2021 43rd Annual International Conference of the
IEEE Engineering in Medicine & Biology Society (EMBC)
Oct 31 - Nov 4, 2021. Virtual Conference

A new Framework for the Spectral Information Decomposition of Multivariate Gaussian Processes

Yuri Antonacci[1], Ludovico Minati[2,3], Gorana Mijatovic[4] and Luca Faes[5]

Abstract— **Different information-theoretic measures are available in the literature for the study of pairwise and higher-order interactions in multivariate dynamical systems. While these measures operate in the time domain, several physiological and non-physiological systems exhibit a rich oscillatory content that is typically analyzed in the frequency domain through spectral and cross-spectral approaches. For Gaussian systems, the relation between information and spectral measures has been established considering coupling and causality measures, but not for higher-order interactions. To fill this gap, in this work we introduce an information-theoretic framework in the frequency domain to quantify the information shared between a target process and two sources, even multivariate, and to highlight the presence of redundancy and synergy in the analyzed dynamical system. Firstly, we simulate different linear interacting processes by showing the capability of the proposed framework to retrieve amounts of information shared by the processes in specific frequency bands which are not detectable by the related time-domain measures. Then, the framework is applied on EEG time series representative of the brain activity during a motor execution task in a group of healthy subjects.**

I. INTRODUCTION

Over the last decade, information-theoretic measures have been extensively used to characterize complex dynamic systems in very different areas such as neuroscience, financial time series and physiology. The framework of information dynamics provides a unifying set of measures which allow to quantify the amount of information produced and stored in a complex system, transferred from a "source" to a "target" and modified as a consequence of the interaction between the sources and the target [1]. Though being intrinsically model-free, the measures of information dynamics can be often computed through the identification of a Vector Autoregressive (VAR) model by exploiting their formulation for Gaussian variables [2]. Specifically, while the information shared among variables can be used to evaluate their coupling, measures of information modification allow to investigate the nature of the interactions among multiple variables. In particular, two sources are redundant if they provide the same or overlapping information about the target, while they are synergistic if their combination provides information that is not available from either of them considered alone [3]. Classical approaches to evaluate synergy and redundancy refer to the decomposition of interaction information and have been successfully used to study cardiovascular and cardiorespiratory interactions [3], as well as for the analysis electroencephalographic (EEG) recordings in a variety of physiological states [4].

While interaction information measures have been introduced in the time-domain, a thorough spectral formulation leading to assess synergistic and redundant interactions between scalar and multivariate processes as a function of frequency is still missing. In many different context, such as the study of brain dynamics from EEG signals the frequency domain analysis of pairwise coupling is usually performed through spectral measures such as the coherence and the block coherence [5] which, however, do not provide information about higher-order interactions like redundancy and synergy. To fill this gap, the present work introduces a new framework for the spectral analysis of the information shared among three blocks on interacting processes (i.e. two sources and a target). The measures in the framework are derived in a simple way from the cross-spectral matrix of a multivariate process. Here, the measures are tested first in a theoretical example, and then in real EEG signals recorded during a motor execution task.

II. MEASURES OF LINEAR ASSOCIATION AND THEIR SPECTRAL DEFINITION

Let us consider two-zero mean stationary multivariate stochastic processes X and Y of dimension M_X and M_Y, and define the random variables that sample the processes at the time n as $X_n = [X_{1,n} \cdots X_{M_X,n}]^T$ and $Y_n = [Y_{1,n} \cdots Y_{M_Y,n}]^T$. In the frequency domain, the multivariate processes can be described considering the power spectral density (PSD) of each constituent scalar process defined as the Fourier Transform (FT) of the autocorrelation function of the process (e.g., $P_{X_1}(\omega) = FT\{r_{X_1}(k)\}, r_{X_1}(k) = \mathbb{E}[X_{1,n}X_{1,n-k}]$) and the cross spectral density between two scalar processes defined as the FT of their cross-correlation function (e.g., $P_{X_1X_2}(\omega) = FT\{r_{X_1X_2}(k)\}, r_{X_1X_2}(k) = \mathbb{E}[X_{1,n}X_{2,n-k}]$), where $\omega \in [-\pi, \pi]$ is the normalized sampling frequency ($\omega = 2\pi\frac{f}{f_s}$ with $f \in [-\frac{f_s}{2}, \frac{f_s}{2}]$, f_s sampling frequency). The spectral densities are collected in the individual PSD matrices $P_X(\omega), P_Y(\omega)$ of dimensions $M_X \times M_X$ and $M_Y \times M_Y$, and in the joint PSD matrix of dimension $(M_X + M_Y) \times (M_X + M_Y)$, $P_{[XY]}(\omega) \triangleq [P_X(\omega)P_{XY}(\omega); P_{YX}(\omega)P_Y(\omega)]$, where the semicolon delimiter stands for rows separation. $P_{XY}(\omega)$ is the

*This work was supported from Ministero dell'Istruzione, dell'Università e della Ricerca PRIN 2017 (PRJ-0167), "Stochastic forecasting in complex systems"

[1]Y. A. is with Department of Physics and Chemistry, University of Palermo, Palermo, Italy e-mail: yuri.antonacci@unipa.it
[2,3]L. M. is with the Center for Mind/Brain Sciences (CIMeC), University of Trento, Trento, Italy and with the Institute of Innovative Research, Tokyo Institute of Technology, Yokohama, Japan
[4] G. M. is with the Faculty of Technical Sciences, University of Novi Sad, Serbia
[5] L. F. is with the Department of Engineering, University of Palermo, Palermo, Italy

$(M_X \times M_Y)$-dimensional matrix that contains the cross PSD between X_i and Y_j as (i,j) element, and $P_{YX}(\omega) = P^*_{XY}(\omega)$ (* stands for Hermitian transpose). A spectral measure of linear association between X and Y can be defined as:

$$f_{X;Y}(\omega) \triangleq \log \frac{|P_X(\omega)||P_Y(\omega)|}{|P_{[XY]}(\omega)|}, \tag{1}$$

where $|\cdot|$ stands for matrix determinant.The measure (1) has an information-theoretic interpretation since, for Gaussian processes, it constitutes the integral of the spectral representation of the Mutual Information Rate (MIR), a well-known information-theoretic measure of the total degree of association between the two processes [6]:

$$MIR_{X;Y} = \frac{-1}{4\pi} \int_{-\pi}^{\pi} \log \frac{|P_{[XY]}(\omega)|}{|P_X(\omega)||P_Y(\omega)|} \mathrm{d}\omega. \tag{2}$$

A similar decomposition was proposed by Geweke [7] who introduced the time domain measure of linear dependence:

$$F_{X;Y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{X;Y}(\omega)\,\mathrm{d}\omega, \tag{3}$$

which obtains an information-theoretic interpretation considering (2), i.e. $F_{X;Y} = 2MIR_{X;Y}$.We note also that the measure of linear association (1) is directly linked to the so-called block coherence [6], a frequency domain measure of coupling which extends to the multivariate case the very-well known magnitude squared coherence:

$$C^{(b)}_{X;Y} \triangleq 1 - \frac{|P_{[XY]}(\omega)|}{|P_X(\omega)||P_Y(\omega)|}, \tag{4}$$

By combining (1) and (3) it is easy to show that $f_{X;Y} = -\log(1 - C^{(b)}_{X;Y})$. Given this interpretation, and using the natural logarithm, the quantity $F_{X;Y}$ is measured in natural units (nats), and the spectral quantity $f_{X;Y}(\omega)$ is measured in nats/rad.

Now, let us consider a multivariate process $Z = [XY]^T = [X_1X_2Y]^T$, where Y is assumed as "target" and X_1 and X_2 are assumed as "sources". Both the target and the sources can be multivariate processes, so that we study the interactions between "blocks" of time series considered as realizations of these processes. Let M, M_Y, M_1, and M_2 be the dimensions of Z, Y, X_1 and X_2 ($M = M_Y + M_1 + M_2$). Then, to study the interactions in the frequency domain, we consider the $(M \times M)$-dimensional spectral density matrix

$$P_Z(\omega) = \begin{bmatrix} P_{X_1}(\omega) & P_{X_1X_2}(\omega) & P_{X_1Y}(\omega) \\ P_{X_2X_1}(\omega) & P_{X_2}(\omega) & P_{X_2Y}(\omega) \\ P_{X_YX_1}(\omega) & P_{YX_2}(\omega) & P_Y(\omega) \end{bmatrix} \equiv P_{[X_1X_2Y]}(\omega) \tag{5}$$

In (5), $P_{X_1}(\omega)$ is an $(M_1 \times M_1)$-dimensional matrix containing the PSD of the scalar process X_{1i} as $i-th$ diagonal element and the cross-PSD between X_{1i} and X_{1j} as (i,j) off-diagonal element, and $P_{X_1X_2}(\omega)$ is an $(M_1 \times M_2)$-dimensional matrix containing the cross-PSD between X_{1i} and X_{2j} as (i,j) element; the same notion follows intuitively for the matrices $P_{X_2}(\omega), P_Y(\omega), P_{X_1Y}(\omega)$ and $P_{X_2Y}(\omega)$, of dimension $M_2 \times M_2$, $M_Y \times M_Y$ and $M_2 \times M_Y$. Considering the overall spectral matrix $P_Z(\omega)$ and its constituent blocks, time and frequency domain measures of linear association between X_1 and Y, between X_2 and Y, and between $X = [X_1X_2]$ and Y, can be defined as in (1) and (3):

$$F_{X_1;Y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{|P_{X_1}(\omega)||P_Y(\omega)|}{|P_{[X_1Y]}(\omega)|} \mathrm{d}\omega, \tag{6a}$$

$$F_{X_2;Y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{|P_{X_2}(\omega)||P_Y(\omega)|}{|P_{[X_2Y]}(\omega)|} \mathrm{d}\omega, \tag{6b}$$

$$F_{X_1X_2;Y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{|P_{[X_1X_2]}(\omega)||P_Y(\omega)|}{|P_{[X_1X_2Y]}(\omega)|} \mathrm{d}\omega, \tag{6c}$$

where the arguments of the integrals correspond respectively to $f_{X_1;Y}$, $f_{X_2;Y}$ and $f_{X_1X_2;Y}$. Then, according to the principles whereby interaction information is defined for random variables [8], we define the following information-theoretic and spectral measures of source interaction:

$$I_{Y;X_1;X_2} \triangleq F_{X_1;Y} + F_{X_2;Y} - F_{X_1X_2;Y} \tag{7}$$

$$i_{Y;X_1;X_2}(\omega) \triangleq f_{X_1;Y}(\omega) + f_{X_2;Y}(\omega) - f_{X_1X_2;Y}(\omega). \tag{8}$$

If the two sources X_1 and X_2 exhibit a stronger additive degree of linear dependence with the target Y when they are considered separately than when they are considered together $(F_{X_1;Y} + F_{X_2;Y} > F_{X_1X_2;Y})$, the time-domain source interaction measure is positive $(I_{Y;X_1;X_2} > 0)$, denoting *redundancy*; if, on the contrary, the linear association between X_1 and X_2 considered jointly and Y is stronger than the sum of the linear association between X_1 and Y and X_2 and Y, $(F_{X_1;Y} + F_{X_2;Y} < F_{X_1X_2;Y})$, the time-domain source interaction measure is negative $(I_{Y;X_1;X_2} < 0)$, denoting *synergy*. The same properties are satisfied by the frequency domain source interaction measure, and hold for each specific frequency. Moreover, the time and frequency domain interaction measures satisfy the property of spectral integration: $I_{Y;X_1;X_2} = \int_{-\pi}^{\pi} i_{Y;X_1;X_2}(\omega)\,\mathrm{d}\omega$.

III. Theoretical Example

To illustrate the methodology implemented for the evaluation of frequency-domain multivariate interactions among coupled systems, we consider a theoretical example of four Gaussian systems whose associated processes are described by the VAR model with the equations:

$$Z_{1,n} = cZ_{2,n-1} + (1-c)Z_{3,n-2} + U_{1,n}, \tag{9a}$$

$$Z_{2,n} = 2\rho_2 \cos(2\pi f_2) Z_{2,n-1} - \rho_2^2 Z_{2,n-2} + \frac{1}{2} Z_{4,n-2} + U_{2,n}, \tag{9b}$$

$$Z_{3,n} = 2\rho_3 \cos(2\pi f_3) Z_{3,n-1} - \rho_3^2 Z_{3,n-2} + U_{3,n}, \tag{9c}$$

$$Z_{4,n} = 2\rho_4 \cos(2\pi f_4) Z_{4,n-1} - \rho_4^2 Z_{4,n-2} + U_{4,n}, \tag{9d}$$

where $\mathbf{U}_n = [U_{1,n} \cdots U_{4,n}]$ is a vector of zero mean white Gaussian noises with unit variance and uncorrelated with each other $(\Sigma_\mathbf{U} = \mathbf{I})$. The parameter design in (9) is chosen to allow autonomous oscillations in the processes $Z_i, i = 2,3,4$, obtained placing complex-conjugate poles with modulus $\rho_2 = 0.8$ and $\rho_3 = \rho_4 = 0.9$ and normalized frequencies $f_2/f_s = 0.1, f_3/f_s = 0.05$ and $f_4/f_s = 0.35$ which, assuming a sampling frequency $f_s = 100$ Hz determine oscillations at

10 Hz, 5 Hz and 35 Hz. Given (9), we assume $X_1 = Z_2$ and $X_2 = [Z_3, Z_4]$ as source processes, and $Y = Z_1$ as target process. Taking the Z-transform of (9) and exploiting the relation between the frequency version of transfer matrix ($\mathbf{H}(\omega)$) and the AR coefficients matrix, the 4×4 spectral density matrix can be obtained as $P_Z(\omega) = \mathbf{H}(\omega)\Sigma_{\mathbf{U}}\mathbf{H}^*(\omega)$ [9]. This leads to compute the exact values of all the time and frequency domain information measures defined in the Methods section for the theoretical process. We do this varying the parameter c which controls the strength of the connection between the sources X_1, X_2 and the target Y.

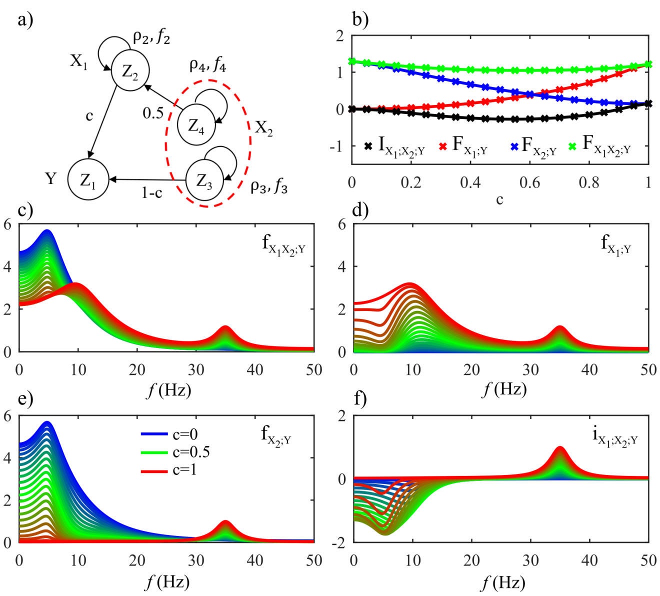

Fig. 1. Simulation scheme of the coupled linear stochastic processes Eq. (9) (panel a) and exemplary computation of time-domain and frequency-domain interaction measures (b,c,d,e and f). Frequency-domain measures are plotted with color coded by the coupling parameter c. Oscillations are introduced in the system at $f_2 = 10$ Hz, $f_3 = 5$ Hz and $f_4 = 35$ Hz.

The results of information decomposition performed in the time and frequency domains for the VAR process (9) are shown in Figure 1 (panel b, panels c-f, respectively). When the coupling parameter c is increased from 0 to 1, the coupling between the first source X_1 and the target Y increases, and the time-domain measure $F_{X_1;Y}$ (red line, panel b) also increases as a direct effect; an opposite trend is obtained for $F_{X_2;Y}$ (blue line, panel b) which decreases when the coupling c increases. Looking at the measure of source interaction ($I_{X_1;X_2;Y}$, black line, panel b), it is null when $c = 0$, while it highlights negative interaction information (denoting synergy) when $c = 0.5$, and becomes positive (denoting redundancy) when $c = 1$. The decomposition of the information measures in the frequency domain highlights interaction mechanisms which are specific for the oscillations simulated at 5 Hz, 10 Hz, and 35 Hz. Specifically, when c is low there is a transmission of the slow oscillation from X_2 to the target through the link $Z_3 \rightarrow Y$, with a peak at 5 Hz in $f_{X_2;Y}$ and $f_{X_1X_2;Y}$ (blue line, panels c,e), with no interaction information ($i_{X_1;X_2;Y} = 0$, panel f). When c = 1, there is a direct transmission of the oscillation at 10 Hz through the link $X_1 \rightarrow Y$ revealed by the peaks in $f_{X_1;Y}$ and $f_{X_1X_2;Y}$ (red line, panels c,d). There is also a less prominent peak at 35 Hz due to the indirect causal link of $Z_4 \in X_2$ with the target, mediated by X_1. In this condition, the interaction is fully redundant as shown by the peak in $i_{X_1;X_2;Y}$ at 35 Hz. When $0 < c < 1$, the coexistence of redundancy and synergy is evident looking at the spectral profile of the interaction information (panel f); importantly, this is not observable with the time-domain measure which highlights only a synergistic effect ($I_{X_1;X_2;Y} < 0$).

IV. Application to EEG Data

The analyzed dataset refers to EEG signals recorded from 64 electrodes and referenced to both mastoids, as per the international 10-10 system with a sampling frequency f_s = 160 Hz in 109 healthy subjects [10]. The conditions analyzed include a motor execution task where subjects were asked to close the right fist (RIGHT) and a resting state condition (REST). The raw signals were firstly detrended and filtered with a second-order Butterworth filter (band-pass, 5-35 Hz), and then segmented to extract ~ 22 trials for each condition and subject with a duration of 4 s each. For our analysis, we selected as sources X_1 and X_2 the multivariate vectors $[FC_2, FC_4, FC_6]$ and $[C_2, C_4, C_6]$ and as a target Y the group composed by $[C_1, C_3, C_5]$ (Figure 2 a). The data of each trial was checked for a restricted form of weak sense stationarity [9]. Then, for each subject, experimental condition and trial, parametric estimates of the spectral information measures were obtained as follows. A VAR model was identified on the nine selected time series, estimating its parameter through the least square method and setting the model order according to Bayesian Information Criterion [9]. The estimated VAR coefficients and covariance matrix of prediction errors were used to estimate the PSD matrix for the three multivariate processes collected in the process Z according to (5). Finally, estimates of the frequency-domain functions measuring the information shared between the target and the two sources taken separately (Eqs. (6a)-(6b)) or jointly (Eq. 6c) and the interaction information between target and sources (Eq.(8)) were obtained from the PSDs.

Fig. 2 depicts the grand-average over subjects and trials of the frequency profiles of each interaction measure, reported separately for the REST and RIGHT conditions (b-e), as well as the distribution across subjects of the measures integrated over the β frequency band (f). For each integrated measure, the distributions obtained during REST and RIGHT are compared with a two-sided Wilcoxon signed rank test for paired data in order to assess the statistical significance of the changes induced by the motor execution task (p-value < 0.05 was considered as statistically significant for each comparison). The trends of each frequency-domain measure show a prominent peak around 10 Hz (α band) and a reduction of the average value in the β band during RIGHT compared to REST . The decrease of the interaction within the β band was detected as statistically significant considering the information shared between the target and the source X_1 ($F_{X_1;Y}$) and between the target and the two

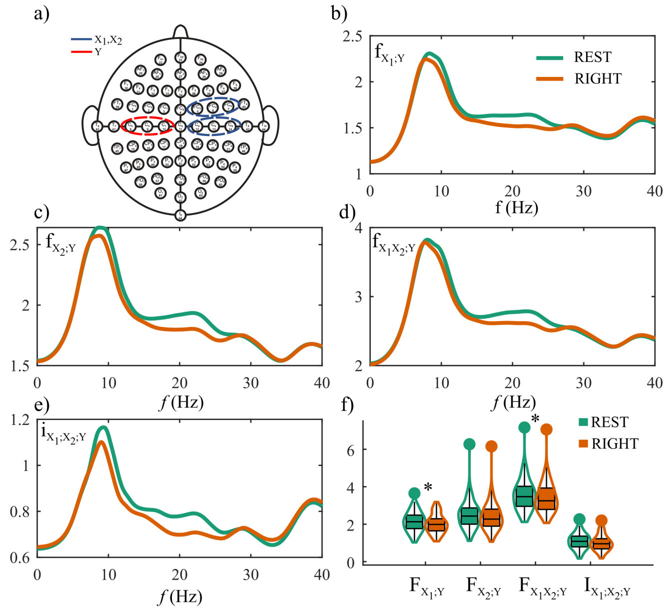

Fig. 2. (a) Overview of the EEG electrode montage highlighting the position of the source and target processes selected for the analysis. (b-e) Average spectral profiles of all the information measures analyzed; (f) violin plots depicting the distributions across subjects of the four different measures integrated over the β frequency range 16.5-20 Hz (probability densities estimated with the kernel density estimator; dots represent outliers). Statistical analysis (Wilcoxon test): *, $p < 0.05$ REST vs. RIGHT).

sources taken jointly ($F_{X_1,X_2;Y}$) (Fig. 2f). The interaction information was positive over the whole frequency profile, thus denoting the presence of redundant information in the system; redundancy was generally reduced across frequency during the execution of the motor task (Fig. 2e).

V. Discussion

We introduced a framework for the spectral decomposition of multivariate information measures of linear association and net redundancy/synergy. The framework was designed to assess the information shared between two sources and one target process, and was tested in both simulated and real EEG signals.

The simulation study was designed to show how redundant and synergistic effects between oscillations with different frequencies arise from the interplay of direct and indirect coupling mechanisms, and can be better elicited looking at the spectral profile of the various measures. We show that redundancy may arise as a consequence of indirect interaction effects such as the coupling between one source and the target mediated by the other source ($c = 1$, red line in Fig. 1f), and that redundancy may coexist with synergy arising from interactions that occur independently between each source and the target at nearby frequencies ($c = 0.5$, green in line Fig. 1f). Importantly, these frequency-specific interaction mechanisms may be hidden in the standard time-domain representation of information as a result of the integration between positive and negative interaction information values occurring at different frequencies (black line, Fig. 1b).

The spectral analysis of the information measures computed for the scalp EEG of 109 subjects during a motor execution task revealed that information is exchanged mainly in two different frequency ranges: the α band associated with the so-called mu-rhythms, displaying a clear peak of the information shared, and the β band, displaying a plateau of the spectral information profiles. The activity in both bands can be related with the physiology of motor execution, for which movement and preparation for movement are typically accompanied by a decrease in mu and β rhythms, particularly in the scalp areas controlateral to the movement [11]. Such expected result was reflected in the decrease of the information measures during the RIGHT condition. The analysis focused on the β band showed a reduction of the interaction measures moving from REST to RIGHT; the decrease was statistically significant for the information shared jointly between the the target Y and the source X_1. This was not the case for the information shared between the source X_2 and the target, which are associated with ipsilateral and controlateral motor cortex activities respectively. Due to the known bilateral desynchronization in the β band associated with right or left hand movement [11], it is reasonable to assume that an overlapped activity between X_2 and Y can result in redundancy. These results are also in line with a previous study analyzing the topology of brain networks estimated through causal spectral measures, showing a strong involvement of both ispilateral and controlateral hemispheres associated with right hand motor imagery task [12].

References

[1] A. B. Barrett, “Exploration of synergistic and redundant information sharing in static and dynamical gaussian systems,” *Physical Review E*, vol. 91, no. 5, p. 052802, 2015.

[2] O. M. Cliff *et al.*, “Assessing the significance of directed and multivariate measures of linear dependence between time series,” *Physical Review Research*, vol. 3, no. 1, p. 013145, 2021.

[3] L. Faes *et al.*, “Multiscale information decomposition: Exact computation for multivariate gaussian processes,” *Entropy*, vol. 19, no. 8, p. 408, 2017.

[4] S. Stramaglia *et al.*, “Synergy and redundancy in the granger causal analysis of dynamical networks,” *New Journal of Physics*, vol. 16, no. 10, p. 105003, 2014.

[5] A. G. Nedungadi *et al.*, “Block coherence: a method for measuring the interdependence between two blocks of neurobiological time series,” *Biological cybernetics*, vol. 104, no. 3, pp. 197–207, 2011.

[6] D. Y. Takahashi, L. A. Baccalá, and K. Sameshima, “Canonical information flow decomposition among neural structure subsets,” *Frontiers in neuroinformatics*, vol. 8, p. 49, 2014.

[7] J. Geweke, “Measurement of linear dependence and feedback between multiple time series,” *Journal of the American statistical association*, vol. 77, no. 378, pp. 304–313, 1982.

[8] W. McGill, “Multivariate information transmission,” *Trans. IRE Prof. Group Inf. Theory*, vol. 4, no. 4, pp. 93–111, 1954.

[9] L. Faes *et al.*, “Measuring connectivity in linear multivariate processes: definitions, interpretation, and practical analysis,” *Computational and mathematical methods in medicine*, vol. 2012, 2012.

[10] A. L. Goldberger *et al.*, “Physiobank, physiotoolkit, and physionet: components of a new research resource for complex physiologic signals,” *circulation*, vol. 101, no. 23, pp. e215–e220, 2000.

[11] J. R. Wolpaw *et al.*, “Brain–computer interfaces for communication and control,” *Clinical neurophysiology*, vol. 113, no. 6, pp. 767–791, 2002.

[12] Y. Antonacci *et al.*, “Single-trial connectivity estimation through the least absolute shrinkage and selection operator,” in *2019 41st Annual International Conference of the IEEE Engineering in Medicine and Biology Society (EMBC)*. IEEE, 2019, pp. 6422–6425.