$.

*Proof:* We have a candidate control barrier function and its Lie-derivative:

$$\dot{h}(\boldsymbol{x}) = L_f h(\boldsymbol{x}) + L_g h(\boldsymbol{x})\boldsymbol{u}(t - D), \tag{39}$$

and with the accurate state prediction, the following equation is obtained:

$$\dot{h}(\boldsymbol{x}_p) = L_f h(\boldsymbol{x}_p) + L_g h(\boldsymbol{x}_p)\boldsymbol{u}(t). \tag{40}$$

We show the following necessary and sufficient condition for forward invariant of $\mathcal{S}$:

$$\dot{h}(\boldsymbol{x}_p) + \alpha(h(\boldsymbol{x}_p)) \geq 0. \tag{41}$$

However, since we do not know the true input delay, it is not possible to know $\boldsymbol{x}_p$. Instead, we can introduce the estimated state prediction, (35) with the error bound, (24). If there exist the the functions $L_f h$, $L_g h$, and $\alpha \circ h$ that are Lipschitz continuous on $\mathcal{S}$ with Lipschitz coefficients $\mathfrak{L}_{L_f h}, \mathfrak{L}_{L_g h}$, and $\mathfrak{L}_{\alpha \circ h}$, then the following inequalities are derived:

$$\begin{aligned} & \dot{h}(\boldsymbol{x}_p) + \alpha(h(\boldsymbol{x}_p)) \\ & \geq -||L_f h(\boldsymbol{x}_p)|| - ||L_g h(\boldsymbol{x}_p)||\boldsymbol{u}(t) - ||\alpha(h(\boldsymbol{x}_p))|| \\ & \geq ||L_f h(\hat{\boldsymbol{x}}_p)|| + ||L_g h(\hat{\boldsymbol{x}}_p)||\boldsymbol{u}(t) + ||\alpha(h(\hat{\boldsymbol{x}}_p))|| - d_e(t) \\ & \geq L_f h(\hat{\boldsymbol{x}}_p) + L_g h(\hat{\boldsymbol{x}}_p)\boldsymbol{u}(t) + \alpha(h(\hat{\boldsymbol{x}}_p)) - d(t) \geq 0 \end{aligned} \tag{42}$$

where $d_e(t) = (\mathfrak{L}_{L_f h} + \mathfrak{L}_{\alpha \circ h})||e_p|| + \mathfrak{L}_{L_g h}||e_p||||\boldsymbol{u}(t)||$, and $d(t) = (\mathfrak{L}_{L_f h} + \mathfrak{L}_{\alpha \circ h}) e_{p,\max} + \mathfrak{L}_{L_g h} e_{p,\max}||\boldsymbol{u}(t)||$.

Since (36) and (38) hold, which ensures the last inequality, (42); thus the system (31) is safe w.r.t $\mathcal{S}$ such that $\forall \boldsymbol{x}_p(t) \in \mathcal{S}, \forall t \geq 0$. This also implies that $\boldsymbol{x}(t) \in \mathcal{S}, \forall t \geq D$ by Theorem 2. ■

We formulate the following Second-Order Cone Programming (SOCP) optimization problem endowing safety guarantees in the presence of the unknown input delay as:

$$\begin{aligned} \boldsymbol{u}_{\text{safe}} = \quad & \arg\min_{\boldsymbol{u} \in \mathcal{U}} \frac{1}{2} ||\boldsymbol{u} - \boldsymbol{u}_{\text{nominal}}||^2 \\ \text{s.t. } & L_f h(\hat{\boldsymbol{x}}_p) + L_g h(\hat{\boldsymbol{x}}_p)\boldsymbol{u}(t) - d(t) \geq -\alpha\big(h(\hat{\boldsymbol{x}}_p)\big), \end{aligned} \tag{43}$$

where $d(t) = (\mathfrak{L}_{L_f h} + \mathfrak{L}_{\alpha \circ h}) e_{p,\max} + \mathfrak{L}_{L_g h} e_{p,\max}||\boldsymbol{u}(t)||$.

## V. Simulation Results

In this section, we verify the proposed method in an application where a connected automated truck should follow a lead vehicle, while keeping the proper distance between the two vehicles. The inter-vehicle communication delay causes an input delay in the truck's system, and the input delay in the system depends on the communication quality as well. Therefore, it is reasonable to assume that we have no prior knowledge of the input delay. To compare between baselines and the proposed method, we follow the system model, safety regulation, and scenario (e.g. the lead car makes the emergency stop) provided by [6].

We carry out two simulations; the existing methods without delay adaptation are simulated where the initial input delay used for the state prediction is assumed under $20\%$ of the true delay, $D = 0.5$ seconds, and the proposed method with delay adaptation is verified, while showing the performance of the delay estimator. In the second simulation we assume that the initial input delay estimate is zero.

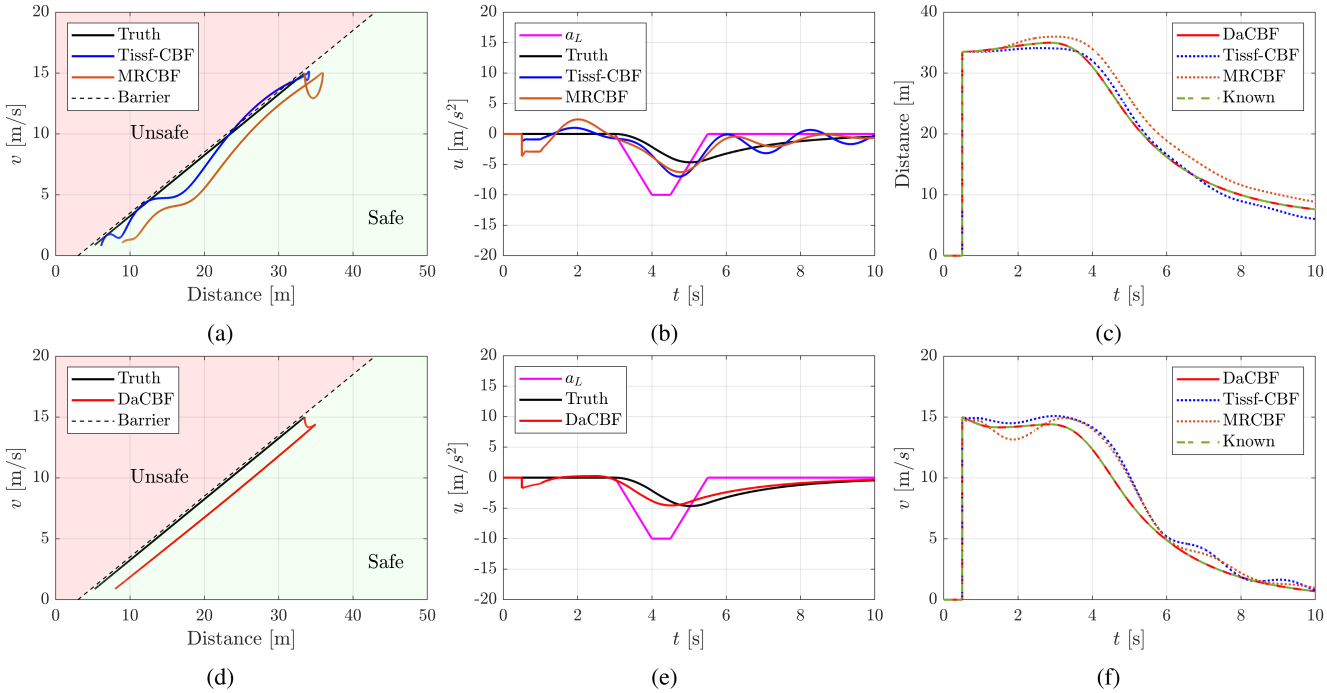


Fig. 2: The plots show the performances of each safety-critical control and delay estimator in the presence of unknown input delay. Truth (solid black line) describes the safety-ensured trajectories for the delay-free system with a normal CBF. (a) and (d) show the performances of existing methods without delay adaptation and ours with delay adaptation, respectively. (b) and (e) show the designed safety control inputs from existing methods and ours, respectively. (c) and (f) present the estimated state predictions of each method, and known (dashed green line) is the true state prediction for all $t \geq D$ when we know the true input delay in the system.

### A. Application to Connected Automated Vehicle Control

Consider the following control affine system dynamics with input delay:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\xi} \\ \dot{v} \\ \dot{v}_L \end{bmatrix} = \underbrace{\begin{bmatrix} v_L - v \\ 0 \\ a_L \end{bmatrix}}_{f(\boldsymbol{x})} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g(\boldsymbol{x})} \boldsymbol{u}(t-D), \tag{44}$$

where $\xi$ is the distance between two vehicles, $v$ is the speed of the follower truck, and $v_L, a_L$ are the speed and acceleration of the lead vehicle, respectively. The safety regulation is defined as follows:

$$h(\boldsymbol{x}) = \xi - \xi_{\text{sf}} - Tv, \tag{45}$$

where $\xi_{\text{sf}}$ is a minimum standstill distance, and $T$ is time headway. We use the following nominal controller provided by [6]:

$$\begin{aligned} \boldsymbol{u}_{\text{nominal}} &= A(V(\xi) - v) + B(W(v_L) - v), \\ V(\xi) &= \min\{k(\xi - \xi_{\text{st}}), v_{\max}\}, \\ W(v_L) &= \min\{v_L, v_{\max}\}, \end{aligned} \tag{46}$$

where $A$ is a distance gain, and $B$ is a velocity gain, and $\xi_{\text{st}}$ is a safe standstill distance, and $V(\cdot), W(\cdot)$ are the range and speed policy, respectively.

Fig. 2 shows the performances of each safety-critical control in the scenario where the lead vehicle makes a sudden emergency brake. We first simulate Tissf-CBF [6] and MRCBF [23] without delay adaptation as baselines. As shown in Fig. 2a, Tissf-CBF violates the safety even if its safety constraint includes robustness against the effect of the input delay. The effect is encapsulated into an input disturbance in the system, so it is not able to deal with the uncertainty caused by the state prediction error in the safety constraint. Furthermore, the putative input delay has 80% of the error from the true delay in the simulation, which leads to a large state prediction error in the prediction feedback controller as shown in Fig. 2c and Fig. 2f. The designed safety-critical control inputs without delay adaptation are shown in Fig. 2b. On the other hand, MRCBF is designed to be robust against the state uncertainty in its safety constraint based on Lipschitz coefficients, which makes the system more conservative than Tissf-CBF. Like Tissf-CBF, MRCBF also degrades the performance of the controller due to inaccurate state prediction. Overall, without delay adaptation, it shows that the state prediction with inaccurate input delay has a large error as shown in Fig. 2c and Fig. 2f, so it is difficult to ensure safety guarantees in Tissf-CBF and achieve the desired operations in both cases.

While the existing methods above fail to satisfy safety and the desired system behaviors simultaneously, the proposed method, DaCBF allows the system to not only ensure the safety guarantees, but also recover quickly the aspect of the delay-free system's behaviors as shown in Fig. 2d and

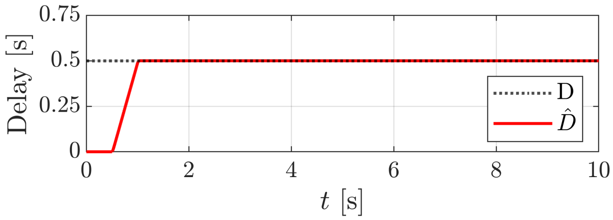


Fig. 3: The plot shows input delay estimation: the estimated input delay (solid red line) and the true input delay (dashed black line).

2e. From $t = 0$ to the time when the estimated delay converges to the true delay as shown in Fig. 3, it is observed that DaCBF does not violate the safety regulation since we have the robustness in the safety constraint. Meanwhile, the estimated state predictions, $\hat{\boldsymbol{x}}_p(t + \hat{D}(t))$ for all $t \geq D$ is quickly getting accurate as shown in Fig. 2c and 2f compared to others. After the estimated input delay converges to the true delay, the effect of the input delay is compensated by the state prediction feedback controller. However, the slight conservative behavior is still observed due to the robustness that we initially enforced in the safety constraint.

## VI. Conclusion

This paper investigated robust safety-critical control design for the system with unknown but bounded constant input delay. We proposed a method for ensuring safety by combining delay adaptation and a robust CBFs-based scheme. The input delay was estimated to predict the future state which was to compensate for the effect of the inherent input delay. We subsequently established the worst case bound of the state prediction error in the safety constraint. We provided the proof of the bounded error coupled with the delay estimation error and simulated the proposed method in the connected automated vehicles system. The results showed that the proposed method allowed for the system to not only be safe, but also compensate the impact of the input delay, outperforming the existing works. Our future work will cover the analysis of conservatism.

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