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An Emulation Approach to Output-Feedback Sampled-Data Synchronization

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***Abstract*— This paper studies state synchronization of homogeneous LTI agents to a trajectory generated by a given exosystem under both spatial and temporal communication constraints. In particular, communication between the agents is assumed to be intermittent and asynchronous, i.e. effectively acting on a time-varying graph at irregular sampling instances. The paper extends our previous state-feedback result to the output feedback setting. This naturally requires the introduction of local state observers, which complement local continuous-time emulators of unconstrained closed-loop dynamics. The observer interaction with emulators is not unique and we propose an architecture that greatly streamlines the analysis of the closed-loop system and simplifies the implementation of the scheme. As a result, the synchronization is proved under mild persistency of connectivity assumption on spacial connectivity under arbitrary uniformly bounded sampling intervals.**

***Index Terms*— Sampled-data systems, network control systems, synchronization, observer-based control.**

I. Introduction

The synchronization problem among multi-agent systems (MAS) is a cornerstone of many coordination tasks [1]–[5]. The challenge of synchronization problems is to design control laws for each agent that rely on *local* information obtained by sensing or communication with other agents. Within the vast literature on synchronization of MAS, a taxonomy of challenges has emerged. The complexity of the general problem is due to three main components: *i*) the dynamics of agents comprising the MAS (linear, nonlinear, homogeneous, et cetera), *ii*) the spatial architecture, i.e., the graph (undirected, directed, switching, et cetera), and *iii*) the temporal architecture (continuous time, discrete time, event-triggered, et cetera). For an overview of these problems, the reader may refer to the following books on the subject, [6]–[8].

Solving the synchronization problem while requiring complex agent dynamics, spatial architectures and temporal architectures has proven to be challenging; see [9] and the references therein. In our previous work [10], we focused on state synchronization for a group of homogeneous LTI agents that have access to their state vectors and exchange information asynchronously. This temporal constraint is also coupled to a spatial constraint, where the neighborhood of each agent is also time-varying. The main conceptual idea behind [10] was for each agent to emulate the behavior of the entire ensemble in-between transmission times. The emulator design is performed by assuming a completely unconstrained version of the problem, i.e., with no spatial constraints (a complete graph), and continuous time information exchange. Each agent then transmits, when the temporal constraints permit, the centroid state of its local emulator, which is used to update the emulators of each agent. The emulator updates are then implemented as a discrete time consensus-like protocol.

In this work, we extend [10] by considering an ensemble of LTI homogeneous agents that are *not* able to measure their complete state. The natural change to the control architecture in this case is to first introduce a local observer for each agent. In the unconstrained case, assuming continuous time information exchange over the complete graph, the observer becomes a part of the closed-loop dynamics. This might call for the inclusion of observers to each emulated closed-loop state. However, we found that keeping emulators practically unchanged from the state-feedback case, i.e. observer-independent, simplifies the convergence analysis. We then show that each agent need only transmit the centroid of the observer states to the other agents. Each agent then implements an emulator for the centroid dynamics of the complete system in between transmission times, and as in [10] uses a consensus-style protocol at sampling instances to drive the emulated centroids to an agreement. The resulting closed-loop dynamics for the analog emulator and observer dynamics and the discrete dynamics at transmission times have a clear block structure simplifying the analysis. Finally, we show that under some standard assumptions on the agent dynamics and the sequence of graphs, the agents synchronize their trajectories.

This paper is organized as follows. Section II defines the problem and key assumptions required for the solution. Section III provides a review of the results from [10] and then outlines the control architecture for the output-feedback case. Section IV presents the main results of the work. Finally, an illustrative example is given in Section V and concluding remarks and future outlook in Section VI.

Notation: The sets of all non-negative integers are denoted as $\mathbb{Z}_+$ and $\mathbb{N}_\nu := \{i \in \mathbb{Z} \mid 1 \le i \le \nu\}$. Sequences with indices from $\mathbb{Z}_+$ are indicated as $\{s_i\}$. The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively, and $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. By e_i we understand the ith standard basis vector in $\mathbb{R}^\nu$ and by $\mathbb{1}_\nu$, or simply $\mathbb{1}$ when the dimension is clear from the context, the all-ones vector from $\mathbb{R}^\nu$. The complex-conjugate transpose of a matrix M is denoted by M'. The orthogonal projection onto the image of $\mathbb{1}_\nu$ (the "agreement space") is $P_{\mathbb{1}} := \mathbb{1}_\nu \mathbb{1}_\nu'/\nu$. Given two matrices (vectors) M and N, $M \otimes N$ denotes their Kronecker product, while $\operatorname{spec}(M)$ refers to the set of all eigenvalues

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of M.

II. Problem setup

Consider ν homogeneous agents, each with linear dynamics given by

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases} \tag{1}$$

for some $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$, where x_i, u_i, and y_i are the ith state, control signal, and measured output, respectively. The global version of the dynamics can be written via Kronecker products as

$$\begin{cases} \dot{x}(t) = (I_\nu \otimes A)x(t) + (I_\nu \otimes B)u(t) \\ y(t) = (I_\nu \otimes C)x(t) \end{cases}. \tag{2}$$

The ensemble is subject to some set of communication constraints, manifesting as restrictions on the information each agent may use to generate its local control signal u_i.

We assume that only the *communication* between agents is limited, meaning that local variables such as the output, $y_i(t)$, or controller states are continuously available for the ith agent. The inter-agent communication is restricted both temporally and spatially. The temporal constraints are represented by a strict monotonically increasing sequence of sampling instances $\{s_k\}$, $k \in \mathbb{Z}_+$, where agents may exchange information only at $t = s_k$. Our convention is that $t = s_k$ corresponds to the time at the *receiving agent*. The spatial constraints are defined through time-varying neighborhood sets, $\mathcal{N}_i(t) \subset \mathbb{N}_\nu \setminus \{i\}$, where each $\mathcal{N}_i(t)$ denotes the neighbors of agent i at time t. When combined, the collection of neighborhoods $\mathcal{N}_i[k]$ induces a directed graph at each $t = s_k$, denoted as $\mathcal{G}[k]$. This graph encodes the available communication channels, where for $t \notin \{s_k\}$ agents are privy only to their local information. These constraints are similar to those in [10], the change being that agents can locally measure only their output, $y_i(t)$, and not their entire state.

We consider the following objective in the spirit of [4].

$\mathcal{P}_\mathbf{s}$: Given $A_0 \in \mathbb{R}^{n\times n}$ such that $\operatorname{spec}(A_0) \cap \mathbb{C}_0 = \varnothing$ and its pure imaginary eigenvalues are all semi-simple, design u_i satisfying the spatio-temporal constraints and ensuring

$$\lim_{t\to\infty} \|x_i(t) - \mathrm{e}^{A_0 t} r_0\| = 0, \quad \forall i \in \mathbb{N}_\nu, \tag{3}$$

for some constant $r_0 \in \mathbb{R}^n$ and all initial conditions $x_i(0)$ of agents (1).

It shall be emphasized that the matrix A_0 does not represent a leader node, but rather *the shape* of required agreement trajectories. Because setting $A_0 = 0$ recovers the consensus problem and setting $A_0 = A$ recovers the classical synchronization [3], $\mathcal{P}_\mathbf{s}$ may be viewed as a generalization of both.

We address $\mathcal{P}_\mathbf{s}$ assuming that

$\mathcal{A}_1$: the triple (C, A, B) is stabilizable and detectable,

$\mathcal{A}_2$: there is $\bar{F}$ such that $A_0 = A + B\bar{F}$,

$\mathcal{A}_3$: there is a strictly increasing sub-sequence of sampling indices $\{k_i\}$ such that (i) the sampling intervals $s_{k_{i+1}} - s_{k_i}$ are uniformly bounded and (ii) $\cup_{k=k_i+1}^{k_{i+1}} \mathcal{G}[k]$ contains a directed rooted tree for all $i \in \mathbb{Z}_+$.

Assumption $\mathcal{A}_1$ is obviously needed for the existence of a stabilizing controller. The matching condition of $\mathcal{A}_2$ is required for the existence of a local feedback law guaranteeing (3) for each agent, at least for all j$\mathbb{R}$ modes of A_0. $\mathcal{A}_3$ is commonly employed in works related to coordination protocols over switching or time-varying graphs [2], [5], [11]. It ensures that information propagates throughout the entire network persistently across bounded sampling intervals, leaving no nodes forever detached from the rest of the network.

Introduce the signals

$$\bar{x} := \frac{1}{\nu}(\mathbb{1}'_\nu \otimes I_n)x \quad \text{and} \quad x_\delta := ((I_\nu - P_\mathbb{1}) \otimes I_n)x,$$

which may be interpreted as the *centroid* and *disagreement* signals, respectively, and satisfy $x = x_\delta + (\mathbb{1}_\nu \otimes I_n)\bar{x}$. The control objective (3) may then be equivalently decomposed into two separate objectives, one for the disagreement,

$$\lim_{t\to\infty} x_\delta(t) = 0, \tag{4a}$$

and one for the centroid,

$$\lim_{t\to\infty} \|\bar{x}(t) - \mathrm{e}^{A_0 t} r_0\| = 0. \tag{4b}$$

This decomposition shall be useful in the analysis.

III. Controller Architecture

A. State-feedback synchronization: review of [10]

We start with reviewing main results of [10], which solves the state-feedback version of $\mathcal{P}_\mathbf{s}$ and constitutes a base for the developments in this paper. This solution hangs on two key elements.

The first one is a solution of the unconstrained version of the problem, where the communication graph is complete and the information exchange is analog. This solution, requiring that each agent has full access to the centroid $\bar{x}$, acts at each $i \in \mathbb{N}_\nu$ as

$$u_i = F_\mathrm{d} x_i + (\bar{F} - F_\mathrm{d})\bar{x} \tag{5}$$

for $\bar{F}$ as in $\mathcal{A}_2$ and some $F_\mathrm{d} \in \mathbb{R}^{m\times n}$ such that

$$A_\mathrm{d} := A + BF_\mathrm{d}$$

is Hurwitz. The resulting closed-loop dynamics are

$$\dot{x}_i = A_\mathrm{d} x_i + B(\bar{F} - F_\mathrm{d})\bar{x} \tag{6}$$

or

$$\dot{x} = \big(I \otimes A_\mathrm{d} + P_\mathbb{1} \otimes (B(\bar{F} - F_\mathrm{d}))\big)x \tag{7}$$

in the aggregate form. The disagreement dynamics

$$\dot{x}_\delta = (I \otimes A_\mathrm{d})x_\delta$$

are then stable, i.e. satisfy (4a), and the centroid dynamics

$$\dot{\bar{x}} = (A + B\bar{F})\bar{x}(t) = A_0\bar{x}$$

satisfy (4b), as required.

The second key element of the approach of [10] is to emulate at each agent the whole "ideal" closed-loop dynamics (7) between sampling instances, when no information about neighboring agents is available. Emulated states are then used to produce local control signals during the intersample and the information transferred to neighbouring agents at sampling instances. Specifically, at the ith agent define the (νn)-dimensional signal μ_i such that $\mu_{ii} := (e_i' \otimes I)\mu_i = x_i$ and $\mu_{ij} := (e_j' \otimes I)\mu_i$ emulates (6) in the intersample as

$$\dot{\mu}_{ij} = A_{\mathrm{d}}\mu_{ij} + B(\bar{F} - F_{\mathrm{d}})\bar{\mu}_i, \quad j \in \mathbb{N}_\nu \setminus \{i\}, \tag{8}$$

where $\bar{\mu}_i := (1/\nu)(\mathbb{1}' \otimes I)\mu_i$ is the emulated centroid. The locally implemented control law is then the emulated counterpart of (5), i.e.,

$$u_i = F_{\mathrm{d}}\mu_{ii} + (\bar{F} - F_{\mathrm{d}})\bar{\mu}_i. \tag{9}$$

In this setup the emulated state μ_i satisfies

$$\dot{\mu}_i(t) = \big(I \otimes A_{\mathrm{d}} + P_{\mathbb{1}} \otimes (B(\bar{F} - F_{\mathrm{d}}))\big)\mu_i(t) \tag{10a}$$

during the intersample. At each sampling instance s_k neighboring agents transmit their emulated centroids and the emulator μ_i is updated by the diffusive jump

$$\mu_i(s_k^+) = \mu_i(s_k) - (\alpha_i \otimes I)\sum_{j \in \mathcal{N}_i[k]}\big(\bar{\mu}_i(s_k) - \bar{\mu}_j(s_k)\big), \tag{10b}$$

for some gain vector $\alpha_i \in \mathbb{R}^\nu$ such that $\mathbb{1}'\alpha_i = 1$ and $e_i'\alpha_i = 0$ (the latter condition reflects the fact that the ith component of μ_i is the actual agent state and thus cannot jump).

As shown in [10], the aggregate closed-loop system, which comprises all agents and all emulators, can be decomposed into two parts connected in series. The evolution of the aggregate centroids, $\bar{\mu} := \sum_{i=1}^{\nu} e_i \otimes \mu_i$, satisfies

$$\begin{cases} \dot{\bar{\mu}}(t) = (I \otimes A_0)\bar{\mu}(t) \\ \bar{\mu}(s_k^+) = ((I - (1/\nu)\Lambda[k]) \otimes I)\bar{\mu}(s_k) \end{cases}, \tag{11}$$

where $\Lambda[k]$ is the out-degree Laplacian matrix associated with $\mathcal{G}[k]$, and does not depend on the disagreements or the gains α_i. If $\mathcal{A}_3$ holds, then these dynamics asymptotically synchronize, in the sense (4b) with $\bar{x}$ replaced with $\bar{\mu}_i$ for all $i \in \mathbb{N}_\nu$. The $(\nu^2 n)$-dimensional aggregate disagreement signal $\mu_\delta := \sum_{i=1}^{\nu}(e_i \otimes (I - P_{\mathbb{1}}) \otimes I)\mu_i$ satisfies

$$\begin{cases} \dot{\mu}_\delta(t) = (I \otimes A_{\mathrm{d}})\mu_\delta(t) \\ \mu_\delta(s_k^+) = \mu_\delta(s_k) + ((B_{\mathrm{d}}[k]\Lambda[k]) \otimes I)\bar{\mu}(s_k) \end{cases} \tag{12}$$

for some $B_{\mathrm{d}}[k]$ which depend on α_i. Since $(\Lambda[k] \otimes I)\bar{\mu}(s_k)$ vanishes asymptotically whenever all $\bar{\mu}_i$ asymptotically synchronize so does $((B_{\mathrm{d}}[k]\Lambda[k]) \otimes I)\bar{\mu}(s_k)$. Therefore, $\mu_\delta \to 0$ and not only the emulators, but also the actual states synchronize.

Also note that we do not need to actually implement the whole (νn)-dimensional vector μ_i at the ith agent. Rather, we may only implement the n-dimensional centroid $\bar{\mu}_i$. By (10), this signal satisfies

$$\begin{cases} \dot{\bar{\mu}}_i(t) = A_0\bar{\mu}_i(t) \\ \bar{\mu}_i(s_k^+) = \bar{\mu}_i(s_k) - \dfrac{1}{\nu}\displaystyle\sum_{j \in \mathcal{N}_i[k]}\big(\bar{\mu}_i(s_k) - \bar{\mu}_j(s_k)\big) \end{cases} \tag{13}$$

and it is sufficient to implement (9).

B. What changes in the output-feedback case

If the state x_i is not measurable, we can no longer realize the dynamics (10) in the intersample, its μ_{ii} component is not available. The use of a state observer is a conventional solution in such situations. There is certain ambiguity in how exactly an observer may be incorporated into the emulation and information exchange procedures. The extension proposed below is motivated mainly by the relative simplicity of analyzing the closed-loop dynamics with it.

Because measurement channels of agents (1) are uncoupled, we construct the local, i.e., uncoupled, analog observer

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) - L(y_i(t) - C\hat{x}_i(t)) \tag{14}$$

for some L such that $A + LC$ is Hurwitz. The observer-based counterpart of (5) is then straightforward, we just need to replace x_i and $\bar{x}$ with $\hat{x}_i$ and the centroid of the observer states of all agents. It is then readily seen that the resulting disagreement x_δ and centroid $\bar{x}$ still satisfy (4), the only change in their evolution is the addition of the aggregate observer error, which vanishes exponentially.

Moving to the spatially distributed sampled-data setting, we now substitute the control law (9) with

$$u_i(t) = F_{\mathrm{d}}\hat{x}_i(t) + (\bar{F} - F_{\mathrm{d}})\bar{\mu}_i(t), \tag{15}$$

where the observed state $\hat{x}_i$ is local and can thus be implemented in continuous time and the emulated centroid $\bar{\mu}_i$ is still generated by the n-dimensional hybrid system (13).

Remark 3.1: It can be shown the centroid of the observer states of all agents, say $\check{x} := (1/\nu)\sum_{i=1}^{\nu}\hat{x}_i$, under the analog observer-based counterpart of (5) satisfies

$$\begin{bmatrix} \dot{\check{x}} \\ \dot{\bar{\epsilon}} \end{bmatrix} = \begin{bmatrix} A_0 & -LC \\ 0 & A + LC \end{bmatrix}\begin{bmatrix} \check{x} \\ \bar{\epsilon} \end{bmatrix},$$

where $\bar{\epsilon} := \bar{x} - \check{x}$ is the centroid observation error. This relation may be used for alternative forms of the emulator. Exploring these alternatives might involve some involved technicalities and is thus left for future research. ▿

IV. Synchronization Analysis

Combining plant (2) with the aggregate versions of (13)–(15), the closed-loop dynamics read as the analog flow

$$\begin{cases} \dot{x} = (I \otimes A)x + (I \otimes (BF_{\mathrm{d}}))\hat{x} + (I \otimes (B(\bar{F} - F_{\mathrm{d}})))\bar{\mu} \\ \dot{\hat{x}} = -(I \otimes (LC))x + (I \otimes (A_{\mathrm{d}} + LC))\hat{x} \\ \qquad\qquad + (I \otimes (B(\bar{F} - F_{\mathrm{d}})))\bar{\mu} \\ \dot{\bar{\mu}} = (I \otimes A_0)\bar{\mu} \end{cases}$$

between sampling instances with the jump

$$\begin{cases} x(s_k^+) = x(s_k) \\ \hat{x}(s_k^+) = \hat{x}(s_k) \\ \bar{\mu}(s_k^+) = (I - (1/\nu)\Lambda[k])\bar{\mu}(s_k) \end{cases}$$

at each s_k, where $\Lambda[k]$ is the Laplacian matrix associated with the network connectivity graph $\mathcal{G}[k]$ at s_k. Note that $\Lambda[k]\mathbb{1} = 0$ for all k.

Introduce now the emulation and observation errors

$$\varepsilon := x - \bar{\mu} \quad \text{and} \quad \epsilon := x - \hat{x},$$

respectively, the closed-loop dynamics can be rewritten in the more transparent form

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{\epsilon}(t) \\ \dot{\bar{\mu}}(t) \end{bmatrix} = \begin{bmatrix} I \otimes A_{\rm d} & -I \otimes (BF_{\rm d}) & 0 \\ 0 & I \otimes (A + LC) & 0 \\ 0 & 0 & I \otimes A_0 \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \epsilon(t) \\ \bar{\mu}(t) \end{bmatrix} \quad (16a)$$

(here $A_{\rm d} = A + BF_{\rm d}$ and $A_0 = A + B\bar{F}$), with the jumps

$$\begin{bmatrix} \varepsilon(s_k^+) \\ \epsilon(s_k^+) \\ \bar{\mu}(s_k^+) \end{bmatrix} = \begin{bmatrix} I & 0 & (1/\nu)\Lambda[k] \otimes I_n \\ 0 & I & 0 \\ 0 & 0 & (I - (1/\nu)\Lambda[k]) \otimes I_n \end{bmatrix} \begin{bmatrix} \varepsilon(s_k) \\ \epsilon(s_k) \\ \bar{\mu}(s_k) \end{bmatrix}. \quad (16b)$$

The signal ε is affected by both ϵ, via flow (16a), and $\bar{\mu}$, via jump (16b). At the same time, ϵ and $\bar{\mu}$ are completely decoupled. As such, we start the analysis with the last two signals.

It shall be clear that ϵ is an exponentially decaying signal. Therefore, it does not affect asymptotic properties of (16) and can be excluded from the analysis. Asymptotic behavior of $\bar{\mu}$ is more complex, as established by the following result (it is formulated in [10], but its proof is not presented there).

Lemma 4.1: If $\mathcal{A}_3$ holds true, then there is r_0 such that

$$\lim_{t\to\infty} \|\bar{\mu}(t) - \mathbb{1} \otimes \bar{\mu}_{\rm ss}(t)\| = 0,$$

where the n-dimensional $\bar{\mu}_{\rm ss}$ is such that $\bar{\mu}_{\rm ss}(t) = {\rm e}^{A_0 t} r_0$.

Proof: It is readily verified that $\bar{\mu}$ from (16) satisfies

$$\bar{\mu}(s_k + \tau) = {\rm e}^{I\otimes(A_0\tau)} \Biggl(\prod_{j=1}^{k} \Bigl(\Bigl(I - \frac{1}{\nu}\Lambda[j] \Bigr) \otimes I \Bigr) {\rm e}^{I\otimes(A_0 h_j)} \Biggr) \bar{\mu}_0$$

for all k and $0 < \tau \le h_{k+1}$, where $h_j := s_j - s_{j-1}$. Because ${\rm e}^{I\otimes(A_0 h_j)} = I \otimes {\rm e}^{A_0 h_j}$ and $N \otimes I$ and $I \otimes M$ commute for all compatibly dimensioned M and N, we have

$$\bar{\mu}(s_k + \tau) = \Biggl(\Biggl(\prod_{j=1}^{k} \Bigl(I - \frac{1}{\nu}\Lambda[j] \Bigr) \Biggr) \otimes {\rm e}^{A_0(s_k+\tau)} \Biggr) \bar{\mu}_0.$$

If the connectivity assumption $\mathcal{A}_3$ holds, then [2, Lem. 2.29 and 2.30] there exists some constant $q \in \mathbb{R}^\nu$ such that

$$\lim_{k\to\infty} \prod_{j=1}^{k} \Bigl(I - \frac{1}{\nu}\Lambda[j] \Bigr) = \mathbb{1} q'. \quad (17)$$

Therefore, if we choose $r_0 = (q' \otimes I)\bar{\mu}_0$ for $\bar{\mu}_{\rm ss}$, then

$$\lim_{k\to\infty} \bigl(\bar{\mu}(s_k + \tau) - \mathbb{1} \otimes \bar{\mu}_{\rm ss}(s_k + \tau) \bigr)$$
$$= \lim_{k\to\infty} \Biggl(\Biggl(\prod_{j=1}^{k} \Bigl(I - \frac{1}{\nu}\Lambda[j] \Bigr) - \mathbb{1} q' \Biggr) \otimes {\rm e}^{A_0(s_k+\tau)} \Biggr) \bar{\mu}_0 = 0$$

whenever ${\rm e}^{A_0 t}$ is bounded. The latter is guaranteed by the assumption that all pure imaginary eigenvalues of A_0 are semi-simple. ■

Remark 4.1: If the second condition of assumption $\mathcal{A}_3$ is replaced with the strong connectivity of $\cup_{k=k_i+1}^{k_{i+1}} \mathcal{G}[k]$, then (17) can be strengthened. Namely, the result of [12, Thm. 1] can be used to show the exponential convergence. In that case we no longer need the assumption that all pure imaginary eigenvalues of A_0 are semi-simple. In other words, we could afford synchronizing around polynomially diverging trajectories then. ▽

Thus, although the νn-dimensional signal $\bar{\mu}$ is not decaying, all its n-dimensional block components are asymptotically equivalent. This leads to the following result, which is the main result of this paper.

Theorem 4.2: If $F_{\rm d}$ and L are such that $A + BF_{\rm d}$ and $A + LC$ are Hurwitz and $\bar{F}$ is such that $A_0 = A + B\bar{F}$, then the control law defined by (15), (14), and (13) solves $\mathcal{P}_{\rm s}$ for any sampling sequence $\{s_k\}$ satisfying $\mathcal{A}_3$.

Proof: By Lemma 4.1 and the fact that $\Lambda[k]\mathbb{1} = 0$ we have that

$$\lim_{k\to\infty} \Lambda[k]\bar{\mu}_{\rm ss}(s_k) = 0,$$

The latter property implies that $\bar{\mu}$ asymptotically decouples from ε in (16b). Because the matrix $A_{\rm d}$ is Hurwitz and because ϵ vanishes exponentially, we have $\lim_{t\to\infty} \varepsilon(t) = 0$. This, in turn, yields

$$\lim_{t\to\infty} \|x(t) - \mathbb{1} \otimes \bar{\mu}_{\rm ss}(t)\| = 0,$$

which leads to (3). ■

A. Directly emulating the observers

It is worth emphasising that despite the simplicity of the control law defined by (15), (14), and (13) it is not merely a reapplication of the methodology from [10]. In fact, repeating the emulation process described in §-III-A with the simple change of $\mu_{ii} \equiv \hat{x}_i$ would result in a significantly different system. It can be shown that this process would result in the following counterpart of (16)

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{\epsilon}(t) \\ \dot{\bar{\mu}}(t) \end{bmatrix} = \begin{bmatrix} I \otimes \bar{\bar{A}} & -I \otimes M & 0 \\ 0 & I \otimes (A + LC) & 0 \\ 0 & -I \otimes (\frac{1}{\nu}LC) & I \otimes A_0 \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \epsilon(t) \\ \bar{\mu}(t) \end{bmatrix} \quad (16a')$$

(here $\bar{\bar{A}} = A + BF_{\rm d} + B\bar{F}$, $A_0 = A + B\bar{F}$, and $M = BF_{\rm d} + (1/\nu)LC$), with the jumps

$$\begin{bmatrix} \varepsilon(s_k^+) \\ \epsilon(s_k^+) \\ \bar{\mu}(s_k^+) \end{bmatrix} = \begin{bmatrix} I & 0 & (1/\nu)\Lambda[k] \otimes I_n \\ 0 & I & 0 \\ 0 & 0 & (I - (1/\nu)\Lambda[k]) \otimes I_n \end{bmatrix} \begin{bmatrix} \varepsilon(s_k) \\ \epsilon(s_k) \\ \bar{\mu}(s_k) \end{bmatrix}. \quad (16b')$$

Since now we do not have access to the actual states, this emulation leads to coupling between the flow of each $\bar{\mu}_i(t)$ and its observation error. Note that A_0 will generally have eigenvalues on the imaginary axis, hence not asymptotically stable, and that the stability of $\bar{\bar{A}}$ is not guaranteed. Thus while the estimation error, $\epsilon(t)$, is LTI and decays exponentially to zero, the same cannot be said for $\bar{\mu}(t)$ and $\varepsilon(t)$. In fact both are hybrid, non-autonomous, with an unstable flow and shift varying jumps.

By forgoing the straightforward derivation via emulation methodology we were able to decouple $\bar{\mu}$ for the state and the observer, cumulating with the simpler (16) rather than (16′). This significantly streamlined the proof and allowed us to avoid analyzing the aforementioned complicated hybrid system. The "price" we pay for the simplified analysis is that

$\bar{\mu}$ is now completely decoupled, and in fact can be thought of as some sort of exosystem without direct feedback from the agents.

Remark 4.2: It is worth mentioning that the results of Theorem 4.2 still hold for (16′), but the proof is significantly longer and more involved. Curiously, (16) consistently outperformed (16′) in simulation despite the latter having continuous feedback from the agents. This is subject to current research. ▽

V. ILLUSTRATIVE EXAMPLE

To illustrate the proposed sampled-data protocol, consider a simple system comprised of $\nu = 3$ agents described by (1) with

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & \end{array}\right] = \left[\begin{array}{cc:c} 4 & 9 & 2 \\ 1 & 4 & 1 \\ \hdashline 1 & 0 & \end{array}\right].$$

The goal is to synchronize to

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which corresponds to harmonic oscillations with the frequency 1 rad/sec. We choose

$$\bar{F} = -\begin{bmatrix} 2 & 4 \end{bmatrix}, \quad F_{\rm d} = -\begin{bmatrix} 7 & 1 \end{bmatrix}, \quad \text{and} \quad L = -\begin{bmatrix} 19 \\ 11 \end{bmatrix},$$

which satisfy the requirements of Theorem 4.2, assigning the spectrum of $A_{\rm d}$ to $\{-3, -4\}$ and that of $A + LC$ to $\{-5, -6\}$.

We assume that communication between agents is intermittent and asynchronous, meaning that each agent transmits only at a subset of sampling instances. At each sampling instance the connectivity graph $\mathcal{G}[k]$ is a union of any nonempty combination of the three graphs in Fig. 1. The

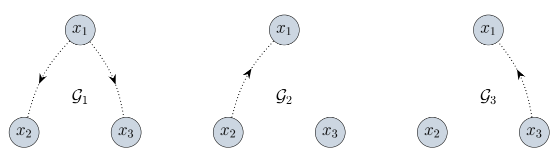


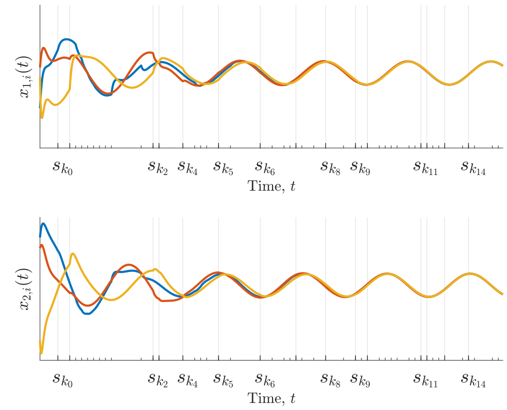
Fig. 1: The three possible graphs.

second condition of assumption $\mathcal{A}_3$ is equivalent in this case to the existence of a subsequence of sampling instances at which $\mathcal{G}[k]$ contains $\mathcal{G}_1$.

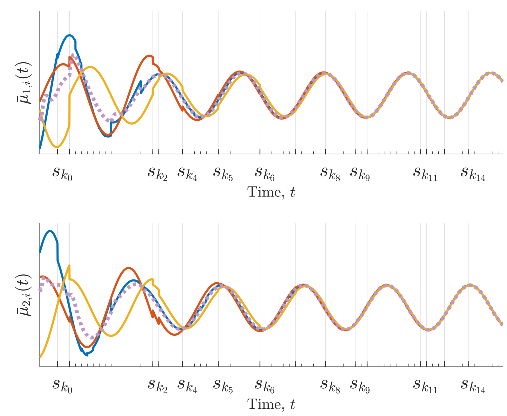
The simulation results, carried out over the time interval $t \in [0, 35]$, are presented in Fig. 2. The sampling instances, shown by abscissa ticks, are a random variable such that $s_{k+1} - s_k \in 0.45\,\mathbb{N}_5$. Major ticks indicate the sampling instances at which $\mathcal{A}_3$ is satisfied.

Fig. 2(a) presents the time evolution of the agents states. It can be seen that each component of the state converges to a common trajectory solving $\mathcal{P}_{\rm s}$.

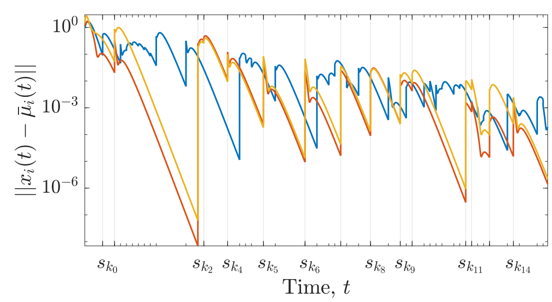
Fig. 2(b) portrays the time evolution of the emulated centroid states, while the real centroid, $\bar{x}(t)$, is plotted in dashed lavender line. This is to be expected, as the agents approach synchronization the only non-zero component of their state is the centroids. Coupled with the fact that Theorem 4.2



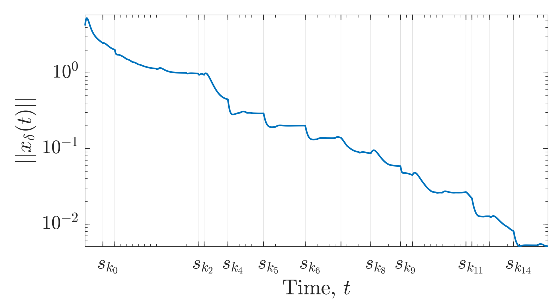
(a) Agents states.



(b) Centroid states.



(c) Components of ε.



(d) Aggregate disagreement.

Fig. 2: Simulations for the example.

established that $\varepsilon(t) \to 0$ as $t \to \infty$, this indicates that $\bar{\mu}(t) - \bar{x}(t) \to 0$ for all $i \in \mathbb{N}_3$.

Fig. 2(c) shows the norm of the components of ε, i.e. the signals $x_i - \bar{\mu}_i$ for $i \in \mathbb{N}_3$, on a logarithmic scale. When no information arrives, these signals decay exponentially fast because each agents tracks the local emulated centroid $\bar{\mu}_i$. When new information about neighboring agents is received, each $\bar{\mu}_i$ updates, as the centroids are drawn together by the jump map. This normally increases $\|x_i - \bar{\mu}_i\|$, for the local target jumps. Yet at the same time these targets at communicating agents approach each other, which is required to satisfy (4b).

Finally, Fig. 2(d) depicts the norm of x_δ on a logarithmic scale. In contrast to the components of ε from Fig. 2(c), the quantity in Fig. 2(d) decreases when new information is received. This behaviour indicates that the agent disagreements consistently decrease during information exchange, as required to satisfy condition (4a).

VI. Concluding remarks

In this work we considered the state synchronization problem for a class of LTI homogeneous agents without the ability to measure their own complete state. The agents exchange information asynchronously and over a time-varying network. Similar to our approach in [10], we proposed an emulation-based strategy that emulates the centroid dynamics of the ensemble. A key technical point is that each agent may implement a local observer and then transmit only the observer centroid to neighboring agents. This effectively decouples the emulators from the observers, greatly simplifying the dynamics. In future work we plan to extend this architecture to handle additional requirements, such as disturbance rejection, delays, and heterogeneous dynamics.

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