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Stability-Preserving Model Reduction of Networked Lur'e Systems

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Abstract— **This paper proposes a model reduction approach for simplifying the interconnection topology of Lur'e network systems. A class of reduced-order models are generated by the projection framework based on graph clustering, which not only preserve the network structure but also ensure absolute stability. Furthermore, we provide an upper bound on the input-output approximation error between the original and reduced-order Lur'e network systems, which is expressed as a function of the characteristic matrix of graph clustering. Finally, the results are illustrated via a numerical example.**

Index Terms— **Lur'e system, model reduction, absolute stability, graph clustering**

I. Introduction

Lur'e systems represent an important class of nonlinear systems that consist of linear dynamics with a static nonlinear feedback. Many physical nonlinear systems, such as mechanical systems, power systems, and hyperchaotic attractors can be modeled as Lur'e systems, see some examples in [1], [2]. The interconnection of Lur'e systems gives rise to networked Lur'e systems, which gain much attention from the literature, see e.g., [3]–[5].

From the perspective of design and optimization, dealing with large-scale models of dynamic systems can often be challenging. Therefore, model reduction serves as an indispensable tool for generating lower-order approximations that facilitate efficient design and optimization processes. Various methods have been developed for the model reduction of Lur'e systems. [6] introduces the dissipativity-based model reduction for Markov jump Lur'e systems using linear matrix inequalities. [1] proposes a balanced truncation approach for Lur'e systems, which preserves the absolute stability of the reduced-order model, while [7] presents a generalized balanced truncation approach for Lur'e networks, focusing on preserving the synchronization property of the networks. In the context of Lur'e network systems, how to reduce the number of interconnected subsystems (or nodes) is also a crucial research problem. In [8], a method based on Kullback-Leibler divergence is applied to simplify the interconnected structure of Lur'e networks. However, the model reduction error is difficult to characterize since only the linear part is considered for the divergence analysis, disregarding the nonlinear component of the system.

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In general, reducing a system containing nonlinear elements is challenging, especially when dealing with nonlinear systems with network structure. Nevertheless, insights can be drawn from model reduction techniques employed in linear network systems. In such systems, a mainstream method for reducing complicated network structures is graph clustering, which is realized by dividing the subsystems into several disjoint clusters [9], [10]. Different from conventional model reduction methods such as balanced truncation [11] or moment matching [12], the reduced order system obtained by clustering can still be represented as a network system with fewer number of nodes than its full-order counter part. How to find suitable clusters becomes the key in this kind of methods. [13] proposes a clustering-based model reduction method for networked passive systems by analyzing the controllability and observability properties of associated edge systems. [14] introduces the notion of clustering reducibility, which is related to the approximation error. The works in [15]–[17] present a dissimilarity-based clustering approach for both undirected and directed network systems, where the dissimilarity between two nodes can be featured in the difference of node behaviors with respect to external inputs. In [18], clustering method is generalized to scale-free networks, where the reduced system is obtained by minimizing the scale-free cost function. However, compared to the reduction of linear network systems, model reduction for nonlinear network systems is remain relatively underdeveloped.

In this paper, we propose a clustering-based model reduction method for reducing the network structure of the Lur'e network, meanwhile preserving the absolute stability. The main framework is based on the clustering-based project using the characteristic matrix of a clustering. Different from [8], we consider both the linear and nonlinear parts of Lur'e networks in the framework, i,e, both parts will determine the upper hound of the approximation error. Moreover, the clustering-based projection framework guarantees the preservation of the absolute stability in reduced-order Lur'e networks. In this paper, an explicit expression of the upper bound on input-to-output error between the original and reduced-order network systems is provided, where the bound can be calculated via a linear system parameterized by the characteristic matrix of a graph clustering.

The remainder of this paper is structured as follows. The preliminaries and problem setting are introduced in Section II. Section III then provides the clustering-based model reduction method for Lur'e networks and analyze the bound on the reduction error. In Section IV, an example is shown to illustrate the results. Finally, in Section V, the conclusion remarks and potential future works are given.

Notation: The symbol $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}_+$ denotes the set of nonnegative real numbers. I_n represents the identity matrix of size n. $\mathbf{1}$ represents a vector with all the elements equal to 1. $\|x(t)\|_2$ denotes the $\mathcal{L}_2$-norm of a signal $x(t)$, and $\|G(s)\|_{\mathcal{H}_\infty}$ denotes the $\mathcal{H}_\infty$-norm of the transfer function $G(s)$ of a linear time-invariant system.

II. Preliminaries and Problem Setting

In this paper, we consider a Lur'e network system composed of N subsystems formulated as follows:

$$\Sigma_i : \dot{x}_i = -a_i x_i + u_i - \phi(x_i),\ i = 1, 2, \cdots, N, \tag{1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and the input of the ith subsystem Σ_i. $v_i \in \mathbb{R}$ is the feedback. $\phi(x_i) \in \mathbb{R} \mapsto \mathbb{R}$ is a continuous nonlinear function. Throughout the paper, we assume that the nonlinearity $\phi(x_i)$ in each subsystem i is *slope-restricted* as

$$0 \leq \frac{\phi(x_a) - \phi(x_b)}{x_a - x_b} \leq \mu_i, \tag{2}$$

for all $x_a, x_b \in \mathbb{R}$ and $x_a \neq x_b$, where $\phi(0) = 0$, $\mu_i > 0$ is known.

Define the input to each subsystem as

$$u_i = \Sigma_{j=1}^N w_{ij}(x_j - x_i) + b_{ik} u_{ei}, \tag{3}$$

where w_{ij} is the weight of the edge between the two nodes i and j, and u_{ek} is the external input with the gain $b_{ik} \in \mathbb{R}$, which is 0 if k-th input does not affect node i. Thus the compact networked system can be expressed as

$$\Sigma : \dot{x} = A_L x + B u_e - \Phi(x) \tag{4}$$

where $x = [x_1, x_2, \cdots, x_N]^T$, $u_e = [u_{e1}, u_{e2}, \cdots, u_{ep}]^T$, and $\Phi(x) = [\phi(x_1), \phi(x_2), \cdots, \phi(x_N)]^T$, with p the number of external inputs, and $B \in \mathbb{R}^{N \times p}$ is a matrix with b_{ik} as its (i,k)-th entry.

$$A_L := -A - L, \tag{5}$$

where $A = \text{diag}(a_1, a_2, \cdots, a_N)$, and L is the Laplace matrix of graph $\mathcal{G}$, and its each element is defined as

$$L_{ij} = \begin{cases} -w_{ij}, & i \neq j \\ \sum_{j=1, j\neq i}^{N} w_{ij}, & \text{otherwise.} \end{cases} \tag{6}$$

Clearly, A_L is symmetric and negative-definite.

This paper aims for structure-preserving model reduction for diffusively coupled Lur'e networks in the form of (4), and the reduced-order model not only approximates the input-output behavior of the original network system with a certain accuracy but also inherits an interconnection structure with diffusive couplings. Specifically, the problem addressed in this paper is as follows.

Given a networked Lur'e system (4), our objective is to derive a simplified model described by:

$$\hat{\Sigma} : \dot{z} = \hat{A}_L z + \hat{B} u_e - \hat{\Phi}(\hat{x}), \quad \hat{x} = \Pi z, \tag{7}$$

where $z \in \mathbb{R}^r$ denotes the state of the reduced-order system, and $\hat{x} \in \mathbb{R}^N$ represents an approximation of x. The matrix $\hat{A}_L$ can be composed of a diagonal matrix $\hat{A}$ and a Laplacian matrix $\hat{L}$ characterizing a reduced undirected graph. Additionally, the reduction error $\|x(t) - \hat{x}(t)\|_2$ remains sufficiently small relative to the external input $\|u(t)\|_2$.

III. Main Results

This section presents the main results of this paper, where we first show the absolute stability can be preserved by the proposed method and then analyze the upper bound of the approximation error.

A. Absolute Stability

To guarantee the approximation error $\|x(t) - \hat{x}(t)\|_2$ to be up-bounded w.r.t. the external inputs $u(t)$, we need to study the condition under which both the original network system (4) and the reduced-order system (7) are stable. Specifically, we consider the concept of absolute stability, which essentially means that the origin of a Lur'e system is globally uniformly asymptotically stable for any nonlinearity in the given sector.

Note that if each subsystem has a slope-restricted nonlinearity as in (2), then it is not hard to show that the nonlinearity $\Phi(x)$ in the Lur'e network system will satisfy the incrementally sector-bounded condition described as:

$$[\Phi(x_a) - \Phi(x_b)]^T [\Phi(x_a) - \Phi(x_b) - K_\mu(x_a - x_b)] \leq 0 \tag{8}$$

for all $x_a,\ x_b \in \mathbb{R}^N$, $x_a \neq x_b$ and $\Phi(0) = 0$, where

$$K_\mu = \text{diag}(\mu_1, \mu_2, \cdots, \mu_N).$$

Based on that we provide a sufficient condition for the absolute stability of the Lur'e network system (4).

Lemma 1. *The unforced original system is absolutely stable if there exist a positive definite symmetric matrix P, a symmetric matrix W and a positive constant ϵ such that*

$$P A_L + A_L^T P = -W^T W - \epsilon P \tag{9a}$$

$$P = K_\mu - \sqrt{2} W. \tag{9b}$$

Proof: Let the external input $u_e = 0$, then the Lur'e network system becomes

$$\dot{x} = A_L x - \Phi(y), \quad y = x \tag{10}$$

From Lemma 10.3 in [19], the system (10) is absolutely stable if $Z(s) = I_N + K_\mu(sI_N - A_L)^{-1}$ is strictly positive real. Apparently, (A_L, I_N) is controllable and (A_L, K_μ) is observable, then according to Lemma 10.3 in [19], $Z(s) = I_N + K_\mu(sI_N - A_L)^{-1}$ is strictly positive real if and only if (9a) and (9b) satisfy. ∎

In the following, we assume that the sector bound K_μ of the original Lur'e network system (4) satisfies (9). Then, in the next section, we study how to generate a reduced-order model (7) such that it can be interpreted as a reduced network system, and moreover, the absolute stability is retained.

B. Clustering-based Model Reduction

Clustering-based methods is a well-studied model reduction method for simplifying network systems [9], allowing the reduced model to be interpreted as a reduced network, where each node in the reduced network corresponds to a cluster of nodes in the original network.

Graph clustering is to partition all the nodes in a graph into several nonempty and disjoint subsets. A graph clustering can be characterized by a binary matrix $\Pi \in \{0,1\}^{N\times r}$, whose element is defined as

$$\Pi_{ij} := \begin{cases} 1 & \text{if node } i \text{ is included in clustering } j, \\ 0 & \text{otherwise.} \end{cases}$$

Since each node can only belong to one cluster, we have $\Pi \mathbf{1}_N = \mathbf{1}_r$, where N and r represent the numbers of nodes and clusters, respectively. Given the reduced order r, clustering-based model reduction is to find a graph clustering with r clusters of nodes such that $x \approx \hat{x} = \Pi z$, where $\Pi \in \mathbb{R}^{N\times r}$ is the characteristic matrix of the clustering. Here, $\hat{x}_i$ represents an approximation of the collective behavior of all the nodes in cluster i.

In this paper, we also resort to graph clustering to reduce the networked Lur'e system (4) for preserving the network structure. Let $\Pi \in \mathbb{R}^{N\times r}$ be the characteristic matrix of a clustering of the underlying graph, and denote:

$$\Pi^\dagger := (\Pi^T K_\mu \Pi)^{-1} \Pi^T K_\mu \tag{11}$$

with K_μ in (8), such that $\Pi^\dagger \Pi = I_r$. Then the coefficient matrices in the reduced-order model (7) are given by

$$\hat{A}_L = \Pi^\dagger A_L \Pi,\ \hat{B} = \Pi^\dagger B,\ \hat{\Phi}(\cdot) = \Pi^\dagger \Phi(\cdot). \tag{12}$$

First, we show that the reduced-order network model in the form of (7) preserves not only the network structure but also the absolute stability.

Theorem 1. *The reduced-order Lur'e network system in* (7) *is absolutely stable and preserves the network structure such that $\hat{A}_L$ is the sum of a positive diagonal matrix and a reduced-dimension Laplacian matrix.*

Proof: For the reduced-order Lur'e network system (13), its unforced system is

$$\dot{z} = \hat{A}_L z - \Pi^\dagger \Phi(\hat{x}), \quad \hat{x} = \Pi z \tag{13}$$

First, we need to prove $(\hat{A}_L, \Pi^\dagger)$ is controllable, $(\hat{A}_L, \Pi)$ is observable. The controllability and observability can be seen from the controllability and observability matrices

$$\hat{\mathcal{C}} = \begin{bmatrix} \Pi^\dagger & \hat{A}_L \Pi^\dagger & \hat{A}_L^2 \Pi^\dagger & \cdots & \hat{A}_L^{r-1} \Pi^\dagger \end{bmatrix},$$
$$\hat{\mathcal{O}} = \begin{bmatrix} \Pi & (\Pi \hat{A}_L)^T & (\Pi \hat{A}_L^2)^T & \cdots & (\Pi \hat{A}_L^{r-1})^T \end{bmatrix}^T,$$

which are full rank due to $\text{rank}(\Pi) = \text{rank}(\Pi^\dagger) = r$.

Following the similar reasoning as the proof of Lemma 1, to show the absolute stability of the above system, we need to prove $\hat{Z}(s) = I_N + K_\mu \Pi (sI_r - \hat{A}_L)^{-1} \Pi^\dagger$ is strictly positive real. Notice that $(\hat{A}_L, K_\mu \Pi)$ is observable since $(\hat{A}_L, \Pi)$ is observable and K_μ is nonsingular, By Lemma 10.2 in [19], $\hat{Z}(s) = I_N + K_\mu \Pi (sI_r - \hat{A}_L)^{-1} \Pi^\dagger$ is positive real if and only if there exist a positive definite symmetric matrix $\hat{P}$, matrix $\hat{W}$ and a positive constant ϵ such that

$$\hat{P}\hat{A}_L + \hat{A}_L^T \hat{P} = -\hat{W}^T \hat{W} - \epsilon \hat{P} \tag{14a}$$
$$\hat{P}\Pi^\dagger = \Pi^T K_\mu - \sqrt{2}\hat{W}^T \tag{14b}$$

From (14a), we have

$$\hat{P}\Pi^\dagger A_L \Pi + \Pi^T A_L^T (\Pi^\dagger)^T \hat{P} = -\hat{W}^T \hat{W} - \epsilon \hat{P} \tag{15}$$

By (14b) and (15), we have

$$\hat{P} = \Pi^T K_\mu \Pi - \sqrt{2}\hat{W}^T \Pi \tag{16}$$

and

$$\begin{aligned} &(\Pi^T K_\mu - \sqrt{2}\hat{W}^T) A_L \Pi + \Pi^T A_L^T (\Pi^T K_\mu - \sqrt{2}\hat{W}^T)^T \\ &= -\hat{W}^T \hat{W} - \epsilon (\Pi^T K_\mu \Pi - \sqrt{2}\hat{W}^T \Pi) \end{aligned} \tag{17}$$

If we choose $\hat{W} = W\Pi$, from (17), we have

$$\begin{aligned} &\Pi^T (K_\mu - \sqrt{2}W) A_L \Pi + \Pi^T A_L^T (K_\mu - \sqrt{2}W) \Pi \\ &= -\Pi^T W^T W \Pi - \epsilon (\Pi^T K_\mu \Pi - \sqrt{2}\Pi^T W \Pi), \end{aligned} \tag{18}$$

From (9), we can obtain

$$\Pi^T P A_L \Pi + \Pi^T A_L^T P \Pi = -\Pi^T W^T W \Pi - \epsilon \Pi^T P \Pi. \tag{19}$$

Thus (14a) and (14b) hold if (9) satisfies, then $\hat{Z}(s) = I_N + K_\mu \Pi (sI_r - \hat{A}_L)^{-1} \Pi^\dagger$ is strictly positive real, which implies the unforced reduced-order system is also absolutely stable. ■

Theorem 1 shows that by using the specific formulation of reduced-order matrices in (12), the reduced-order Lur'e network system (7) is guaranteed to be absolutely stable, regardless of the choice of the characteristic matrix Π. Then, in the following section, we will study how the matrix Π can affect the approximation error between the original and reduced-order systems.

C. Analysis of Reduction Error

In this section, we derive an upper bound on the reduction error between the original and reduced-order Lur'e networks, represented as a function of Π, i.e., the characteristic matrix of graph clustering.

Before proceeding, the following lemma first presents the upper bounds on the approximation errors on the linear parts.

Lemma 2. *Denote the following transfer matrices for the linear parts in the original and reduced-order systems:*

$$\begin{aligned} &g_{u_e}(s) = (sI_N - A_L)^{-1} B,\ g_v(s) = (sI_N - A_L)^{-1}, \\ &\hat{g}_{u_e}(s) = \Pi (sI_r - \hat{A}_L)^{-1} \hat{B},\ \hat{g}_v(s) = \Pi (sI_r - \hat{A}_L)^{-1} \Pi^\dagger. \end{aligned} \tag{20}$$

Assume (9) *holds, then the following error bounds hold:*

$$\|g_{u_e}(s) - \hat{g}_{u_e}(s)\|_{\mathcal{H}_\infty} \le \gamma_H \|g_{u_e}(s)\|_{\mathcal{H}_\infty}, \tag{21}$$
$$\|g_v(s) - \hat{g}_v(s)\|_{\mathcal{H}_\infty} \le \gamma_H \|g_v(s)\|_{\mathcal{H}_\infty}, \tag{22}$$

where γ_H is the $\mathcal{H}_\infty$ norm of the following linear system

$$H(s, \Pi) = C_H (sI_r - A_H)^{-1} B_H + D_H, \tag{23}$$

with $A_H = \Pi^\dagger A_L \Pi$, $B_H = \Pi^\dagger A_L (I_N - \Pi\Pi^\dagger)$, $C_H = \Pi$, *and* $D_H = I_N - \Pi\Pi^\dagger$. ■

The proof of Lemma 2 is shown in Appendix A. Since $\Pi^\dagger A_L \Pi = (\Pi^T K_\mu \Pi)^{-1} \Pi^T K_\mu A_L \Pi$, which is Hurwitz for any full rank characteristic matrix Π, the $\mathcal{H}_\infty$ norm of $H(s)$ always exists. Based on Lemma 2, we present the approximation error bound in the following.

Theorem 2. *Assume* (9) *holds. If* $\|g_v(s)\|_{\mathcal{H}_\infty} < \frac{1}{(\gamma_H+1)\mu_{max}}$, *the* $\mathcal{H}_\infty$ *norm of the approximation error is bounded by*

$$\|x(t) - \hat{x}(t)\|_2 \leq \Gamma(\gamma_H)\|u_e(t)\|_2, \tag{24}$$

where

$$\Gamma(\gamma_H) = \frac{\gamma_H \kappa_{u_e}}{[1 - (\gamma_H + 1)\mu_{max}\kappa_v]\,(1 - \mu_{max}\kappa_v)}, \tag{25}$$

and $\kappa_{u_e} = \|g_{u_e}(s)\|_{\mathcal{H}_\infty}$, $\kappa_v = \|g_v(s)\|_{\mathcal{H}_\infty}$. ■

The proof of Theorem 2 is provided in the Appendix B.

Remark 1. *The assumption of the theorem* $\kappa_v = \|g_v(s)\|_{\mathcal{H}_\infty} < \frac{1}{(\gamma_H+1)\mu_{max}}$ *implies that the two terms in the denominator of* $\Gamma(\gamma_H)$, *i.e.* $1 - (\gamma_H + 1)\mu_{max}\kappa_v$ *and* $(1 - \mu_{max}\kappa_v)$, *are both positive. Hence the error bound* (24) *is well-defined.*

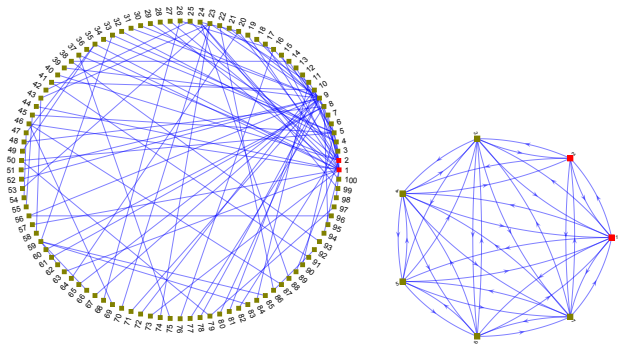
Observe that in (25), only γ_H is dependent on the choice of Π, or equivalently, graph clustering of the original network. The other parameters, $\mu_{\max}$, κ_{u_e} and κ_v, are priori since they are determined by the original network system.

Furthermore, it can be verified that $\Gamma(\gamma_H)$ is a monotonically increasing function with respect to γ_H, i.e. a smaller γ_H will lead to lower $\Gamma(\gamma_H)$. As γ_H is the $\mathcal{H}_\infty$-norm of the linear system $H(s)$, we can use a Riccati inequality or an LMI to characterize $\mathcal{H}_\infty$, then Π can be selected to minimize γ_H. This would also lead to a lower error bound on the approximation the nonlinear Lur'e network. In particular, if $\Pi = I_N$, then $\gamma_H = 0$, which yields $\Gamma(\gamma_H) = 0$, meaning that the reduced-order model has exactly the same outputs as the original system with the same external inputs applied.

It is also worth mentioning that since the dimension of A_H in (23) is r, that is the dimension of the reduced-order system, to obtain γ_H does not require an expensive computation. Therefore it will be beneficial for the subsequent optimization procedure that is to find an optimal Π to minimize γ_H. However, we leave the detailed discussion to our future work.

IV. Simulation Results

To illustrate the proposed model reduction approach for Lur'e networks, we consider a netwrok example of 100 nodes which is shown as Fig. 1a. The network is generated by the B-A Scale-Free Network Generation algorithm [20]. The 100 nodes are divided into 7 clusters: nodes 1, 2, 3-22, 23-42, 43-62, 63-81, and 82-100. The reduced-order network resulting from the given clustering is shown in Fig. 1b. Note that the reduced graph is now bidirectional, but it is not undirected, as $\Pi^\dagger L \Pi$ is no longer symmetric.



(a) (b)

Fig. 1: (a) The topology of the original network. (b) The topology of the reduced network. The input nodes are highlighted by the red color.

For the original system in (4) and (5), we choose $A = 2I_{100}$, $F = [I_2, 0_{98\times 98}]^T$. The nonlinearity of each subsystem is

$$\phi(x_i) = |x_i + 0.1| - |x_i - 0.1|,$$

thus $K_\mu = 0.2I_{100}$.

First, we show both the unforced original system and the reduced-order system are absolutely stable. Set the external input $u_e = 0$ and choose random values ranging from -1 to 1 as initial states of both the original and reduced-order systems. From Fig. 2a and Fig. 2b, we observe that the states of both the original and the reduced-order Lur'e network systems asymptotically converge to 0, which implies the stability.

Then we choose both the external inputs as $\sin(t)$, and set the initial states of both systems are zero. The state trajectories are plotted in Fig. 3a and Fig. 3b, respectively, where the nodes in the same cluster are indicated by the same color. Note that nodes 1 and 2 form two clusters, and the approximation errors are shown to be relative small. In contrast, the other clusters are formed without any optimization, leading to larger approximation errors.

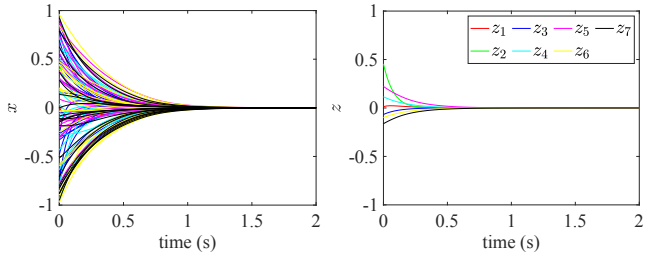
To validate the error bound in Theorem 2, we estimate the input-to-output error of the model reduction as follows:

$$\gamma_\epsilon^2 = \frac{\int_0^T x^T(t)x(t)dt}{\int_0^T u^T(t)u(t)dt} \approx \frac{\sum_{k=0}^{T/\delta t} x^T(k)x(k)\delta t}{\sum_{k=0}^{T/\delta t} u^T(k)u(k)\delta t},$$

where T is the length of the simulation time, and δt is the stepsize. In this simulation, we obtain $\gamma_\epsilon = 0.0761$. Meanwhile, using Lemma 2, we compute $\|g_{u_e}(s)\|_{\mathcal{H}_\infty} = 0.1372$, $\|g_v(s)\|_{\mathcal{H}_\infty} = 0.5$, and $\gamma_H = 1.2607$. Therefore, $\|g_v(s)\|_{\mathcal{H}_\infty} < \frac{1}{(\gamma_H+1)\mu_{\max}}$ holds. It then leads to the error bound $\Gamma(\gamma_H) = 0.2484$, according to Theorem 2.

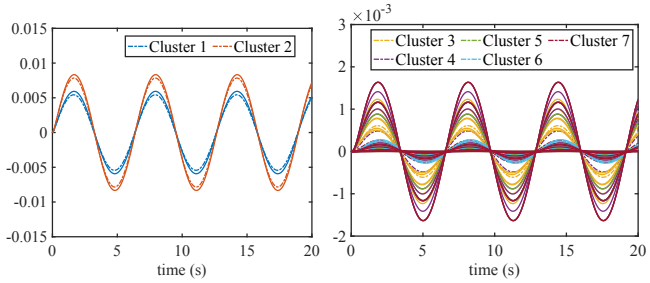
V. Conclusions

In this paper, we have introduced a clustering-based model reduction technique aimed at preserving the network structure of Lur'e network systems while ensuring the crucial



(a) (b)

Fig. 2: (a) State trajectories of the original Lur'e network system with random initial states and $u_e = 0$. (b) State trajectories of the reduced-order system with random initial states and $u_e = 0$.



(a) (b)

Fig. 3: (a) State trajectories of the original and reduced-order Lur'e network system for Clusters 1 and 2. (b) State trajectories of the original and reduced-order Lur'e network system for Clusters 3 to 7. The solid curves represent the trajectories of the original network, and the dashed ones represent the trajectories of the reduced network.

property of absolute stability. We have provided the input-output error bound, which is determined by the $\mathcal{H}_\infty$-norm of a linear system parameterized in the characteristic matrix of a graph clustering. As for our future works, we will extend the proposed model reduction approach to Lur'e network systems with non-scalar subsystems. Additionally, we will explore the preservation method of other important properties, such as synchronization and passivity.

APPENDIX

A. Proof of Lemma 2

The transfer function of the error system for u_e is

$$g_{u_e}(s) - \hat{g}_{u_e}(s) = C_e \left(sI_{N+r} - A_e\right)^{-1} B_e, \tag{26}$$

where $A_e = \begin{bmatrix} A_L & \\ & \hat{A}_L \end{bmatrix}$, $B_e = \begin{bmatrix} F \\ \hat{F} \end{bmatrix}$, $C_e = \begin{bmatrix} I & -\Pi \end{bmatrix}$. Inspired by [14], we introduce a pair of transfer matrices

$$T = \begin{bmatrix} -\Pi^\dagger & I_r \\ I_N & 0 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} 0 & I_N \\ I_r & \Pi^\dagger \end{bmatrix} \tag{27}$$

such that

$$\begin{aligned} \tilde{A}_e &= T A_e T^{-1} = \begin{bmatrix} \Pi^\dagger A_L \Pi & \Pi^\dagger A_L(-I_N + \Pi\Pi^\dagger) \\ 0 & A_L \end{bmatrix}, \\ \tilde{B}_e &= T B_e = \begin{bmatrix} 0 \\ F \end{bmatrix}, \\ \tilde{C}_e &= C_e T^{-1} = \begin{bmatrix} -\Pi & I_N - \Pi\Pi^\dagger \end{bmatrix}. \end{aligned} \tag{28}$$

Thus, from (28), we have

$$\begin{aligned} &g_{u_e}(s) - \hat{g}_{u_e}(s) = \tilde{C}_e \left(sI_{N+r} - \tilde{A}_e\right)^{-1} \tilde{B}_e \\ &= \left[\Pi(sI_r - \Pi^\dagger A_L \Pi)^{-1}\Pi^\dagger A_L + I_N\right](I_N - \Pi\Pi^\dagger) \\ &= H(s) g_{u_e}(s), \end{aligned} \tag{29}$$

where $H(s)$ is shown as (23). Similarly, we obtain

$$g_v(s) - \hat{g}_v(s) = \tilde{C}_{ve} \left(sI_{N+r} - \tilde{A}_{ve}\right)^{-1} \tilde{B}_{ve} = H(s) g_v(s), \tag{30}$$

where $H(s)$ is shown as (23).

Then, we show that $\Pi^\dagger A_L \Pi$ is Hurwitz for any full rank characteristic matrix Π. Note that there exists a positive definite matrix $P_H := \Pi^T K_\mu \Pi$ such that

$$\begin{aligned} &(\Pi^\dagger A_L \Pi)^T P_H + P_H \Pi^\dagger A_L \Pi \\ =&\Pi^T A_L^T K_\mu \Pi + \Pi^T K_\mu A_L \Pi \\ =&\Pi^T (A_L^T K_\mu + K_\mu A_L)\Pi < 0. \end{aligned} \tag{31}$$

Therefore, $H(s)$ is asymptotically stable, and its $\mathcal{H}_\infty$-norm is well defined.

Finally, according to (29) and (30), the inequality (21) and (22) hold.

B. Proof of Theorem 2

In the complex frequency domain, we have

$$X(s) = g_{u_e}(s) U_e(s) + g_v(s) V(s), \tag{32a}$$

$$\hat{X}(s) = \hat{g}_{u_e}(s) U_e(s) + \hat{g}_v(s) \hat{V}(s), \tag{32b}$$

where $X(s)$ and $\hat{X}(s)$ are the Laplace transforms of the time domain signals $x(t)$ and $\hat{x}(t)$, respectively, assuming zero initial conditions. All the transfer matrices are defined in (20). Then, the approximation error in the complex frequency domain is given as

$$\begin{aligned} &X(s) - \hat{X}(s) \\ &= [g_{u_e}(s) - \hat{g}_{u_e}(s)] U_e(s) + [g_v(s) - \hat{g}_v(s)] V(s) \\ &\quad + \hat{g}_v(s)[V(s) - \hat{V}(s)], \end{aligned} \tag{33}$$

which leads to the following upper bound in the time domain:

$$\begin{aligned} \|x(t) - \hat{x}(t)\|_2 \leq &\|g_{u_e}(s) - \hat{g}_{u_e}(s)\|_{\mathcal{H}_\infty} \|u_e(t)\|_2 \\ &+ \|g_v(s) - \hat{g}_v(s)\|_{\mathcal{H}_\infty} \|v(t)\|_2 \\ &+ \|\hat{g}_v(s)\|_{\mathcal{H}_\infty} \|v(t) - \hat{v}(t)\|_2. \end{aligned} \tag{34}$$

Then, we analyze each terms in the above error bound.

First, we discuss the bound of $\|v(t) - \hat{v}(t)\|_2$ and $\|v(t)\|_2$. Following a same procedure in [7], we make use of the

incremental sector bounded condition (8) and obtain

$$\begin{aligned}
&[v(t)-\hat{v}(t)]^T[v(t)-\hat{v}(t)]\\
&\le [v(t)-\hat{v}(t)]^T[v(t)-\hat{v}(t)]\\
&\quad -[v(t)-\hat{v}(t)]^T[v(t)-\hat{v}(t)-K_\mu(\hat{x}(t)-x(t))]\\
&= [v(t)-\hat{v}(t)]^T K_\mu[\hat{x}(t)-x(t)]\\
&\le \frac{1}{2}[v(t)-\hat{v}(t)]^T[v(t)-\hat{v}(t)\\
&\quad +\frac{1}{2}[x(t)-\hat{x}(t)]^T K_\mu^2[x(t)-\hat{x}(t)].
\end{aligned}$$

Thus, $[v(t)-\hat{v}(t)]^T[v(t)-\hat{v}(t)] \le \mu_{\max}^2[x(t)-\hat{x}(t)]^T[x(t)-\hat{x}(t)]$, where $\mu_{\max}>0$ is the largest element of K_μ. This also implies

$$\|v(t)-\hat{v}(t)\|_2 \le \mu_{\max}\|x(t)-\hat{x}(t)\|_2. \tag{35}$$

Since $\Phi(0)=0$, which leads to

$$\begin{aligned}
\|v(t)\|_2 \le &\mu_{\max}\|x(t)\|_2 \le \mu_{\max}\|g_{u_e}(s)\|_{\mathcal{H}_\infty}\|u_e(t)\|_2\\
&+\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}\|v(t)\|_2.
\end{aligned} \tag{36}$$

If $\|g_v(s)\|_{\mathcal{H}_\infty} < \frac{1}{\mu_{\max}}$, then

$$\|v(t)\|_2 \le \frac{\mu_{\max}\|g_{u_e}(s)\|_{\mathcal{H}_\infty}}{1-\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}}\|u_e(t)\|_2. \tag{37}$$

According to (21), it has

$$\begin{aligned}
&\|g_{u_e}(s)-\hat{g}_{u_e}(s)\|_{\mathcal{H}_\infty}\|u(t)\|_2\\
&\qquad \le \gamma_H\|g_{u_e}(s)\|_{\mathcal{H}_\infty}\|u(t)\|_2.
\end{aligned} \tag{38}$$

By (37) and (22),

$$\begin{aligned}
&\|g_v(s)-\hat{g}_v(s)\|_{\mathcal{H}_\infty}\|v(t)\|_2\\
&\le \frac{\gamma_H\mu_{\max}\|g_{u_e}(s)\|_{\mathcal{H}_\infty}\|g_v(s)\|_{\mathcal{H}_\infty}}{1-\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}}\|u_e(t)\|_2.
\end{aligned} \tag{39}$$

According to (35) and (22), it can be obtained that

$$\begin{aligned}
&\|\hat{g}_v(s)\|_{\mathcal{H}_\infty}\|v(t)-\hat{v}(t)\|_2\\
&\le \|g_v(s)-\hat{g}_v(s)\|_{\mathcal{H}_\infty}\|v(t)-\hat{v}(t)\|_2\\
&\quad +\|g_v(s)\|_{\mathcal{H}_\infty}\|v(t)-\hat{v}(t)\|_2\\
&\le (\gamma_H+1)\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}\|x(t)-\hat{x}(t)\|_2.
\end{aligned} \tag{40}$$

Thus, by (38), (39) and (40), we obtain

$$\begin{aligned}
\|x(t)-\hat{x}(t)\|_2 \le &(\gamma_H+1)\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}\|x(t)-\hat{x}(t)\|_2\\
&+\gamma_H\|g_{u_e}(s)\|_{\mathcal{H}_\infty}\|u_e(t)\|_2\\
&+\frac{\gamma_H\mu_{\max}\|g_{u_e}(s)\|_{\mathcal{H}_\infty}\|g_v(s)\|_{\mathcal{H}_\infty}}{1-\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}}\|u_e(t)\|_2.
\end{aligned} \tag{41}$$

If $\|g_v(s)\|_{\mathcal{H}_\infty} < \frac{1}{\mu_{\max}(\gamma_H+1)}$, then $\|g_v(s)\|_{\mathcal{H}_\infty} < \frac{1}{\mu_{\max}}$ also holds. The error bound is obtained as

$$\begin{aligned}
&\|x(t)-\hat{x}(t)\|_2\\
&\le \frac{\gamma_H\|g_{u_e}(s)\|_{\mathcal{H}_\infty}+\frac{\gamma_H\mu_{\max}\|g_{u_e}(s)\|_{\mathcal{H}_\infty}\|g_v(s)\|_{\mathcal{H}_\infty}}{1-\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}}}{1-(\gamma_H+1)\mu_{\max}\|g_v(s)\|_{\mathcal{H}_\infty}}\|u_e(t)\|_2,
\end{aligned}$$

which can be simplified to the inequality (24) with $\Gamma(\gamma_H)$ defined in (25).

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