2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Mean-Covariance Steering of a Linear Stochastic System with Input Delay and Additive Noise

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***Abstract*—In this paper, we introduce a novel approach to solve the (mean-covariance) steering problem for a fairly general class of linear continuous-time stochastic systems subject to input delays. Specifically, we aim at steering delayed linear stochastic differential equations to a final desired random variable with given mean and covariance. We first establish a controllability result for these control systems, revealing the existence of a lower bound under which the covariance of the control system cannot be steered. This structural threshold covariance stems from a unique combined effect due to stochastic diffusions and delays. Next, we propose a numerically cheap approach to reach any neighbor of this threshold covariance in finite time. Via an optimal control-based strategy, we enhance the aforementioned approach to keep the system covariance small at will in the whole control horizon. Under some additional assumptions on the dynamics, we give theoretical guarantees on the efficiency of our method. Finally, numerical simulations are provided to ground our theoretical findings, showcasing the ability of our methods in optimally approaching the covariance threshold.**

***Index Terms*—Stochastic systems, Delay systems, Linear systems, Covariance steering.**

I. Introduction

Dynamical processes are often affected by disturbances stemming from various factors, such as imprecise measurements, parameter uncertainties, and external disturbances. Such disturbances may considerably alter the dynamics, as in several applications such as automated vehicle steering, traffic network control, or building heat regulation. Therefore, effectively mitigating these uncertainties is crucial to establish the reliability and security of such controlled systems. Stochastic Differential Equations (SDEs) provide broad and accurate modelization of a large class of uncertain systems [1]. Stochastic control enables the effective design of stabilizing controllers for SDEs, which are also robust against random fluctuations. In particular, such robustness may be reliably achieved by seeking controllers that keep the state variance relatively small, see [2], [3] , and references therein.

Another crucial consideration in the modeling of dynamical systems is the integration of delays into the dynamics. Delays in the system state or the control input stem from various sources, including physical constraints or transmission times [4]. When delays take large values, neglecting them may lead to critical stability issues, hindering robust control of the system dynamics. For deterministic systems, the generation of predictive state models has been suggested to handle delays [5], [6]. However, these methods often depend on prior knowledge of the system dynamics, a challenging limitation when dealing with stochastic systems. Certain methods have been proposed to handle systems with unknown perturbations [7], [8]. However, their implementation still requires some prior knowledge of the noise structure. Consequently, such approaches can hardly be adjusted to stabilize delayed SDEs.

All the aforementioned hindrances show the urgency in developing novel methods to efficiently and robustly control stochastic systems with delays. In this regard, some stabilization methods have been proposed [9], [10]. Yet, state covariance minimization, often key to mitigating uncertainty, is generally disregarded. To effectively compute strategies seeking minimal covariance, optimal control methods have been alternatively investigated. These approaches are supported by necessary conditions for optimality, which are, however, efficiently implementable only in specific settings [11]. Notably, in Linear Quadratic (LQ) settings, i.e., linear dynamics and quadratic costs, conditions for optimality may be efficiently solved by seeking solutions to Riccati-type ordinary differential equations [12], a numerically cheap method. Nevertheless, to the best of our knowledge, as efficient as they may be, these approaches do not support final state constraints. Importantly, estimates bounding the state covariance are generally underrated and not investigated, although these are crucial to establish the system's overall safety [13].

In this paper, by merging methodologies from deterministic delayed control and stochastic control, we start bridging these gaps. Specifically, for the first time, we introduce a novel approach to control linear SDEs with delays under guarantees, ensuring the state covariance is kept small throughout the control horizon. Our controls are uniquely cheap to implement numerically. Our contribution is threefold:

1) First, we investigate the controllability, in mean and covariance, of delayed SDEs. For this, we extend Arstein-type transformations [5], [14]. Unlike in the non-delayed settings, our analysis shows that the covariance of linear delayed SDEs cannot be steered to any symmetric definite positive matrix. In particular, the system can never be steered beyond some *minimal covariance* (minimal with respect to the order in the space of symmetric matrices). Still, under classical rank-type controllability conditions, we prove an open-loop control exists, enabling the steering of delayed SDEs close to this minimal covariance at will.
2) To achieve the design of numerically tractable control laws, we leverage covariance steering techniques [3], [15]. This enables computing feedback controls that steer linear delayed SDEs from an initial covariance to any final covariance as close as wanted to the aforementioned minimal covariance while minimizing the control effort.
3) Finally, we leverage optimal control techniques to minimize both the final state covariance and the covariance along the whole trajectory. Importantly, under some additional assumptions on the dynamics, we provide estimates of these minimal covariances in the autonomous case. These bounds show the covariance of delayed SDEs can be forced to evolve within any neighbor of the minimal covariance.

The paper is organized as follows. In Section II, we outline the problem formulation. Our first controllability result is stated and proved in Section III. In Section IV, we introduce a numerically efficient methodology to steer the system covariance. Upon these results, in Section V, we develop a technique to minimize the covariance throughout the control horizon, developing error estimates. Finally, in Section VI, we present numerical simulations on a real-world system model for building temperature control, subject to realistic temperature transmission delays and random external temperature fluctuations.

II. Problem formulation

A. Notations

Let us consider n and m two positive integers. We assume state variables take values in $\mathbb{R}^n$, while control variables take values in $\mathbb{R}^m$. We assume we are given a filtered probability space $(\Omega, \mathcal{F} \triangleq (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$. For the sake of clarity in the exposition and without loss of generality, from now on, we assume stochastic perturbations are due to a one-dimensional Wiener process W_t, which is adapted to the filtration $\mathcal{F}$. Let $T > 0$ be some given time horizon, while $0 < h < T$ is some fixed delay. For any $r \in \mathbb{N}$, we denote by $L^2_{\mathcal{F}}([0,T],\mathbb{R}^r)$ the set of square integrable processes $P : [0,T] \times \Omega \to \mathbb{R}^r$ that are adapted to $\mathcal{F}$. The spaces of semi-definite and definite positive symmetric matrices in $\mathbb{R}^n$ are denoted by $\mathcal{S}_n^+$ and $\mathcal{S}_n^{++}$, respectively. If $M(t)$ is an $L^\infty([0,T],\mathbb{R}^{n\times n})$ matrix function, we denote $\Phi_M(t,s)$ the fundamental matrix associated to it, which is by definition the unique solution to the system

$$\begin{cases} \dfrac{d\Phi_M}{dt}(t,s) & = M(t)\Phi_M(t,s), \\ \Phi_M(s,s) & = I. \end{cases}$$

If $X \in L^2_{\mathcal{F}}([0,T],\mathbb{R}^r)$, we denote by $\Sigma_X(\cdot)$ its covariance (matrix), which is defined as $\Sigma_X(t) \triangleq \mathbb{E}[(X(t) - \mathbb{E}[X(t)])(X(t) - \mathbb{E}[X(t)])^T] \in \mathcal{S}_n^+$.

B. Control system and assumptions

In this paper, we consider delayed-input SDEs of the form

$$\begin{cases} dX(t) = (A(t)X(t) + B(t)U(t-h) + r(t))\,dt + \sigma(t)dW_t, \\ X(0) = X_0, \\ U(s) = 0 \quad \text{for } s \in [-h, 0[, \end{cases} \tag{1}$$

where $X_0 \in \mathbb{R}^n$ is a fixed initial condition, whereas the control U lies in the control space $\mathcal{U} \triangleq L^2_{\mathcal{F}}([0,T],\mathbb{R}^m)$. The constant $h > 0$ is a positive delay acting on the control input. Given $X_T \in \mathbb{R}^n$ and $\Sigma_T \in \mathcal{S}_n^{++}$, our goal is to find $U \in \mathcal{U}$ that steers the corresponding solution X of (1) to $\begin{pmatrix} \mathbb{E}[X(T)] \\ \Sigma_X(T) \end{pmatrix} = \begin{pmatrix} X_T \\ \Sigma_T \end{pmatrix}$.We call this problem the *(mean-covariance) steering problem*. It boils down to finding the control that displaces the initial probability distribution of the state to a more desirable final distribution. To fulfill this control objective, we make the following assumption.

Assumption 1. *Throughout the paper, we assume the following properties hold true:*

- $A \in L^\infty([0,T],\mathbb{R}^{n\times n})$, $B \in L^\infty([0,T],\mathbb{R}^{n\times m})$, $r \in L^\infty([0,T],\mathbb{R}^n)$ *and* $\sigma \in L^\infty([0,T],\mathbb{R}^n)$. *In particular, these mappings are deterministic.*
- *The Grammian associated with the non-delayed deterministic system, defined as*
  $$G_\tau^{T+h} \triangleq \int_\tau^{T+h} \Phi_A(T+h,s)B(s)B(s)^T\Phi_A(T+h,s)^T dt,$$
  is invertible for all $0 < \tau < T+h$.

The first assumption on the dynamics is classical as it guarantees the well-posedness of the system [16]. The second assumption allows for the total controllability of the deterministic non-delayed system, meaning it can be controlled in any sub-interval of $[0, T+h]$ [17]. Furthermore, we assume that the initial state X_0 is deterministic. While our approach can be extended straightforwardly to Gaussian random variables (as in [15]), we maintain a null initial covariance for the sake of simplicity. The above assumptions are generic and usually satisfied by a wide class of systems.

C. System reduction

We can reduce the steering problem to an easier one, where X_0 and X_T are both 0 and where the drift $r(t)$ is the zero function. For this, let $X_r(t) \triangleq X_T\frac{t}{T} + X_0\frac{T-t}{T}$

and $\overline{X}(t) \triangleq X(t) - X_r(t)$. The difference $\overline{X}(t)$ follows the dynamic:

$$\begin{cases} d\overline{X}(t) = \left(A(t)\overline{X}(t) + B(t)U(t-h) + \overline{r}(t)\right)dt + \sigma(t)dW_t, \\ \overline{X}(0) = 0, \\ U(s) = 0 \quad \text{for } s \in [-h, 0). \end{cases} \tag{2}$$

where $\overline{r}(t) \triangleq r(t) + \dot{X}_r(t) - A(t)X_r(t)$. Steering the mean and the covariance of X from $\begin{pmatrix} X_0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} X_T \\ \Sigma_T \end{pmatrix}$ is thus equivalent to steering the mean and covariance of $\overline{X}(t)$ from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ \Sigma_T \end{pmatrix}$. Indeed, adding a deterministic term to the controller compensates for the effect induced by the drift and the change of variables [3]. More precisely, we consider $U(t) = U_{feed}(t) + U_{drift}(t)$, with a deterministic U_{drift}.By leveraging similar computations to the ones detailed in [17], it can be verified that we can choose U_{drift} such that controlling system (2) as required above boils down to steering the mean and covariance of the reduced system

$$\begin{cases} dX(t) = \left(A(t)X(t) + B(t)U_{feed}(t-h)\right)dt + \sigma(t)dW_t, \\ X(0) = 0, \\ U(s) = 0 \quad \text{for } s \in [-h, 0), \end{cases} \tag{3}$$

from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ \Sigma_T \end{pmatrix}$, with a control $U = U_{feed} \in \mathcal{U}$ of zero mean. Therefore, from now on, we only focus on this latter problem. Of particular interest is the case where Σ_T is the smallest possible.

III. Covariance steering: open-loop controls

In this section, we study which state covariances can be reached by the solution of (3) within a finite time frame. In particular, we show that there exists a lower bound on the state covariance under which (3) cannot be steered: we call this *minimal covariance*. We then establish the existence of open-loop controls that achieve state covariances that are arbitrarily close to this lower bound.

A. Extending the Artstein transformation

We achieve the aforementioned goal by leveraging the Artstein transformation [5] to our stochastic setting.

Definition 2 (Artstein transform). *Let X be the process solving equation* (2)*. The Artstein transform of X is the following process, which is adapted to $\mathcal{F}$,*

$$Y(t) \triangleq X(t) + \int_{t-h}^{t} \Phi_A(t, s+h)B(s+h)U(s)ds. \tag{4}$$

It can be easily checked that the Artstein transform solves the following non-delayed stochastic equation

$$\begin{cases} dY(t) = & \left(A(t)Y(t) + \overline{B}(t)U(t)\right)dt + \sigma(t)dW_t, \\ Y(0) = & 0, \end{cases} \tag{5}$$

where $\overline{B}(t) \triangleq \Phi_A(t, t+h)B(t+h)$. We can now apply known results from non-delayed stochastic control, e.g., [3], [16], to the Artstein transform Y and steer (3) as desired. However, it is unclear how to compute the covariance of X, which we aim to estimate once the covariance of Y is available. For this, we make use of the following:

Lemma 3. *Let X be the process solving equation* (2)*, and Y its Artstein transform. Then for all $t \in [h, T]$*

$$X(t) = \Phi_A(t, t-h)Y(t-h) + \int_{t-h}^{t} \Phi_A(t,s)\sigma(s)dW_s. \tag{6}$$

Proof. The proof of the lemma is omitted for conciseness. More details can be found on the online version [18]. □

Equation (6) provides important insights for the controllability of the process X. In particular, it states $X(t)$ can only be controlled through $Y(t-h)$, in that the additional noise term $\int_{t-h}^{t} \Phi_A(t,s)\sigma(s)dW_s$ cannot be controlled. Equation (6) can be further manipulated to express Σ_X as a function of Σ_Y as explained in the following lemma.

Lemma 4. *Let X be the process solving equation* (3)*, and let Y be its Artstein transform. For all $t \in [h, T]$,*

$$\Sigma_X(t) = \Phi_A(t, t-h)\Sigma_Y(t-h)\Phi_A(t, t-h)^T + \Sigma_{\min}(t), \tag{7}$$

where $\Sigma_{\min}(t) \triangleq \int_{t-h}^{t} \Phi_A(t,s)\sigma(s)\sigma(s)^T\Phi_A(t,s)^T ds$.

Proof. Due to space constraints, the proof of this lemma is omitted here but provided in the online version [18]. □

The covariance of the state X is made of two components: one determined by the covariance of Y, which we may control, and another determined by the covariance of $\int_{t-h}^{t} \Phi_A(t,s)\sigma(s)dW_s$ which remains unaffected by the control term. Consequently, the state covariance is inherently greater than the latter covariance, presenting a limitation as there is no means to reduce it.

B. Controllability of the Artstein transform

Our goal is now to minimize the covariance as much as possible and propose control strategies that can approach the lower covariance bound as closely as wanted. The following result is classical [19, Theorem 13]:

Theorem 5 (Controllability of non-delayed SDEs)**.** *If the Grammian $\overline{G}_\tau^T$ associated to the deterministic part of equation* (5)*, defined as*

$$\overline{G}_\tau^T \triangleq \int_\tau^T \Phi_A(T,s)\overline{B}(s)\overline{B}(s)^T\Phi_A(T,s)^T ds \tag{8}$$

is invertible for all τ in $[0, T]$, then for every $\Sigma_T \in \mathcal{S}_n^{++}$ there exists $U \in \mathcal{U}$ such that the solution Y to (5) *associated with U is such that $\mathbb{E}[Y(T-h)] = 0$ and $\Sigma_Y(T-h) = \Sigma_T$.*

We may combine Lemma 4 with Theorem 5 to infer open-loop controllability of the original control system (3).

Theorem 6 (Controllability of delayed SDEs)**.** *Let $\Sigma_T \in \mathcal{S}_n^{++}$. Under Assumption 1, there exists $U \in \mathcal{U}$ such that the solution X to* (2) *associated with U verifies $\mathbb{E}[X(T)] = 0$ and $\Sigma_X(T) = \Sigma_T + \Sigma_{\min}(T)$.*

Proof. Details of the proof are provided in the online version [18]. □

Thanks to Theorem 6, we can steer the covariance of the state X to a final covariance that can be arbitrarily close to the limit $\Sigma_{\min}(T)$, by selecting Σ_T arbitrarily small. The control derived in this section to steer the system is possibly a general open-loop stochastic control, posing challenges for numerical implementation [20]. Consequently, we need to explore the possibility of addressing the steering problem using a feedback controller. Another limitation to keep in mind is that, despite its theoretical advantages, Artstein's transformation encounters some obstacles in practice. Discretizing the integral during its computation can potentially render the closed-loop system unstable, as discussed in [21]. To address this issue, solutions such as filtering have been proposed to implement the controller safely [21]. However, exploring these solutions further is beyond the scope of our study.

IV. Covariance steering: close-loop controls

In the previous section, we solved the steering problem by means of open-loop controls. Such control laws are known to be very expensive to compute. To fix this hindrance, we show in this section, the existence of tractable feedback controls that solve the steering problem. We achieve this result by leveraging and combining recent advances in covariance steering techniques, e.g., [2], [3], with the previous Artstein transformation. In particular, the covariance of solutions to equation (3) can be steered to $\Sigma_T + \Sigma_{\min}$, with $\Sigma_T \in \mathcal{S}_n^{++}$, via feedback controls.

In [3], the authors prove that under the controllability conditions of Theorem 5, there exists a unique feedback control $U \in \mathcal{U}$ that steers the covariance of Y solution to (5) to a desired covariance, while minimizing the functional cost $J(U) \triangleq \mathbb{E}\left[\int_0^T U(t)^T R U(t) dt\right]$, where $R \in \mathcal{S}_m^{++}$. We can show the existence of tractable feedback controls that solve the steering problem by combining [3] with our previous results as follows:

Theorem 7. *Let $\Sigma_T \in \mathcal{S}_n^{++}$ and Y denote solutions to* (5)*. Under Assumption 1, there exists a unique feedback control U^* of the form*

$$U^*(t) = -R(t)^{-1}\overline{B}(t)^T \Pi(t) Y(t), \tag{9}$$

that minimizes the functional cost J under the constraint $\Sigma_Y(T-h) = \Sigma_T$. Moreover, the gain $\Pi(t)$ is the solution to the following Riccati-type ordinary differential equation

$$\begin{cases} \dot{\Pi} & = -A^T\Pi - \Pi A + \Pi B R^{-1} B^T \Pi \\ \Pi(0) & = \Pi_0, \end{cases} \tag{10}$$

where Π_0 is the unique solution of the system

$$\Sigma_T = f(\Pi_0), \tag{11}$$

where f is a homeomorphism[1].

Proof. The proof of this theorem is a direct consequence of [3, Lemma 6], whose assumptions are satisfied thanks to Theorem 6. □

Similarly to what has been done in the previous section, once we can control the state covariance of the process Y, we can leverage the relation given by Lemma 4 and steer the covariance of X to any symmetric positive definite matrix above the lower bound Σ_M.

This method still presents some limitations, mainly:

- Computing the controller requires solving equation (11), which, although it can be done through root-finding algorithms [3], can be expensive as the function f is not easy to compute.
- There are no guarantees in controlling the value of the covariance along the trajectory.

In the next section, we therefore introduce an optimal control-based approach to minimizing the covariance along the trajectory.

[1]For conciseness, we do not provide the full formula of f, see [3] for more details.

V. Global-in-time covariance steering via close-loop controls

In the previous section, we showed the existence of tractable feedback controls that solve the steering problem. Such control laws may not guarantee that the covariance remains small during the whole control horizon $[0, T]$. In this section, our objective is to establish an efficiently implementable control strategy that keeps the covariance of the state small along the whole trajectory and not just at the final time. For this, we propose to compute controllers that enable approaching any neighbor of the covariance threshold via techniques from Linear Quadratic (LQ) control [16]. The goal consists of minimizing the quadratic functional cost J_R, defined as

$$J_R(U) \triangleq \mathbb{E}\left[\int_h^T X(t)^T Q(t) X(t) dt + X(T)^T G X(T)\right] + \mathbb{E}\left[\int_0^T U(t)^T R(t) U(t) dt\right], \quad X \text{ solves SDE (2).} \tag{12}$$

where $Q \in L^\infty([0,T], \mathcal{S}_n^+)$, $G \in \mathcal{S}_n^+$ and $R \in L^\infty([0,T], \mathcal{S}_n^{++})$. This functional cost penalizes the covariance along the trajectory, the final covariance, and the control effort. Our problem thus states:

$$\min_{U \in \mathcal{U}} J_R(U). \tag{13}$$

To effectively solve (13), using equation (6), we replace the process X with the process Y in J_R. The benefit of this transformation is that optimal control techniques for non-delayed systems may be leveraged, e.g., [16]. Under some additional assumptions on the dynamics, we prove next that when the penalization on the control effort via R approaches zero, the control solutions to (13) enable the covariance of the delayed SDE to track any neighbor of the threshold covariance.

A. Optimal control for delayed SDEs.

We may change the variable in (13) as follows. For $t > h$, let us denote by $V_{min,Q}(t)$ the following positive real number:

$$V_{min,Q}(t) \triangleq \int_{t-h}^t \sigma(s)^T \Phi_A(t,s)^T Q(t) \Phi_A(t,s) \sigma(s) ds. \tag{14}$$

Lemma 8. *The cost function J_R can be written in terms of the state Y as follows:*

$$J_R(U) = \mathbb{E}\left[\int_0^{T-h} Y(t)^T \overline{Q}(t) Y(t) dt + Y(T-h)^T \overline{G} Y(T-h)\right] + \mathbb{E}\left[\int_0^T U(t)^T R(t) U(t) dt\right] + \int_h^T V_{min,Q}(t) dt + V_{min,G}(T),$$

where $\overline{Q}(t) \triangleq \Phi_A(t+h,t)^T Q(t+h) \Phi_A(t+h,t)$ and $\overline{G} \triangleq \Phi_A(T+h,T)^T G \Phi_A(T+h,T)$ are symmetric positive matrices.

Proof. Due to space constraints, the proof of this lemma is provided in the online version [18]. □

At this step, we can apply the results from [16, Chapter 6.2] to obtain optimal controls that minimize (12) in the form of feedback controls:

Theorem 9. *The control U^* solution to(13) is given by*
$$U^* = -R(t)^{-1}\overline{B}(t)^T P(t)Y(t),$$
with $P(\cdot)$ solution to the deterministic Riccati equation
$$\begin{cases} \dot{P} = -A^T P - PA - \overline{Q} + P\overline{B}R^{-1}\overline{B}^T P \\ P(T-h) = \overline{G}. \end{cases} \tag{15}$$

Proof. Details of the proof are provided in the online version [18]. □

Thanks to Lemma 8, we may note that (12) is lower-bounded by the quantity $\int_h^T V_{min,Q}(t)dt + V_{min,G}(T)$. We will prove that, under some additional assumptions, when R tends to zero it holds that $J_R(U^*) \underset{R\to 0}{\to} V_{min}$, meeting our desired goal.

B. Convergence equivalence

From now on, we select the specific control weight $R(t) = \rho I$, where $\rho > 0$ is a user-defined parameter that characterizes the control effort in (12). We accordingly denote the associated cost functional J_R by J_ρ.We denote by U_ρ the corresponding optimal control, whose existence is granted by Theorem 9. Finally, we denote X_ρ the corresponding state trajectory. Our objective is to give some conditions on the dynamics under which the weighted variance of X_ρ approaches the minimal quantity as ρ approaches zero. The following lemma plays a crucial role in achieving this goal:

Lemma 10. *Let U_n be a sequence of controls in $\mathcal{U}$ such that, for any matrix S in $\mathcal{S}_n^+$, the associated state Y_n obtained through equation* (5) *verifies*
$$\forall t \in [0, T-h], \quad \mathbb{E}\left[Y_n(t)^T S Y_n(t)\right] \underset{n\to\infty}{\to} 0. \tag{16}$$
Then,
$$J_\rho(U_\rho) \underset{\rho\to 0}{\to} \int_h^T V_{min,Q}(t)dt + V_{min,G}(T). \tag{17}$$

Proof. Details of the proof can be found in the online version [18]. □

Establishing the convergence (17) consists of proving the existence of a sequence of controls that steers the state variance toward the minimum threshold. However, showing the existence of such sequence is not straightforward, as larger controllers may force the integral of the weighted variance to divergence. Below, we provide examples of SDEs for which finding such a sequence of controls is feasible.

C. Sufficient conditions for convergence equivalence

1) Pole placement with a normal matrix: Let us consider the following assumption:

Assumption 11. *The followings hold true:*

1) *The matrices A and B are time independent.*
2) *There exists a sequence of matrices $K_n \in \mathbb{R}^{m\times n}$ such that every $H_n \triangleq A - \overline{B}K_n$ is normal, and $\max(\Re(Sp(H_n))) \le -n$.*

Lemma 12. *Let $S \in \mathcal{S}_n^+$. Under Assumption 11, the sequence of controls $U_n(t) = K_n Y_n(t)$ satisfies*
$$\mathbb{E}\left[Y_n(t)^T S Y_n(t)\right] \le \frac{\|S\|\|\sigma\|_\infty^2}{2n}. \tag{18}$$

Proof. The proof is provided in the online version [18]. □

Characterizing systems that satisfy Assumption 11 can be difficult. Yet, (18) is in particular satisfied by the reasonably large class of fully actuated systems, as shown in the next section. Note that assuming full actuation is not particularly restricting, in that, if this assumption is not satisfied, one can prove the existence of random states that can not be reached by any open-loop control, see, e.g., [22].

2) Fully actuated systems:

Assumption 13. *$B(t)$, $t \in [0, T-h]$, has rank n.*

Lemma 14. *Let $S \in \mathcal{S}_n^+$. Under Assumption 13, there exists a matrix $K(t) \in \mathbb{R}^{m\times n}$ such that $\overline{B}(t)K(t) = I$. In particular, the control sequence defined by $U_n(t) = -K(t)(A(t) + nI)Y_n(t)$ yields the estimate* (18).

Proof. Again, the proof is provided in the online version [18]. □

Summing up, thanks to Theorem 9, we come up with a closed-loop-control-based method to steer the delayed stochastic system while keeping the variance-in-time of the state close to the threshold variance at will. Next, we demonstrate the efficiency of this method via numerical simulations.

VI. Numerical results

To validate our theoretical findings, we implemented our optimal control-based method presented in Section V to regulate the temperature of a building in realistic settings, where the heat source delivers thermal energy up to some delay. We leveraged and enhanced the dynamical models outlined in [23]. In particular, we made these models more realistic by adding noise. This originates from various sources: 1) the stochastic nature of external temperature fluctuations (modeled using an Ornstein-Uhlenbeck process, see, e.g., [24]), and 2) the unpredictable usage of the building (modeled via the coefficient σ_i below). Then, the dynamics are given by
$$dT(t) = \left[\begin{pmatrix} -R_e & R_e \\ 0 & -\theta \end{pmatrix} T(t) + \begin{pmatrix} R_u \\ 0 \end{pmatrix} U(t-h)\right] dt + \begin{pmatrix} -R_e T_{eq} \\ \theta T_p(t) + \dot{T}_p(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_i \\ \sigma_e \end{pmatrix} dW_t,$$
where $R_e = 5\ 10^{-4}, R_u = 2\ 10^{-4}, \theta = 3.5\ 10^{-4}, T_{eq} = 20, \sigma_e = 0.1, \sigma_i = 0.05$, and with baseline (i.e., "predicted") external temperature $T_p(t) = 5 + 5\cos(0.004t)$. The simulation horizon is five days. This enables to stress test our control strategy in mitigating the fluctuations of the external temperature, that are induced by the day-night cycle. The threshold variance of the temperature of the building amounts to $V_{min} = 1.76$.

Figure 1 shows the trajectory of the system using our optimal control-based feedback. The temperature is efficiently stabilized even under poor knowledge of the external temperature. Figure 2 shows the evolution of the variance of the temperature of the building. Our method enables to successfully track the threshold variance under controls with high gains, as granted by Theorem 6.

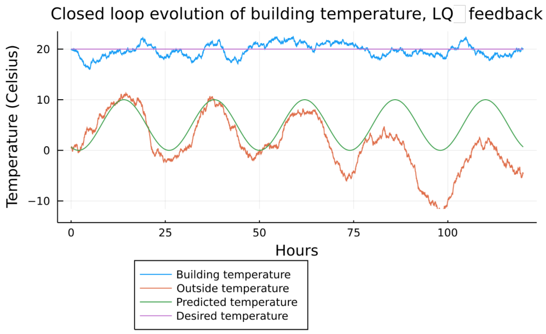

Fig. 1. Optimal control-based feedback trajectory with high gain.

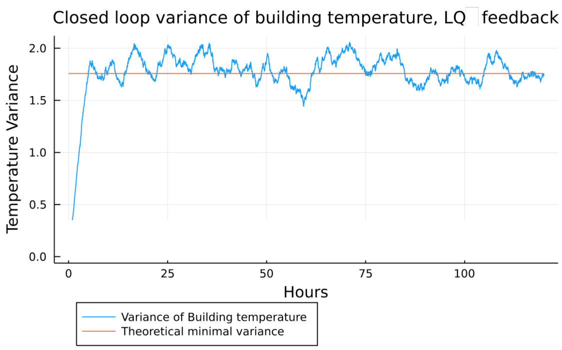

Fig. 2. State variance through time and lower bound.

VII. Conclusion

In this paper, we solved the steering problem for a general class of linear SDEs subject to delays. Our objective consisted of steering in finite time the state of the system to a given final probability distribution. In particular, we proposed a method to minimize the covariance of the final state. We did so by leveraging the Artstein transformation, thanks to which we derived a "predictor" of the state that tracks a non-delayed SDE. We then established a linear relationship between the predictor covariance and the original state covariance. In particular, this relationship revealed the existence of a structural minimal covariance below which the system cannot be steered, and which is essentially due to the presence of the delay. Nevertheless, we proved the system can be steered to any covariance that is greater than this minimal covariance. Finally, we introduced an optimal control-based approach to minimize the system covariance throughout the whole control horizon, computing upper bounds for this minimal variance.

Several exciting research directions are listed hereafter. Extending our approach to linear SDEs with multiplicative noise would enable the modeling of more sophisticated systems. The main challenge would lie in deriving an explicit expression for the variance threshold. Additionally, it would be interesting to explore systems with multiple input delays. One could build upon previous studies on optimal control of ODEs with multiple input delays, and then try to extend such results to more general SDEs.

References

[1] R. Bonalli, T. Lew, and M. Pavone, "SCP for non-linear stochastic optimal control," *ESAIM: COCV*, vol. 28, p. 64, 2022.

[2] Y. Chen, T. T. Georgiou, and M. Pavon, "Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part II," *IEEE Transactions on Automatic Control*, vol. 61, pp. 1170–1180, May 2016.

[3] F. Liu and P. Tsiotras, "Optimal Covariance Steering for Continuous-Time Linear Stochastic Systems With Additive Noise," Feb. 2023.

[4] S.-I. Niculescu, E. I. Verriest, L. Dugard, and J.-M. Dion, "Stability and robust stability of time-delay systems: A guided tour," in *Stability and Control of Time-delay Systems* (L. Dugard and E. I. Verriest, eds.), pp. 1–71, Springer, 1998.

[5] Z. Artstein, "Linear systems with delayed controls: A reduction," *IEEE Transactions on Automatic Control*, vol. 27, pp. 869–879, Aug. 1982.

[6] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*. SIAM, Sept. 2008.

[7] V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, and A. Chriette, "New predictive scheme for the control of LTI systems with input delay and unknown disturbances," *Automatica*, vol. 52, pp. 179–184, Feb. 2015.

[8] T. Alves Lima, V. V. Viana, B. C. Torrico, F. G. Nogueira, and D. d. S. Madeira, "Newton-series-based observer–predictor control for disturbed input-delayed discrete-time systems," *Automatica*, vol. 146, p. 110624, Dec. 2022.

[9] L. Shaikhet, *Lyapunov Functionals and Stability of Stochastic Functional Differential Equations*. Heidelberg: Springer International Publishing, 2013.

[10] F. Cacace, A. Germani, C. Manes, and M. Papi, "Predictor-Based Output-Feedback Control of Linear Stochastic Systems With Large I/O Delays," *IEEE Transactions on Automatic Control*, vol. 66, pp. 307–313, Jan. 2021.

[11] B. Øksendal and A. Sulem, "A maximum principle for optimal control of stochastic systems with delay, with applications to finance," pp. 64–79, 2001. ISSN: ,.

[12] X. Liang, J. Xu, and H. Zhang, "Solution to stochastic LQ control problem for Itô systems with state delay or input delay," *Systems & Control Letters*, vol. 113, pp. 86–92, Mar. 2018.

[13] Y. Wang and M. P. Chapman, "Risk-averse autonomous systems: A brief history and recent developments from the perspective of optimal control," *Artificial Intelligence*, vol. 311, p. 103743, Oct. 2022.

[14] A. Manitius and A. Olbrot, "Finite spectrum assignment problem for systems with delays," *IEEE Transactions on Automatic Control*, vol. 24, pp. 541–552, Aug. 1979.

[15] Y. Chen, T. T. Georgiou, and M. Pavon, "Optimal Steering of a Linear Stochastic System to a Final Probability Distribution, Part I," *IEEE Transactions on Automatic Control*, vol. 61, pp. 1158–1169, May 2016.

[16] J. Yong and X. Y. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*. No. 43 in Applications of mathematics, New York: Springer, 1999.

[17] J.-M. Coron, *Control and Nonlinearity*, vol. 136 of *Mathematical Surveys and Monographs*. Providence, Rhode Island: American Mathematical Society, Aug. 2009.

[18] G. Velho, R. Bonalli, J. Auriol, and I. Boussaada, "Mean-Covariance Steering of a Linear Stochastic System with Input Delay and Additive Noise," Nov. 2023. arXiv version.

[19] N. Mahmudov, "Controllability of linear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 46, pp. 724–731, May 2001.

[20] N. Touzi, "Introduction," in *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE* (N. Touzi, ed.), pp. 1–4, Springer, 2013.

[21] S. Mondie and W. Michiels, "Finite spectrum assignment of unstable time-delay systems with a safe implementation," *IEEE Trans. on Automatic Control*, vol. 48, pp. 2207–2212, Dec. 2003.

[22] Y. Wang, D. Yang, J. Yong, and Z. Yu, "Exact controllability of linear stochastic differential equations and related problems," *Mathematical Control and Related Fields*, vol. 7, pp. 305–345, Mar. 2017. Publisher: Mathematical Control and Related Fields.

[23] S. van der Zwan and I. Pothof, "Operational optimization of district heating systems with temperature limited sources," *Energy and Buildings*, vol. 226, p. 110347, Nov. 2020.

[24] S. Prabakaran, I. C. Garcia, and J. U. Mora, "A temperature stochastic model for option pricing and its impacts on the electricity market," *Economic Analysis and Policy*, vol. 68, pp. 58–77, Dec. 2020.